HARDY'S INEQUALITY FOR $W_0^{1,p}$ -FUNCTIONS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove that for every Riemannian manifold $\mathcal{X}$ with the isoperimetric profile of particular type there holds an inequality of Hardy type for functions of the class $W_0^{1,p}(\mathcal{X})$. We also study manifolds satisfying Hardy's inequality and, in particular, we establish an estimate for the rate of growth of the weighted volume of the noncompact part of such a manifold.

1. MAIN RESULTS

Let $\mathcal{X}$ be a connected noncompact Riemannian C^2 -manifold without boundary. We denote by $\rho(x', x'')$ the distance between two points $x', x'' \in \mathcal{X}$. For an arbitrary point $x \in \mathcal{X}$ we set

$$\varepsilon(x) = \inf_{(y_k)} \liminf \rho(x, y_k),$$

where the infimum is taken over all sequences of points $(y_k) \subset \mathcal{X}$, which do not have points of accumulation in $\mathcal{X}$.

Let $\alpha(t), \beta(t) : [0, \infty) \rightarrow [0, \infty)$ be positive continuous functions. We fix constants p, q such that $1 < p \leq q < \infty$. Let $\Omega \subset \mathcal{X}$ be an arbitrary domain, and let $\partial\Omega$ be its boundary. We consider a weighted volume

$$V_{\mathcal{X}}(\Omega) = \int_{\Omega} \alpha(\varepsilon(x))^q dv.$$

Here, dv is the element of volume on the manifold $\mathcal{X}$. Below we shall assume that $I_{\mathcal{X}} = V_{\mathcal{X}}(\mathcal{X}) \leq \infty$.

We denote a weighted area

$$A_{\mathcal{X}}(\partial\Omega) = \int_{\partial\Omega} \beta(\varepsilon(x)) \alpha(\varepsilon(x))^{(p-1)q/p} dH^{n-1},$$

where dH^{n-1} is the element of the $(n-1)$ -dimensional Hausdorff measure on $\partial\Omega$.

We consider *isoperimetric profiles* of the Riemannian manifold $\mathcal{X}$ with weight functions α and β . An isoperimetric profile of the manifold $\mathcal{X}$ is the function

$$(1.1) \quad \theta_{\mathcal{X}} : [0, I_{\mathcal{X}}) \rightarrow R_+, \quad \theta(0) = 0,$$

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defined by

$$\theta_{\mathcal{X}}(\tau) = \inf \{ A_{\mathcal{X}}(\partial\Omega) : \Omega \subset \mathcal{Y} \text{ a compact domain} \\ \text{with } H^{n-1}(\partial\Omega) < \infty, \quad V_{\mathcal{X}}(\Omega) = \tau \},$$

i.e. the isoperimetric profile $\theta_{\mathcal{X}}$ is the best function among the functions θ satisfying

$$(1.2) \quad \theta(V_{\mathcal{X}}(\Omega)) \leq A_{\mathcal{X}}(\partial\Omega).$$

In the special case of surfaces this definition goes back to Ahlfors [1, p.188]. In the general case see, for example, [4].

In general, the isoperimetric profile $\theta_{\mathcal{X}}(\tau)$ is difficult to compute. It is also difficult to estimate the isoperimetric profile in terms of curvature and other geometric data. We give some of these cases below.

1.3. Example. Let $\mathcal{X}$ be a complete, simply connected, n -dimensional Riemannian manifold with nonpositive sectional curvature. We write

$$V_{\mathcal{X}}(\Omega) = \text{vol}(\Omega) \quad \text{and} \quad A_{\mathcal{X}}(\partial\Omega) = \text{area}(\partial\Omega).$$

