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FIXED POINTS FOR OPERATORS IN A SPACE OF CONTINUOUS FUNCTIONS AND APPLICATIONS

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ABSTRACT. This paper investigates the fixed points for self-maps of a closed set in a space of abstract continuous functions. Our main results essentially extend the Banach contracting mapping principle. An application to integro-differential equations is given.

1. Introduction

Let E be a real Banach space with norm $||\cdot||$, $I=[0,T]$ $(T>0)$. Denote $C[I,E]=\{u:I\to E \mid u(t) \text{ is continuous on } I\}$. It is easy to see that $C[I,E]$ is a Banach space with the norm $\|u\|_C=\max\limits_{t\in I}\|u(t)\|$ for $u\in C[I,E]$. In this paper we investigate the fixed points for self-maps of a closed set in $C[I,E]$. We show that our main theorem extends the Banach contracting mapping principle in $C[I,E]$. Finally, an application to integro-differential equations is given.

2. Main results

Theorem 2.1. *Let F be a closed subset of $C[I,E]$ and $A:F\to F$ an operator. If there exist $\alpha,\ \beta\in[0,1),\ K\geq 0$ such that for any $u,v\in F$,*

(2.1)
$$\|Au(t)-Av(t)\|\leq\beta\|u(t)-v(t)\|+\frac{K}{t^\alpha}\int_0^t\|u(s)-v(s)\|ds,\quad \forall\ t\in(0,T],$$

then A has exactly one fixed point u^* in F. For any $x_0\in F$, the iterative sequence $x_n=Ax_{n-1}$ $(n=1,2,3,\cdots)$ converges to u^* in F and for all $s>0$,

$$\|x_n-u^*\|_C=o(n^{-s})\qquad (as\ n\to\infty).$$

Proof. For any $u_0\in F$, set $u_n=Au_{n-1}$ $(n=1,2,3,\cdots)$. By (2.1) we get

$$\|u_2(t)-u_1(t)\|\leq(\beta+Kt^{1-\alpha})\|u_1-u_0\|_C,\qquad\forall\ t\in(0,T].$$

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It follows by induction and (2.1) that, for any $t \in (0,T]$,

$$\begin{aligned} \|u_{n+1}(t) - u_n(t)\| \quad \le \quad & \left(\beta^n + \begin{pmatrix} n \\ 1 \end{pmatrix} \beta^{n-1} K t^{1-\alpha} + \frac{\begin{pmatrix} n \\ 2 \end{pmatrix} \beta^{n-2} K^2 t^{2-2\alpha}}{2-\alpha} + \cdots \right. \\ & \left. + \frac{K^n t^{n-n\alpha}}{(2-\alpha)(3-2\alpha)\cdots(n-(n-1)\alpha)} \right) \|u_1 - u_0\|_C, \end{aligned}$$

$n = 1, 2, 3, \cdots$. Therefore,

(2.2)

$$\|u_{n+1} - u_n\|_C \le \left(\beta^n + \begin{pmatrix} n \\ 1 \end{pmatrix} \beta^{n-1} h + \frac{\begin{pmatrix} n \\ 2 \end{pmatrix} \beta^{n-2} h^2}{2!} + \cdots + \frac{h^n}{n!} \right) \|u_1 - u_0\|_C,$$

where $h = KT^{1-\alpha}(1-\alpha)^{-1}$. It is easy to see that

$$\lim_{k \to \infty} \left(\beta^{k-1} k \left(\frac{k}{k-1} \right)^{k-1} \right)^{1/k} = \beta < 1,$$

hence we can choose a fixed integer $k > 2$ such that

$$\left(\beta^{k-1} k \left(\frac{k}{k-1} \right)^{k-1} \right)^{1/k} \equiv g < 1.$$

For any n, set $n = km + j$ $(0 \le j < k)$, where k is given as above. Then whenever n is sufficiently large, it follows from the Stirling formula that

$$\begin{aligned} S_1 \quad & \equiv \quad \beta^n + \begin{pmatrix} n \\ 1 \end{pmatrix} \beta^{n-1} h + \frac{\begin{pmatrix} n \\ 2 \end{pmatrix} \beta^{n-2} h^2}{2!} + \cdots + \frac{\begin{pmatrix} n \\ m \end{pmatrix} \beta^{n-m} h^m}{m!} \\ & \le \quad \beta^{n-m} \begin{pmatrix} n \\ m \end{pmatrix} \left(1 + h + \frac{h^2}{2!} + \cdots + \frac{h^m}{m!} \right) = O(1) \beta^{n-m} \begin{pmatrix} n \\ m \end{pmatrix} \\ & = \quad \frac{O(1) \beta^{n-m} n^n \sqrt{2\pi n} (1 + O(\frac{1}{m}))}{m^m \sqrt{2\pi m} \sqrt{2\pi (n-m)} (n-m)^{n-m}} = O\left(\frac{k^m}{\sqrt{m}} \right) \left(\frac{\beta n}{n-m} \right)^{n-m} \\ & = \quad O\left(\frac{\left(\beta^{k-1} k \left(\frac{k}{k-1} \right)^{k-1} \right)^m}{\sqrt{m}} \right) = O\left(\frac{g^{km}}{\sqrt{m}} \right) \quad = \quad O\left(\frac{g^n}{\sqrt{n}} \right). \end{aligned}$$

