ON SUMS AND PRODUCTS OF INTEGERS

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(Communicated by David E. Rohrlich)

ABSTRACT. Erdős and Szemerédi proved that if A is a set of k positive integers, then there must be at least $ck^{1+\delta}$ integers that can be written as the sum or product of two elements of A , where c is a constant and $\delta > 0$. Nathanson proved that the result holds for $\delta = \frac{1}{31}$. In this paper it is proved that the result holds for $\delta = \frac{1}{5}$ and $c = \frac{1}{20}$.

1. INTRODUCTION

Let $h \geq 2$, and let A be a finite set of positive integers. Let

$$hA = \{a_1 + a_2 + \cdots + a_h \mid a_i \in A \text{ for } i = 1, \dots, h\},$$

$$A^h = \{a_1 a_2 \cdots a_h \mid a_i \in A \text{ for } i = 1, \dots, h\}.$$

We let

$$E_h(A) = hA \cup A^h.$$

It is not difficult to see that (see [3])

$$|E_h(A)| \leq \frac{2}{h!} k^h + O(k^{h-1}).$$

Erdős and Szemerédi [1, 2] conjectured that for every $\varepsilon > 0$,

$$|E_h(A)| \gg k^{h-\varepsilon}.$$

For $h = 2$, Nathanson and Tenenbaum [4] have proved that if $|A| = k$ and $|2A| \leq 3k - 4$, then

$$|A^2| \gg k^{2-\varepsilon}.$$

Erdős and Szemerédi [2] showed that there exists a real number $\delta > 0$ such that

$$|E_2(A)| \gg |A|^{1+\delta}.$$

Recently Nathanson [3] proved that the result holds for $\delta = \frac{1}{31}$. In this paper I show that the result holds for $\delta = \frac{1}{5}$.

Received by the editors September 24, 1997.

1991 *Mathematics Subject Classification*. Primary 11B05, 11B13, 11B75, 11P99, 05A17.

Key words and phrases. Additive number theory, sumsets, sums and products of integers.

This research was supported by the Fok Ying Tung Education Foundation and the National Natural Science Foundation of China.

2. THE MAIN RESULT

Lemma 1. *Let a_i ($i \in I$) be nonnegative real numbers and $\beta > 1$. Then*

$$\sum_{i \in I} a_i^\beta \geq |I|^{1-\beta} \left(\sum_{i \in I} a_i \right)^\beta.$$

In particular,

$$\sum_{i \in I} a_i^2 \geq |I|^{-1} \left(\sum_{i \in I} a_i \right)^2.$$

Proof. By the Hölder inequality we have

$$\sum_{i \in I} a_i \leq \left(\sum_{i \in I} 1^{\beta/(\beta-1)} \right)^{(\beta-1)/\beta} \left(\sum_{i \in I} a_i^\beta \right)^{1/\beta}.$$

Then the lemma follows immediately.

Lemma 2. *Let B be a nonempty, finite set of positive integers such that*

$$\max(B) \leq 2 \min(B).$$

Then

$$|E_2(B)| \geq \frac{7}{5} \sqrt{2} \left(\frac{|B|}{5} \right)^{5/4}.$$

Proof. If $\min(B) \leq 4$, it is trivial, so we may assume that $\min(B) > 4$. Thus

$$(1) \quad B^2 \cap (2B) = \emptyset.$$

Let $|B| = k$ and

$$l = \left\lceil \left(\frac{k}{5} \right)^{1/2} \right\rceil.$$

If $|B| < 12500$, then

$$|E_2(B)| \geq 2(2|B| - 1) \geq 3|B| \geq 2 \left(\frac{|B|}{5} \right)^{5/4},$$

so we may also assume that $|B| \geq 12500$. Thus $l \geq 49$. Let

$$\begin{aligned} B &= \{b_1, b_2, \dots, b_k\}, & b_1 &< b_2 < \dots < b_k \leq 2b_1, \\ d_i &= b_{i+l-1} - b_i, & d_{i_0} &= \min d_i, \\ B^* &= \{b_{i_0}, b_{i_0+1}, \dots, b_{i_0+l-1}\}, \\ B_i &= \{b_{2il+1}, b_{2il+2}, \dots, b_{2il+2l}\}, \\ E(B^*, B_i) &= (B^* + B_i) \cup B^* B_i. \end{aligned}$$