By [12], [7] for every n there is a constant $c_n > 0$ such that

$$c_n (\text{vol}(\Omega))^{(n-1)/n} \leq \text{area}(\partial\Omega)$$

and it follows that

$$\theta_{\mathcal{X}}(\tau) \geq c_n \tau^{(n-1)/n}.$$

1.4. Example. Let $\mathcal{X}$ be a complete, simply connected Riemannian manifold, $\dim \mathcal{X} = n$. If the sectional curvature $K_{\mathcal{X}}$ of $\mathcal{X}$ satisfies $K_{\mathcal{X}} \leq k < 0$, $k = \text{const}$, then

$$(n-1)\sqrt{-k} \text{vol}(\Omega) \leq \text{area}(\partial\Omega)$$

([19], p.504; [6], 34.2.6) and thus

$$\theta_{\mathcal{X}}(\tau) \geq (n-1)\sqrt{-k}\tau.$$

Furthermore, we shall consider only isoperimetric profiles $\theta = \theta_{\mathcal{X}}(t)$ of a special type. In fact, we shall assume that

$$(1.5) \quad B = \sup_{r \in (0, I_{\mathcal{X}})} r^{1/q} \left(\int_r^{I_{\mathcal{X}}} \frac{dt}{\theta(t)^{p/(p-1)}} \right)^{(p-1)/p} < \infty.$$

Below by functions of the class $W_0^{1,p}(\mathcal{X})$, $p > 1$, we shall understand the closure of the class of the $C_0^\infty(\mathcal{X})$ -functions in the norm

$$\|u\|_{W^{1,p}} = \|u\|_p + \|\nabla u\|_p$$

where $\|\cdot\|_p$ stands for the standard norm in $L^p(\mathcal{X})$.

We have the following theorem.

1.6. Theorem. *Let $\mathcal{X}$ be an n -dimensional connected noncompact Riemannian C^2 -manifold without boundary. If the manifold $\mathcal{X}$ has an isoperimetric profile with the properties (1.1), (1.2) and (1.5), then for an arbitrary function $u \in W_0^{1,p}(\mathcal{X})$, $p > 1$, we have*

$$(1.7) \quad \left(\int_{\mathcal{X}} |\alpha(\varepsilon(x))u(x)|^q dv \right)^{1/q} \leq \lambda \left(\int_{\mathcal{X}} (\beta(\varepsilon(x))|\nabla u(x)|)^p dv \right)^{1/p},$$

where $q \geq p$ is arbitrary and λ is such that

$$(1.8) \quad B \leq \lambda \leq B q^{1/q} \left(\frac{q}{q-1} \right)^{(p-1)/p}.$$

In the case when $\mathcal{X} = D$ is a proper subdomain of R^n , the function

$$\varepsilon(x) = \rho(x, \partial D)$$

is the distance from a point $x \in D$ to the boundary ∂D , the inequality (1.7) yields a generalized Hardy inequality. We record a widely known variant of it ($\alpha(t) = 1/t$, $\beta(t) = 1$ and $p = q$)

$$(1.9) \quad \int_D \left(\frac{u(x)}{\rho(x, \partial D)} \right)^p dv \leq \lambda^p \int_D |\nabla u(x)|^p dv, \quad u \in W_0^{1,p}(D), \quad p > 1.$$

A one-dimensional version of Hardy's classical inequality can be found in [10, p. 175]; for its generalization to the case of domains $D \subset R^n$ see [2], [15], [14], [17], [18], and [13].

For $\alpha(t) = \beta(t) \equiv 1$ and $p = q$ the inequality (1.7) can be written as the well-known Poincaré-Sobolev inequality (cf. [19], [11, p. 48], and the literature cited there)

$$(1.10) \quad \int_D |u(x)|^p dv \leq \lambda^p \int_D |\nabla u(x)|^p dv, \quad u \in W_0^{1,p}(D), \quad p > 1.$$

1.11. Example. Let $\mathcal{X} = D$ be a domain in R^2 , and let $p = q = 2$, $\beta \equiv 1$. We use the metric

$$ds^2 = \alpha^2(\rho(x, \partial D)) (dx_1^2 + dx_2^2),$$

and consider the manifold $\mathcal{X} = (D, ds)$. We have for an arbitrary domain $\Omega \subset D$ with $H^1(\partial\Omega) < \infty$:

$$V_{(D, ds)}(\Omega) = \int_{\Omega} \alpha(\rho(x, \partial D))^2 dx_1 dx_2, \quad A_{(D, ds)}(\partial\Omega) = \int_{\partial\Omega} \alpha(\rho(x, \partial\Omega)) dH^1.$$

