ON A QUESTION IN THE THEORY OF ALMOST PERIODIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We show that there exists a real homogeneous differential equation of order n with classical almost periodic coefficients such that all solutions are uniformly bounded on the real line yet no non-trivial solution is almost periodic. This now appears to make the search for a Floquet theory of such equations a futile enterprise.

1. INTRODUCTION

One of the greatest impediments to a thorough understanding of the nature of solutions of linear differential equations with (Bohr) almost periodic coefficients [1] is the lack of a Floquet theory.

It is well-known [3, p. 101] that if the $p_i(t)$, $i = 1, 2, \dots, n$, are piecewise continuous periodic functions on $\mathbb{R}$, then every bounded solution of

$$(1.1) \quad x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0$$

is almost periodic on account of Floquet theory. The analogous result for almost periodic coefficients $p_i(t)$ is now known to be *false*. For example, Conley and Miller [2] gave an example of an equation (1.1) with $n = 1$ where a bounded solution is not almost periodic; however, this result did not extend to $n > 1$. In [4], Mingarelli, Pu and Zheng constructed an example, for each $n > 1$, of an equation (1.1) with almost periodic coefficients in which there existed a bounded solution which is not almost periodic. For each such case $n > 1$ there is always another unbounded solution. As a result of this peculiarity the second author raised the following question:

If $p_i(t)$ are all almost periodic and all the solutions of (1.1) are bounded, does it necessarily follow that all solutions are almost periodic?

This is a question whose answer has eluded us for some time. In this paper we answer this simple question in the negative thus annihilating any hope of a Floquet-type theory for linear almost periodic differential equations. We will show that, for each $n \geq 2$, there exists an equation of form (1.1) for which *every* solution is bounded on $\mathbb{R}$ but yet no solution (except the trivial solution) is almost periodic. Indeed, the situation is worse than we thought originally, as we had hoped for a dichotomy, at the very least.

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2. GENERAL RESULT

First, we give two lemmas in order to establish our general result.

Lemma 2.1. *Suppose that $a_i(t)$ ($i = 1, 2, \dots, n$) and $g(t)$ have continuous $(n - 1)$ th-order derivatives. Then the following equation holds:*

$$(2.1) \quad \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a'_1 + a_1g & a'_2 + a_2g & \cdots & a'_n + a_ng \\ (a'_1 + a_1g)' & (a'_2 + a_2g)' & \cdots & (a'_n + a_ng)' \\ \cdots & \cdots & \cdots & \cdots \\ (a'_1 + a_1g)^{(n-2)} & (a'_2 + a_2g)^{(n-2)} & \cdots & (a'_n + a_ng)^{(n-2)} \end{vmatrix} = W(a_1, \dots, a_n)(t)$$

where $W(a_1, \dots, a_n)(t)$ is the Wronskian determinant of $a_1, \dots, a_n$.

Proof. We use an induction argument. For $n = 2$, we have

$$\begin{vmatrix} a_1 & a_2 \\ a'_1 + a_1g & a'_2 + a_2g \end{vmatrix} = a_1a'_2 + a_1a_2g - a_2a'_1 - a_1a_2g = \begin{vmatrix} a_1 & a_2 \\ a'_1 & a'_2 \end{vmatrix}.$$

We assume that (2.1) is true for some n . Now, we verify that (2.1) is true for $n + 1$. In fact,