If $b_1^*, b_2^* \in B^*$ and $j - i \geq 2l - 1$, then

$$\begin{aligned} b_2^* + b_j - (b_1^* + b_i) &= b_2^* - b_1^* + b_j - b_i > -d_{i_0} + d_i \geq 0, \\ b_2^* b_j - b_1^* b_i &= \left(b_1^* - \frac{1}{2}b_j\right)(b_j - b_i) + b_j \left(\frac{b_j - b_i}{2} + b_2^* - b_1^*\right) \\ &> b_j \left(\frac{d_i + d_{i+l-1}}{2} - d_{i_0}\right) \geq 0. \end{aligned}$$

Hence, if $j - i \geq 2$, then

$$\begin{aligned} (B^* + B_i) \cap (B^* + B_j) &= \emptyset, \\ (2) \quad B^* B_i \cap B^* B_j &= \emptyset. \end{aligned}$$

Let

$$\begin{aligned} B(i, u, v) &= \{(b_1^*, b_2^*, b_3^*, b_4^*) \in (B^*)^4 : b_1^* \neq b_2^* \\ &\text{and there exist } b'_3, b'_4 \in B_i \text{ such that} \\ &b_1^* + b'_3 = b_2^* + b'_4 = u, b_3^* b'_3 = b_4^* b'_4 = v\}. \end{aligned}$$

Suppose that

$$B(i_1, u_1, v_1) \cap B(i_2, u_2, v_2) \neq \emptyset,$$

say

$$(b_1^*, b_2^*, b_3^*, b_4^*) \in B(i_1, u_1, v_1) \cap B(i_2, u_2, v_2).$$

Then

$$(3) \quad b_1^* + x = b_2^* + y, \quad b_3^* x = b_4^* y$$

has at least one solution in integers x, y . It is easy to see that (3) has at most one solution. So (3) has a unique solution x_0, y_0 . Thus

$$x_0 \in B_{i_1} \cap B_{i_2}, \quad u_1 = b_1^* + x_0 = u_2, \quad v_1 = b_3^* x_0 = v_2,$$

whence $i_1 = i_2, u_1 = u_2, v_1 = v_2$. Hence

$$(4) \quad \sum_{i, u, v} |B(i, u, v)| \leq |B^*|^4 - |B^*|^3 = l^4 - l^3.$$

Let

$$\begin{aligned} B_i^{(v)} &= \{b : b \in B_i \text{ and there exists a } b^* \in B^* \text{ such that } b^* b = v\}, \\ \rho_{i, v}(u) &= |\{b^* : b^* \in B^* \text{ and there exists a } b' \in B_i^{(v)} \text{ such that } b^* + b' = u\}|. \end{aligned}$$

Then

$$\begin{aligned} |B(i, u, v)| &\geq \rho_{i, v}(u)(\rho_{i, v}(u) - 1), \\ (5) \quad \sum_{u \in B^* + B_i^{(v)}} \rho_{i, v}(u) &= |B^*| |B_i^{(v)}| = l |B_i^{(v)}|. \end{aligned}$$