The Gaussian curvature K of $\mathcal{X} = (D, ds)$ is equal (see, for example, [8, §13 of Chapter 2])

$$K = -\frac{1}{\alpha^2(\rho)} \Delta \log \alpha(\rho).$$

It is nonpositive if and only if

$$\Delta \log \alpha(\rho(x, \partial D)) \geq 0, \quad \forall x \in D.$$

Since $|\nabla \rho(x, \partial D)| \equiv 1$, this inequality is equivalent to the following inequality

$$(1.12) \quad \frac{\alpha'(\rho)}{\alpha(\rho)} \Delta \rho + \left(\frac{\alpha'(\rho)}{\alpha(\rho)} \right)' \geq 0, \quad \forall x \in D.$$

Therefore, if the condition (1.12) is true then we have for the isoperimetric profile of the manifold (D, ds) (see Example 1.3)

$$\theta_{(D, ds)}(\tau) \geq c_2 \tau^{1/2},$$

and the property (1.5) holds.

Consequently, we obtain for such domains

$$\int_D \alpha(\rho(x, \partial D))^2 u^2(x) dx \leq \lambda^2 \int_D |\nabla u(x)|^2 dx, \quad u \in W_0^{1,2}(D).$$

In the special case $\alpha(\tau) = \frac{1}{\tau}$, we can conclude that if the domain $D \subset \mathbb{R}^2$ has the property

$$\Delta\rho(x, \partial D) \leq \frac{1}{\rho(x, \partial D)},$$

then the Hardy's inequality (1.9) (with $p = 2$) is correct for this domain. For convex domains the function $\rho(x, \partial D)$ is superharmonic ([3]) and hence the above inequality holds.

The next theorem characterizes manifolds, on which a generalized Hardy's inequality is valid.

1.13. Theorem. *Let $\mathcal{X}$ be an n -dimensional connected noncompact Riemannian C^2 -manifold without boundary, satisfying the condition*

$$(1.14) \quad \forall t \in (0, \varepsilon_0), \quad \{x \in \mathcal{X} : \varepsilon(x) \geq t\} \subset\subset \mathcal{X}, \quad \varepsilon_0 = \sup_{x \in \mathcal{X}} \varepsilon(x).$$

If for every function $u \in W_0^{1,p}(\mathcal{X})$, $p > 1$, the inequality (1.7) holds with the constant $\lambda < \infty$, then

$$(1.15) \quad \sup_{r \in (0, \varepsilon_0)} \left(\int_r^{\varepsilon_0} \alpha(t)^q S(t) dt \right)^{1/q} \left(\int_0^r \left(\frac{1}{\beta(t)^p S(t)} \right)^{1/(p-1)} dt \right)^{(p-1)/p} \leq \lambda,$$

where

$$S(t) = \int_{\varepsilon(x)=t} dH^{n-1}, \quad t \in (0, \varepsilon_0),$$

is the $(n-1)$ -dimensional Hausdorff measure of the set $\{x \in \mathcal{X} : \varepsilon(x) = t\}$.

A relation of the type of (1.15) contains important information on the structure of the manifold $\mathcal{X}$. For the purpose of illustration we give two particular cases of Theorem 1.13.

In the case $\alpha(t) = 1/t, \beta(t) \equiv 1, p = q$, we have:

1.16. Corollary. *Let $\mathcal{X}$ be an n -dimensional connected noncompact Riemannian C^2 -manifold without boundary, satisfying the condition (1.14). If for all functions $u \in W_0^{1,p}(\mathcal{X})$, $p > 1$, the inequality (1.9) holds, then*

$$(1.17) \quad \limsup_{r \rightarrow 0} \left(\log \frac{1}{r} \right)^{p/(p-1)} \int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt < \infty.$$

and for each pair of numbers $r', r'', 0 < r' < r'' < \varepsilon_0$,

$$(1.18) \quad \int_{\varepsilon(x) > r''} \frac{dv}{\varepsilon(x)^p} \bigg/ \int_{r' < \varepsilon(x) < r''} \frac{dv}{\varepsilon(x)^p} \leq \frac{\lambda^p}{(\log \frac{r''}{r'})^p}.$$

Setting $\alpha(t) = \beta(t) \equiv 1, p = q$, we arrive at the following characterization of noncompact manifolds, for which the Poincaré-Sobolev inequality is fulfilled.