$$\begin{aligned} & \begin{vmatrix} a_1 & a_2 & \cdots & a_{n+1} \\ a'_1 + a_1g & a'_2 + a_2g & \cdots & a'_{n+1} + a_{n+1}g \\ \cdots & \cdots & \cdots & \cdots \\ (a'_1 + a_1g)^{(n-2)} & (a'_2 + a_2g)^{(n-2)} & \cdots & (a'_{n+1} + a_{n+1}g)^{(n-2)} \\ (a'_1 + a_1g)^{(n-1)} & (a'_2 + a_2g)^{(n-1)} & \cdots & (a'_{n+1} + a_{n+1}g)^{(n-1)} \end{vmatrix} \\ &= \sum_{i=1}^{n+1} (-1)^{n+1+i} (a'_i + a_i g)^{(n-1)} \\ & \times \begin{vmatrix} a_1 & \cdots & a_{i-1} \\ a'_1 + a_1g & \cdots & a'_{i-1} + a_{i-1}g \\ \cdots & \cdots & \cdots \\ (a'_1 + a_1g)^{(n-2)} & \cdots & (a'_{i-1} + a_{i-1}g)^{(n-2)} \end{vmatrix} \\ & \quad \begin{vmatrix} a_{i+1} & \cdots & a_{n+1} \\ a'_{i+1} + a_{i+1}g & \cdots & a'_{n+1} + a_{n+1}g \\ \cdots & \cdots & \cdots \\ (a'_{i+1} + a_{i+1}g)^{n-2} & \cdots & (a'_{n+1} + a_{n+1}g)^{(n-2)} \end{vmatrix} \\ &= \sum_{i=1}^{n+1} (-1)^{n+1+i} (a'_i + a_i g)^{(n-1)} \cdot W(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1})(t) \\ &= \begin{vmatrix} a_1 & a_2 & \cdots & a_{n+1} \\ a'_1 & a'_2 & \cdots & a'_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{(n-1)} & a_2^{(n-1)} & \cdots & a_{n+1}^{(n-1)} \\ (a'_1 + a_1g)^{(n-1)} & (a'_2 + a_2g)^{(n-1)} & \cdots & (a'_{n+1} + a_{n+1}g)^{(n-1)} \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
&= W(a_1, \dots, a_{(n+1)})(t) + \begin{vmatrix} a_1 & \cdots & a_{n+1} \\ a'_1 & \cdots & a'_{n+1} \\ \cdots & \cdots & \cdots \\ a_1^{(n-1)} & \cdots & a_{n+1}^{(n-1)} \\ (a_1 g)^{(n-1)} & \cdots & (a_{n+1} g)^{(n-1)} \end{vmatrix} \\
&= W(a_1, \dots, a_{n+1})(t).
\end{aligned}$$

So, by induction, (2.1) holds for any n . $\square$

The following lemma also follows by induction on account of Lemma 2.1.

Lemma 2.2. Suppose that $a_i(t)$ ($i = 1, 2, \dots, n$) and $g(t)$ are the same as in Lemma 2.1. If

$$\phi_i(t) = a_i(t) \exp\left(\int_0^t g(s) ds\right), i = 1, 2, \dots, n,$$

then

$$W(\phi_1, \dots, \phi_n)(t) = W(a_1, \dots, a_n)(t) \exp\left(n \int_0^t g(s) ds\right).$$

We now state our general result.

Theorem. If there exist functions $a_i(t)$ ($i = 1, 2, \dots, n$) and $g(t)$ such that

- (i) $a_i(t) a_i^{(k)}(t)$ ($i = 1, 2, \dots, n$; $k = 1, 2, \dots, n-1$) are all almost periodic on $\mathbb{R}$;
- (ii) $\inf_{t \in \mathbb{R}} |W(a_1, \dots, a_n)(t)| \neq 0$ for all $t \in \mathbb{R}$;
- (iii) $g(t), g^{(k)}(t)$ ($k = 1, 2, \dots, n-1$) are all almost periodic on $\mathbb{R}$;
- (iv) $\int_0^t g(s) ds \leq 0$ for all $t \in \mathbb{R}$;
- (v) $\inf_{t \in \mathbb{R}} \int_0^t g(s) ds = -\infty$,

then there exist almost periodic functions $p_i(t)$ ($i = 1, 2, \dots, n$) such that the all solutions of the equation

$$(2.2) \quad x^{(n)} + p_n(t)x^{(n-1)} + \cdots + p_1(t)x = 0$$

are bounded on $\mathbb{R}$, but any non-trivial solution of (2.2) is not almost periodic.

Proof. Let

$$\phi_i(t) = a_i(t) \exp\left(\int_0^t g(s) ds\right), i = 1, 2, \dots, n.$$

By Lemma 2.2 and condition (ii), we have

$$W(\phi_1, \dots, \phi_n)(t) = W(a_1, \dots, a_n)(t) \exp\left(n \int_0^t g(s) ds\right) \neq 0$$

for all $t \in \mathbb{R}$. Therefore, the $p_i(t)$ ($i = 1, 2, \dots, n$) can be determined by the following linear system:

$$(2.3) \quad \begin{cases} \phi_1 p_1(t) + \phi'_1 p_2(t) + \cdots + \phi_1^{(n-1)} p_n(t) = -\phi_1^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_n p_1(t) + \phi'_n p_2(t) + \cdots + \phi_n^{(n-1)} p_n(t) = -\phi_n^{(n)} \end{cases}$$

and

$$p_i(t) = W_i(t)/W(a_1, \dots, a_n)(t) \exp \left(n \int_0^t g(s) ds \right),$$

where

$$W_i(t) = \begin{vmatrix} \phi_1 & \cdots & \phi_1^{(i-1)} & -\phi_1^{(n)} & \phi_1^{(i+1)} & \cdots & \phi_1^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_n & \cdots & \phi_n^{(i-1)} & -\phi_n^{(n)} & \phi_n^{(i+1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}, i = 1, 2, \dots, n.$$