By Lemma 1 and (5) we have

$$\begin{aligned}
\sum_{u,v} |B(i, u, v)| &\geq \sum_{v \in B^* B_i} \sum_{u \in B^* + B_i^{(v)}} |B(i, u, v)| \\
&\geq \sum_{v \in B^* B_i} \sum_{u \in B^* + B_i^{(v)}} \rho_{i,v}(u) (\rho_{i,v}(u) - 1) \\
&\geq \sum_{v \in B^* B_i} \left(|B^* + B_i^{(v)}|^{-1} \left(\sum_{u \in B^* + B_i^{(v)}} \rho_{i,v}(u) \right)^2 - \sum_{u \in B^* + B_i^{(v)}} \rho_{i,v}(u) \right) \\
&\geq \sum_{v \in B^* B_i} (|B^* + B_i^{(v)}|^{-1} l^2 |B_i^{(v)}|^2 - l |B_i^{(v)}|) \\
&\geq l^2 |B^* + B_i|^{-1} \sum_{v \in B^* B_i} |B_i^{(v)}|^2 - l \sum_{v \in B^* B_i} |B_i^{(v)}| \\
&\geq l^2 |B^* + B_i|^{-1} |B^* B_i|^{-1} \left(\sum_{v \in B^* B_i} |B_i^{(v)}| \right)^2 - l \sum_{v \in B^* B_i} |B_i^{(v)}| \\
&\geq 4l^6 |B^* + B_i|^{-1} |B^* B_i|^{-1} - 2l^3 \\
&\geq 16l^6 (|B^* + B_i| + |B^* B_i|)^{-2} - 2l^3 \\
&\geq 16l^6 |E(B^*, B_i)|^{-2} - 2l^3.
\end{aligned}$$

Let

$$I_1 = \{i : |E(B^*, B_i)| < 2l^{3/2}\} \quad \text{and} \quad I_2 = \{i : |E(B^*, B_i)| \geq 2l^{3/2}\}.$$

If $I_1 \neq \emptyset$, then by (4) and the inequality just above we have

$$\begin{aligned}
2l^3 |I_1| &< \sum_{i \in I_1} (16l^6 |E(B^*, B_i)|^{-2} - 2l^3) \\
&\leq \sum_{i \in I_1} \sum_{u,v} |B(i, u, v)| \leq l^4 - l^3.
\end{aligned}$$

So

$$\begin{aligned}
|I_1| &< \frac{1}{2}l - \frac{1}{2}, \quad |I_1| \leq \frac{1}{2}l - 1, \\
|I_2| &\geq \left\lceil \frac{k}{2l} \right\rceil - \frac{1}{2}l + 1 \geq \frac{k}{2l} - \frac{1}{2}l.
\end{aligned}$$

By (1), (2) and $l \geq 49$ we have

$$\begin{aligned}
|E_2(B)| &\geq \left| \bigcup_{i \in I_2} E(B^*, B_i) \right| \geq \frac{1}{2} \sum_{i \in I_2} |E(B^*, B_i)| \geq \frac{1}{2} \cdot 2 |I_2| l^{3/2} \\
&\geq \frac{1}{2} (k - l^2) (l + 1)^{1/2} \left(\frac{l}{l + 1} \right)^{1/2} \geq \frac{1}{2} \left(k - \frac{1}{5}k \right) \left(\frac{1}{5}k \right)^{1/4} \left(\frac{49}{50} \right)^{1/2} \\
&\geq \frac{7}{5} \sqrt{2} \left(\frac{k}{5} \right)^{5/4}.
\end{aligned}$$

This completes the proof of Lemma 2.

Theorem. *Let A be a nonempty, finite set of positive integers. Then*

$$|E_2(A)| \geq \frac{1}{20}|A|^{6/5}.$$

Proof. Let

$$|A| = k, \quad c_1 = \frac{7}{5}\sqrt{2}5^{-5/4}, \quad A_j = [2^{j-1}, 2^j) \cap A.$$