1.19. Corollary. *Let $\mathcal{X}$ be an n -dimensional noncompact connected Riemannian C^2 -manifold without boundary, satisfying condition (1.14). If for all $u \in W_0^{1,p}(\mathcal{X})$, $p > 1$, the inequality (1.10) holds, then*

$$(1.20) \quad \int_{\varepsilon(x) > r} dv \bigg/ \int_{\varepsilon(x) < r} dv \leq \frac{\lambda^p}{r^p}, \quad \forall r \in (0, \varepsilon_0).$$

2. PROOF OF THEOREM 1.6

We shall apply a standard scheme of proof which originates from a paper of Federer and Fleming [9, Remark 6.6] and which later found frequent applications to similar questions (cf., for instance, [16], [5]).

We fix an arbitrary function $u \in C_0^\infty(\mathcal{X})$. We can assume that $|\nabla u(x)| > 0$ everywhere in the set $\{x \in \mathcal{X} : |u(x)| > 0\}$ except possibly at finitely many points. This can always be achieved on the expense of a small perturbation of the function u (see, for example [8, Theorem 5, p.488]).

We set

$$E_t = \{x \in \mathcal{X} : |u(x)| = t\},$$

$$\mathcal{X}_t = \{x \in \mathcal{X} : |u(x)| > t\}.$$

On the basis of the Kronrod-Federer co-area formula we can write

$$I(t) \equiv \int_{\mathcal{X}_t} \alpha(\varepsilon(x))^q dv = \int_t^\infty d\tau \int_{E_\tau} \alpha(\varepsilon(x))^q \frac{dH^{n-1}}{|\nabla u|}.$$

Therefore for almost all $t \geq 0$ we have

$$I'(t) = - \int_{E_t} \alpha(\varepsilon(x))^q \frac{dH^{n-1}}{|\nabla u|}.$$

Using similar arguments we have

$$(2.1) \quad \begin{aligned} \int_{\mathcal{X}} |\alpha(\varepsilon(x))u(x)|^q dv &= \int_0^\infty \tau^q d\tau \int_{E_\tau} \alpha(\varepsilon(x))^q dH^{n-1} \\ &= - \int_0^\infty \tau^q I'(\tau) d\tau = \int_0^{I_{\mathcal{X}}} \tau^q(I) dI, \end{aligned}$$

where $\tau : [0, I_{\mathcal{X}}) \rightarrow [0, \infty)$ is the inverse function of $I(\tau)$.

Furthermore, we have

$$\begin{aligned} &\left(\int_{E_t} \beta(\varepsilon(x)) \alpha(\varepsilon(x))^{(p-1)q/p} dH^{n-1} \right)^p \\ &\leq \left(\int_{E_t} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1} \right) \left(\int_{E_t} \alpha(\varepsilon(x))^q \frac{dH^{n-1}}{|\nabla u|} \right)^{p-1}, \end{aligned}$$

and therefore

$$(2.2) \quad \int_{E_t} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1} \geq \frac{1}{|I'(t)|^{p-1}} \left(\int_{E_t} \beta(\varepsilon) \alpha(\varepsilon(x))^{(p-1)q/p} dH^{n-1} \right)^p.$$

Because u has a compact support in $\mathcal{X}$, the open set $\mathcal{X}_t \subset \subset \mathcal{X}$. On the basis of the isoperimetric relation (1.2) we can write

$$\theta(I(t)) \leq \int_{E_t} \beta(\varepsilon) \alpha(\varepsilon(x))^{q(p-1)/p} dH^{n-1}.$$

Thus we get by (2.2)

$$(2.3) \quad \frac{\theta(I(t))^p}{|I'(t)|^{p-1}} \leq \int_{E_t} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1}.$$