We also have

$$\phi_i^{(k)}(t) = b_{i,k}(t) \exp \left(\int_0^t g(s) ds \right), i = 1, 2, \dots, n, k = 1, 2, \dots, n-1,$$

where $b_{i,k}(t)$ are some algebraic combination of $a_i^{(k)}(t)$ and $g^{(k)}(t)$. In fact $b_{i,k}(t) = a_i^{(k)}(t) + c_{1,k}(t)a_i^{(k-1)}(t) + \cdots + c_{k,k}(t)a_i(t)$ for $i = 1, 2, \dots, n$, where the $c_{j,k}(t)$ are given by $c_{j,k}(t) = d_{1,j,k}g^{(j-1)} + d_{2,j,k}g^{(j-2)}g + \cdots + d_{m_j,j,k}g^j$ and $d_{l,j,k}$ are constants ($j = 1, 2, \dots, k, k = 1, 2, \dots, n-1$). So,

$$W_i(t) = B_i(t) \exp \left(n \int_0^t g(s) ds \right), i = 1, 2, \dots, n,$$

where $B_i(t)$ ($i = 1, 2, \dots, n$) are still some algebraic combinations of $a_i^{(k)}(t)$ and $g^{(k)}(t)$ ($i = 1, 2, \dots, n; k = 1, 2, \dots, n-1$). It follows that

$$p_i(t) = B_i(t)/W(a_1, \dots, a_n)(t), i = 1, 2, \dots, n,$$

are almost periodic [1]. For these $p_i(t)$, (2.2) is an almost periodic differential equation and $\{\phi_2, \dots, \phi_n\}$ is a fundamental system of solutions. From the assumptions on $a_i(t)$ and $g(t)$, it is obvious that $\phi_i(t)$ ($i = 1, 2, \dots, n$) are all bounded on $\mathbb{R}$, and thus all solutions of (2.2) are bounded.

Next, we show that any non-trivial solution of (2.2) is not almost periodic. Let $x(t)$ be a non-trivial solution; then there are constants $C_1, \dots, C_n$ such that

$$\begin{aligned} x(t) &= C_1\phi_1(t) + \cdots + C_n\phi_n(t) \\ &= (C_1a_1(t) + \cdots + C_na_n(t)) \exp \left(\int_0^t g(s) ds \right). \end{aligned}$$

Let $b(t) = C_1a_1(t) + \cdots + C_na_n(t)$; then $b(t) \not\equiv 0$ ($t \in \mathbb{R}$) and it inherits the same properties as $a_i(t)$. We write $x^{(k)}(t)$ in the form

$$x^{(k)}(t) = b_k(t) \exp \left(\int_0^t g(s) ds \right), k = 1, 2, \dots, n-1,$$

where $b_k(t)$ ($k = 1, 2, \dots, n-1$) are some algebraic combinations of $b(t), b^{(k)}(t), g(t)$ and $g^{(k)}(t)$. We see that the $b_k(t)$ are almost periodic and thus bounded ($k = 1, 2, \dots, n-1$).

Let $X(t) = \text{Col}(x(t), x'(t), \dots, x^{(n-1)}(t))$; then $X(t)$ is a solution of the system

$$(2.4) \quad X'(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_1(t) & -p_2(t) & -p_3(t) & \cdots & -p_n(t) \end{pmatrix} X(t).$$

Now, (2.4) is an almost periodic system and

$$\inf_{t \in \mathbb{R}} |X(t)|^2 = \inf_{t \in \mathbb{R}} \sum_{k=1}^n |b_k(t)|^2 \exp(2 \int_0^t g(s) ds) = 0,$$

$$X(t) \neq 0, \quad \text{for all } t \in \mathbb{R},$$

since $b_k(t)$ ($k = 1, 2, \dots, n-1$) are bounded and $\inf_{t \in \mathbb{R}} \int_0^t g(s) ds = -\infty$. So, $x(t)$ and $x^{(k)}(t)$ ($k = 1, 2, \dots, n-1$) are all not almost periodic because if $x(t)$ is almost periodic, so is $x^{(k)}(t)$ for each k (because of the form of $x(t)$) and thus $X(t)$ is almost periodic, contradicting Theorem 5.7 in [3] (see [3, p. 85]), since $X(t)$ is a non-trivial solution. This ends the proof of the theorem.