Since

$$\left| \bigcup_{j=1}^{\infty} A_j \right| = k,$$

without loss of generality, we may assume that

$$\left| \bigcup_{j=1}^{\infty} A_{2j-1} \right| \geq \frac{1}{2}k.$$

Let $j_1, j_2, \dots, j_T$ be all positive odd integers with $A_{j_i} \neq \emptyset$. Let

$$\sum_{t \leq 0} |A_{j_t}| = 0.$$

Since

$$\sum_{t \leq T} |A_{j_t}| \geq \frac{1}{2}k,$$

there exists an integer t_0 such that

$$\sum_{t \leq t_0} |A_{j_t}| \geq \frac{1}{12}k, \quad \sum_{t \leq t_0-1} |A_{j_t}| < \frac{1}{12}k.$$

Thus

$$\sum_{t \geq t_0} |A_{j_t}| = \sum_{t=1}^T |A_{j_t}| - \sum_{t \leq t_0-1} |A_{j_t}| \geq \frac{5}{12}k.$$

If $i < u$, then $\max(A_{j_i}) < \min(A_{j_u})$, whence

$$A_{j_i}^2 \cap A_{j_u}^2 = \emptyset, \quad 2A_{j_i} \cap 2A_{j_u} = \emptyset.$$

Hence each integer n belongs to at most two of the sets $E_2(A_{j_i})$. Therefore, by Lemma 1,

(6)

$$\begin{aligned} E_2(A) &\geq \left| \bigcup_{t=t_0}^T E_2(A_{j_t}) \right| \geq \frac{1}{2} \sum_{t=t_0}^T |E_2(A_{j_t})| \geq \frac{1}{2} c_1 \sum_{t=t_0}^T |A_{j_t}|^{5/4} \\ &\geq \frac{1}{2} c_1 (T - t_0 + 1)^{-1/4} \left(\sum_{t \geq t_0} |A_{j_t}| \right)^{5/4} \geq \frac{1}{2} c_1 (T - t_0 + 1)^{-1/4} \left(\frac{5}{12}k \right)^{5/4}. \end{aligned}$$

Suppose that $i < u$ and

$$\begin{aligned} a'_i &\in \bigcup_{t \leq i} A_{j_t}, & a_i &\in A_{j_i}, \\ a'_u &\in \bigcup_{t \leq u} A_{j_t}, & a_u &\in A_{j_u}. \end{aligned}$$

Then

$$a'_i + a_i \leq 2^{j_i+1} \leq 2^{j_{i+1}-1} \leq a_u < a'_u + a_u.$$

Hence

$$\left(\bigcup_{t \leq i} A_{j_t} + A_{j_i} \right) \cap \left(\bigcup_{t \leq u} A_{j_t} + A_{j_u} \right) = \emptyset, \quad \text{if } i \neq u.$$

Therefore

$$\begin{aligned} (7) \quad E_2(A) &\geq \left| \bigcup_{i=1}^T \left(\bigcup_{t \leq i} A_{j_t} + A_{j_i} \right) \right| \geq \sum_{i=1}^T \left| \bigcup_{t \leq i} A_{j_t} + A_{j_i} \right| \\ &\geq \sum_{i=1}^T \left| \bigcup_{t \leq i} A_{j_t} \right| = \sum_{i=1}^T \sum_{t \leq i} |A_{j_t}| \\ &\geq (T - t_0 + 1) \sum_{t \leq t_0} |A_{j_t}| \geq \frac{1}{12} (T - t_0 + 1) k. \end{aligned}$$

Therefore, by (6) and (7) we have

$$\begin{aligned} E_2(A) &\geq \max \left\{ \frac{1}{2} c_1 (T - t_0 + 1)^{-1/4} \left(\frac{5}{12} k \right)^{5/4}, (T - t_0 + 1) \frac{1}{12} k \right\} \\ &\geq (6c_1)^{4/5} \frac{5}{144} k^{6/5} \\ &> \frac{1}{20} k^{6/5}. \end{aligned}$$

This completes the proof of the theorem.

ADDED IN PROOF

One should also refer to Elekes, Gy., *Acta Arith.* **81** (1997), 365–367, for a very different method and Kevin Ford, *The Ramanujan J.* **2** (1998), 59–66.

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