Similarly,

$$\begin{aligned} S_2 &\equiv \frac{\binom{n}{m+1}\beta^{n-m-1}h^{m+1}}{(m+1)!} + \cdots + \frac{h^n}{n!} \\ &\leq \frac{\binom{n}{[\frac{n}{2}]}}{(m+1)!}\left(\beta^{n-m-1}h^{m+1} + \cdots + h^n\right) \\ &= \frac{O\left(\frac{2^n}{\sqrt{n}}\right)e^{m+1}\left(\beta^{n-m-1}h^{m+1} + \cdots + h^n\right)}{\sqrt{2\pi(m+1)}(m+1)^{m+1}(1+O(\frac{1}{m+1}))} \\ &= o\left(\frac{1}{(m+1)^s}\right) \quad = \quad o\left(\frac{1}{n^s}\right) \qquad (\text{as} \ \ n \to \infty), \end{aligned}$$

where $s > 1$ can be any real constant.

Consequently, by (2.2) we have

$$\begin{aligned} \|u_{n+1} - u_n\|_C &\leq (S_1 + S_2)\|u_1 - u_0\|_C \\ &= O\left(\frac{g^n}{\sqrt{n}}\right) + o\left(\frac{1}{n^s}\right) = o\left(\frac{1}{n^s}\right) \quad (\text{as} \ \ n \to \infty), \end{aligned} \tag{2.3}$$

which implies that, for any fixed $s > 0$, there exists $n_0 > 0$ such that

$$\|u_{n+1} - u_n\|_C < \frac{1}{n^{s+1}}, \qquad \forall \ n > n_0.$$

Therefore, for any $q > 0, \ n > n_0$, we have

$$\|u_n - u_{n+q}\|_C \leq \|u_n - u_{n+1}\|_C + \cdots + \|u_{n+q-1} - u_{n+q}\|_C < \sum_{i=n}^{\infty} \frac{1}{i^{s+1}}.$$

Since (see, e.g. [1])

$$\sum_{i=n}^{\infty} \frac{1}{i^{s+1}} = \frac{1}{s(n-1)^s} + o\left(\frac{1}{(n-1)^{s+1}}\right) \quad (\text{as} \ \ n \to \infty),$$

we have $\|u_n - u_{n+q}\|_C = O\left(\frac{1}{n^s}\right)$ $(\forall \ s > 0)$. Hence $\{u_n\}$ is a Cauchy sequence and there exists $u^* \in F$ such that $\|u_n - u^*\|_C \to 0$ as $n \to \infty$. By (2.1),

$$\begin{aligned} \|Au^*(t) - u^*(t)\| &\leq \|Au^*(t) - Au_n(t)\| \ + \ \|Au_n(t) - u^*(t)\| \\ &\leq (\beta + Kt^{1-\alpha})\|u_n - u^*\|_C + \|u_{n+1} - u^*\|_C, \quad \forall \ t \in (0, T], \end{aligned}$$

and so

$$\|Au^* - u^*\|_C \leq (\beta + KT^{1-\alpha})\|u_n - u^*\|_C + \|u_{n+1} - u^*\|_C,$$

which implies by $||u_n - u^*||_C \to 0$ $(n \to \infty)$ that $Au^* = u^*$.

For any $x_0 \in F$, set $x_n = Ax_{n-1}$ $(n = 1, 2, 3, \cdots)$. By (2.1) and using a similar way as establishing (2.3) we can get, for any $s > 0$,

$$\|x_n - u^*\|_C = o\left(\frac{1}{n^s}\right) \quad (\text{as} \ \ n \to \infty),$$

which means that u^* is the unique fixed point of A since $x_0 \in F$ is arbitrary. This completes the proof.

Remark 2.1. We show that Theorem 2.1 is a generalization of the Banach contraction mapping principle in $C[I, E]$.