Now we can transform the integral on the right side of (1.7). Applying once again the Kronrod-Federer co-area formula, we have

$$\int_{\mathcal{X}} \left(\beta(\varepsilon(x)) |\nabla u| \right)^p dv = \int_0^\infty d\tau \int_{E_\tau} \beta(\varepsilon(x))^p |\nabla u|^{p-1} dH^{n-1}.$$

Therefore the inequality (2.3) yields

$$(2.4) \quad \int_{\mathcal{X}} \left(\beta(\varepsilon(x)) |\nabla u| \right)^p dv \geq \int_0^\infty \frac{\theta(I(\tau))^p}{|I'(\tau)|^{p-1}} d\tau = \int_0^{I_{\mathcal{X}}} \theta(I)^p |\tau'(I)|^p dI.$$

Combining the relations (2.1) and (2.4), we get

$$\frac{\left(\int_{\mathcal{X}} \left(\beta(\varepsilon(x)) |\nabla u| \right)^p dv \right)^{1/p}}{\left(\int_{\mathcal{X}} \left(\alpha(\varepsilon(x)) |\nabla u| \right)^q dv \right)^{1/q}} \geq \inf \frac{\left(\int_0^{I_{\mathcal{X}}} \theta(I)^p |\tau'(I)|^p dI \right)^{1/p}}{\left(\int_0^{I_{\mathcal{X}}} \tau(I)^p dI \right)^{1/q}} \equiv \frac{1}{\lambda},$$

where the infimum is taken over all nonincreasing AC -functions τ with $\tau(I)$, $\tau(I_{\mathcal{X}}) = 0$.

The one-dimensional functional on the right side of the above inequality has been extensively studied. Applying Theorem 4 from [16, the inequality (13) §1.3.1], we arrive at the dual inequality (1.8), where the quantity B is defined by the relation (1.5).

What was said above proves the validity of the inequality (1.7) for $C_0^\infty(\mathcal{X})$ -functions. For the class $W_0^{1,p}(\mathcal{X})$ the proof follows after a passage to limit in the W_p^1 -norm.

3. PROOFS OF THEOREM 1.13 AND COROLLARIES

The proofs are very simple. The condition (1.14) guarantees the precompactness of the open sets

$$\mathcal{X}_t = \{x \in \mathcal{X} : \varepsilon(x) > t\}, \quad t \in (0, \varepsilon_0).$$

Let $\varphi(t)$ be an arbitrary AC -function on $[0, \varepsilon_0)$ which vanishes for all $t \in [0, t_0)$ for some $t_0 > 0$. Each of the functions $\varphi^*(x) = \varphi(\varepsilon(x))$ has a support contained in one of the sets $\mathcal{X}_t$ and is therefore a member of the class $W_0^{1,p}(\mathcal{X})$.

Choosing in the inequality (1.7) the function $u(x) = \varphi^*(x)$, we obtain

$$\left(\int_{\mathcal{X}} |\alpha(\varepsilon(x)) \phi(\varepsilon(x))|^q dv \right)^{1/q} \leq \lambda \left(\int_{\mathcal{X}} (\beta(\varepsilon(x)) |\phi'(\varepsilon(x))| |\nabla \varepsilon(x)|)^p dv \right)^{1/p}.$$

Integrating along the level sets of $\varepsilon(x)$, we get

$$\begin{aligned} & \left(\int_0^{\varepsilon_0} \alpha(\tau)^q \phi(\tau)^q d\tau \int_{\varepsilon(x)=\tau} \frac{dH^{n-1}}{|\nabla \varepsilon(x)|} \right)^{1/q} \\ & \leq \lambda \left(\int_0^{\varepsilon_0} \beta(\tau)^p |\phi'(\tau)|^p d\tau \int_{\varepsilon(x)=\tau} |\nabla \varepsilon(x)|^{p-1} dH^{n-1} \right)^{1/p}. \end{aligned}$$

Because $\varepsilon(x)$ is the intrinsic distance function on $\mathcal{X}$ we have $|\nabla \varepsilon(x)| \equiv 1$ almost everywhere.