3. CONSTRUCTION OF $a_i(t)$, ($i = 1, 2, \dots, n$) AND $g(t)$

In this section, we will give the construction of the functions $a_i(t)$ ($i = 1, 2, \dots, n$) and $g(t)$ for any n .

For any n , we define the function $g(t)$ as

$$(3.1) \quad g(t) = \sum_{k=1}^{\infty} f_k(t)$$

where

$$f_k(t) = -\frac{1}{k^n} \sin \frac{t}{k^{n+1}}, t \in \mathbb{R}.$$

Then $g(t)$ satisfies the conditions (iii)–(v) (see [4]) of the theorem.

In order to construct the functions $a_i(t)$, ($i = 1, 2, \dots, n$) we give the following lemma, easily proved by induction.

Lemma 3.1. *Let λ_i ($i = 1, 2, \dots, m$) be any real numbers. Then the Wronskian determinant of the functions*

$$\{\cos \lambda_1 t, \sin \lambda_1 t, \dots, \cos \lambda_m t, \sin \lambda_m t\}$$

satisfies

$$(3.2) \quad W(\cos \lambda_1 t, \sin \lambda_1 t, \dots, \cos \lambda_m t, \sin \lambda_m t) = \prod_{i=1}^m \lambda_i \prod_{1 \leq i < j \leq m} (\lambda_i^2 - \lambda_j^2)^2.$$

Corollary. *Let λ_i ($i = 1, 2, \dots, m$) be any real numbers. Then*

$$W(\cos \lambda_1 t, \sin \lambda_1 t, \dots, \cos \lambda_m t, \sin \lambda_m t) = 0$$

if and only if there are i and j such that $i \neq j$ but $\lambda_i = \pm \lambda_j$ or there is an i such that $\lambda_i = 0$.

Now, we can construct the functions $a_i(t)$ ($i = 1, 2, \dots, n$) satisfying the conditions (i) and (ii) of the theorem for any n .

If n is even, we let

$$a_{2k-1}(t) = \cos \lambda_k t, a_{2k}(t) = \sin \lambda_k t, k = 1, 2, \dots, n/2,$$

where $\lambda_k \neq 0$ ($k = 1, 2, \dots, n/2$) and $\lambda_i \neq \pm \lambda_j$ as $i \neq j$. By the corollary, we have that the $a_k(t)$ ($k = 1, 2, \dots, n$) satisfy the conditions (i) and (ii) of the theorem.

If n is odd, we still take the λ_i such that $\lambda_i \neq 0, i = 1, 2, \dots, \frac{n-1}{2}$ and $\lambda_i \neq \lambda_j$ as $i \neq j$. And we let

$$a_1(t) = 1, a_{2k}(t) = \cos \lambda_k t, a_{2k+1}(t) = \sin \lambda_k t, k = 1, 2, \dots, \frac{n-1}{2}.$$

Since $W(a_1, a_2, \dots, a_n)(t) = W(a_2, \dots, a_n)(t)$ and $\inf_{t \in \mathbb{R}} |W(a_2, \dots, a_n)(t)| \neq 0$, the $a_k(t)$ still satisfy the conditions (i) and (ii) of the theorem. In particular, when $n = 2$, we take $\lambda_1 = 1$, i.e., $a_1(t) = \cos(t), a_2(t) = \sin t$, and we get

$$\phi_1(t) = \cos t \exp\left(\int_0^t g(s) ds\right), \phi_2(t) = \sin t \exp\left(\int_0^t g(s) ds\right)$$

and $W(\phi_1, \phi_2)(t) \equiv 1$ for all $t \in \mathbb{R}$. $p_1(t) = 1 + g^2 - g', p_2(t) = -2g$. We consider the second-order linear equation

$$(3.3) \quad x'' - 2g(t)x' + (1 + g^2(t) - g'(t))x = 0.$$

Obviously, (3.3) is an almost periodic equation and has the two linear independent solutions $\phi_1(t)$ and $\phi_2(t)$. So, all solutions are bounded, but no non-trivial solution is almost periodic.

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