On one hand, it is easy to give some self-maps of a closed subset of $C[I,E]$, which satisfy (2.1) but are not contractions. For example, operator $A : C[J,E] \to C[J,E]$ ($J = [0,1]$) defined by

$$Au(t) = \frac{1}{2}u(t) + 2t^{-\frac{1}{2}} \int_0^t u(s)ds, \quad \forall\, t \in (0,1], \qquad Au(0) = \frac{1}{2}u(0)$$

is such a map.

On the other hand, if F is a closed subset of a Banach space E, operator $A : F \to F$ satisfies

$$\|Au - Av\| \le \alpha \|u - v\|, \quad \forall\, u, v \in F, \tag{2.4}$$

where $\alpha \in [0,1)$. Then Banach's theorem shows that A has exactly one fixed point in F. We assert that this conclusion can also be obtained by Theorem 2.1. In fact, we can embed F into $C[I,E]$ by regarding the elements of F as constant-value functions of $C[I,E]$. Then F is a closed set in $C[I,E]$ and $A : F \to F$ can be regarded as a map in $C[I,E]$. So (2.4) implies that A satisfies (2.1) for $K = 0$ and then, in the subset F of $C[I,E]$, A has exactly one fixed point by Theorem 2.1, which is the unique fixed point of A in the subset F of E.

Remark 2.2. Considering the inequality (2.1), it seems that the right side of (2.1) may induce some new norms of $C[I,E]$ such that the contraction mapping principle can be applied in terms of such a new norm. We show that, even in special cases when new norms can be found, Theorem 2.1 cannot yet be replaced by the contraction mapping principle.

For example, let $E = R^1, \beta > 0, \alpha = 0, K = 1$. Then a natural norm of $C[I, R^1]$ relative to the right side of (2.1) is $||\cdot||_X$ defined by

$$||u||_X = \frac{\beta}{\theta}||u||_C + \frac{1}{\theta}\int_0^{t_0} |u(s)|ds,$$

where $0 < \theta < 1$ may be any fixed real, $0 < t_0 \le 1$ is a constant. (Although other norms can also be defined, the analogues of the following discussion are valid for them.) There are examples to show that operator A may satisfy (2.1) and consequently,

$$||Au - Av||_C \le \theta ||u - v||_X, \tag{2.5}$$

but does not satisfy

$$||Au - Av||_X \le \theta ||u - v||_X. \tag{2.6}$$

Hence the contraction mapping principle cannot be applied to A in terms of $||\cdot||_X$, but Theorem 2.1 can. The following is such an example:

$$Au(t) = \beta u(t) + \int_0^t u(s)ds, \qquad u \in C[I, R^1],$$

where $1 > \beta > ((4t_0 - t_0^2)^{1/2} - t_0)/2$. Clearly, A satisfies (2.1). But for any $u(t), v(t) \in C[I, R^1]$ with $u(t) \equiv u, v(t) \equiv v$ and $u > v$, we have

$$Au(t) - Av(t) = (\beta + t)(u - v).$$

So

$$||Au - Av||_X = \frac{1}{\theta}(\beta^2 + \beta + \beta t_0 + \frac{t_0^2}{2})(u - v), \qquad ||u - v||_X = \frac{\beta + t_0}{\theta}(u - v).$$

Hence (2.6) is not satisfied for A in $C[I, R^1]$ since $\beta > ((4t_0 - t_0^2)^{1/2} - t_0)/2$.

As we proved Theorem 2.1, we can similarly prove

Theorem 2.2. *Let $F \subset C[I, E]$ be a closed set and $A : F \to F$ an operator. If there exist $\alpha,\ \beta \in [0, 1),\ K \geq 0$, where α satisfies $(-1)^\alpha = -1$, such that, for some fixed $\eta \in I = [0, T]$ and for any $u, v \in F$,*

$$\|Au(t) - Av(t)\| \leq \beta \|u(t) - v(t)\| + \frac{K}{(t-\eta)^\alpha} \int_\eta^t \|u(s) - v(s)\| ds, \quad \forall\ t \in I \backslash \{\eta\},$$

then the conclusions of Theorem 2.1 hold.

3. An application

Consider the integro-differential equation of mixed type:

$$u'(t) = f(t, u, Tu, Su), \quad t \in J \equiv [0, 1]; \qquad u(0) = u_0, \tag{3.1}$$

where $f \in C[J \times R^1 \times R^1 \times R^1, R^1]$, $u_0 \in R^1$ and

$$Tu(t) = \int_0^t k(t, s)u(s)ds, \quad Su(t) = \int_0^1 h(t, s)u(s)ds,$$

with $k \in C[\Omega, R_+], \Omega = \{(t, s) \in R^2 | 0 \leq s \leq t \leq 1\}$, $h \in C[J \times J, R_+]$. Set $k_0 = \max\limits_{(t,s)\in\Omega} k(t, s)$, $h_0 = \max\limits_{t,s\in J} h(t, s)$. We will use the following conditions:

(H_1) There exist $p, q \in C^1[J, R^1]$, $p(t) \leq q(t)$ $(t \in J)$ such that

$$p' \leq f(t, p, Tp, Sp), \quad p(t) \leq u_0; \qquad q' \geq f(t, q, Tq, Sq), \quad q(t) \geq u_0.$$

(H_2) There exist $M > 0, R \geq 0$ and $Q \geq 0$ such that

$$f(t, u, v, w) - f(t, \overline{u}, \overline{v}, \overline{w}) \geq -M(u - \overline{u}) - R(v - \overline{v}) - Q(w - \overline{w})$$

for $t \in J, p(t) \leq \overline{u} \leq u \leq q(t)$, $Tp(t) \leq \overline{v} \leq v \leq Tq(t)$, $Sp(t) \leq \overline{w} \leq w \leq Sq(t)$.

Theorem 3.1. *Suppose that* (H_1) *and* (H_2) *are satisfied and that*

$$(Rk_0 + Qh_0)(e^M - 1) \leq M, \quad Qh_0(e^M - 1) < M. \tag{3.2}$$

Then there exist monotone sequences $\{p_n(t)\}, \{q_n(t)\} \subset C^1[J, R^1]$ such that

$$p(t) = p_0(t) \leq p_1(t) \leq \cdots \leq p_n(t) \leq \cdots \leq q_n(t) \leq \cdots \leq q_1(t) \leq q_0(t) = q(t)$$

and $p_n(t) \to u_*(t)$, $q_n(t) \to u^*(t)$ as $n \to \infty$ uniformly in $t \in J$, u_*, $u^* \in C^1[J, R^1]$. Moreover, u_* and u^* are minimal and maximal solutions of IVP (3.1) *on the interval $[p, q]$, respectively.*

Proof. For any $\eta \in U \equiv \{\eta(t) \in C[J, R^1] | p \leq \eta \leq q\}$, consider the linear IVP

$$u' = \sigma(t) - Mu - RTu - QSu, \quad u(0) = u_0, \tag{3.3}$$

where $\sigma(t) = f(t, \eta(t), T\eta(t), S\eta(t)) + M\eta(t) + RT\eta(t) + QS\eta(t)$. It is known that $u \in C^1[J, R^1]$ is a solution of (3.1) if and only if u is a solution in $C[J, R^1]$ of the integral equation

$$u(t) = e^{-Mt} \left\{ u_0 + \int_0^t e^{Ms}(\sigma(s) - RTu(s) - QSu(s))ds \right\} \equiv Bu(t). \tag{3.4}$$

For any $u, v \in C[J, R^1]$,

$$\begin{aligned}
|Bu(t) - Bv(t)| &= e^{-Mt}\int_0^t e^{Ms}|RTv(s) - RTu(s) + QSv(s) - QSu(s)|ds \\
&\le Re^{-Mt}\int_0^t e^{Ms}\left[\int_0^s k(s,r)|v(r) - u(r)|dr\right]ds \\
&\quad +Q\left|e^{-Mt}\int_0^1 (v(r) - u(r))H(t,r)dr\right| \\
&\le Rk_0\frac{e^{Mt}-1}{M}\int_0^t |u(r) - v(r)|dr + |L(u(t) - v(t))| \\
&\le K\int_0^t |u(r) - v(r)|dr + |L(u(t) - v(t))|, \quad \forall\ t \in J,
\end{aligned}$$

where

$$H(t,r) = \int_0^t e^{Ms}h(s,r)ds, \qquad Lu(t) = Q\int_0^1 H(t,r)u(r)dr$$

and

$$K = Rk_0(e^M - 1)M^{-1}.$$

By (3.2) we know that $\|L\| < 1$, and consequently Theorem 2.1 shows that B has exactly one fixed point in $C[J, R^1]$, that is, (3.3) has exactly one solution $u \in C^1[J, R^1]$.

Define $A\eta = u$, where u is the unique solution of (3.3). Then $A : U \to C^1[J, R^1] \subset C[J, R^1]$ and η is a solution of IVP (3.1) if and only if $\eta = A\eta$.

Finally, a standard argument (see, e.g. [2, 3]) shows that the conclusions of Theorem 3.1 hold. This completes the proof.

Remark 3.1. In order to guarantee the existence and uniqueness of the fixed point of B defined by (3.4), we use Theorem 2.1 instead of Banach's theorem, which is widely used in most published papers (see, e.g. [2, 3]) but is invalid here.

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