Thus we get for $1 < p \leq q < \infty$

$$(3.1) \quad \left(\int_0^{\varepsilon_0} \alpha(\tau)^q S(\tau) \phi(\tau)^q d\tau \right)^{1/q} \leq \lambda \left(\int_0^{\varepsilon_0} \beta(\tau)^p S(\tau) |\phi'(\tau)|^p d\tau \right)^{1/p},$$

valid for every AC -function $\varphi(t)$ on $(0, \varepsilon_0)$ vanishing close to 0.

This relation between one-dimensional integrals has been extensively studied in the literature, see e.g. [16, p. 40]. The validity of the inequality (3.1) for all φ with the given properties implies the validity of the inequality (1.15).

We next prove Corollary 1.16. We choose in (1.15) the weight functions $\alpha(t) = 1/t$, $\beta(t) \equiv 1$ and $p = q$. We arrive at the estimate

$$(3.2) \quad \sup_{r \in (0, \varepsilon_0)} \left(\int_r^{\varepsilon_0} \frac{S(t)}{t^p} dt \right)^{1/p} \left(\int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{(p-1)/p} \leq \lambda.$$

However, by Hölder's inequality we get for all $r', r'', 0 < r' < r'' < \varepsilon_0$,

$$(3.3) \quad \left(\log \frac{r''}{r'} \right)^p = \left(\int_{r'}^{r''} \frac{dt}{t} \right)^p \leq \left(\int_{r'}^{r''} \frac{S(t)}{t^p} dt \right) \left(\int_{r'}^{r''} \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{p-1}.$$

Let $r' < r'' < \varepsilon_0$. Then we have from (3.2)

$$\begin{aligned} \left(\log \frac{r''}{r'}\right)^p \bigg/ \left(\int_{r'}^{r''} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt\right)^{p-1} &\leq \int_{r'}^{r''} \frac{S(t)}{t^p} dt \\ &\leq \int_{r'}^{\varepsilon_0} \frac{S(t)}{t^p} dt \leq \lambda^p \bigg/ \left(\int_0^{r'} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt\right)^{p-1} \end{aligned}$$

and

$$\left(\log \frac{r''}{r'}\right)^p \left(\int_0^{r'} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt\right)^{p-1} \bigg/ \left(\int_{r'}^{r''} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt\right)^{p-1} \leq \lambda^p.$$

Therefore,

$$\left(\log \frac{r''}{r'}\right)^{-p} \left(\frac{\int_0^{r''} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt - \int_0^{r'} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt}{\int_0^{r'} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt}\right)^{p-1} \geq \lambda^{-p}$$

and further

$$\left(\log \frac{r''}{r'}\right)^{-p/(p-1)} \left(\frac{\int_0^{r''} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt}{\int_0^{r'} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt} - 1\right) \geq \lambda^{-p/(p-1)}$$

or, equivalently,

$$(3.4) \quad \int_0^{r''} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt \geq [1 + \lambda^{-\frac{p}{p-1}} (\log \frac{r''}{r'})^{p/(p-1)}] \int_0^{r'} \left(\frac{1}{S(t)}\right)^{1/(p-1)} dt.$$

The inequality (3.4) holds for all $r' < r''$. Passing to the limit with $r' \rightarrow 0$, we arrive at (1.17).

Further from (3.2) and (3.3)

$$(3.5) \quad \left(\log \frac{r''}{r'}\right)^p \int_{r''}^{\varepsilon_0} \frac{S(t)}{t^p} dt \bigg/ \int_{r'}^{r''} \frac{S(t)}{t^p} dt \leq \lambda^p.$$

We observe that

$$\int_{r'}^{r''} \frac{S(t)}{t^p} dt = \int_{r'}^{r''} \frac{dt}{t^p} \int_{\varepsilon(x)=t} \frac{dH^{n-1}}{|\nabla \varepsilon|} = \int_{r' < \varepsilon(x) < r''} \frac{dv}{\varepsilon(x)^p}$$

and

$$\int_{r''}^{\varepsilon_0} \frac{S(t)}{t^p} dt = \int_{\varepsilon(x) > r''} \frac{dv}{\varepsilon(x)^p}.$$

On the basis of the inequality (3.5) we convince ourselves of the validity of the inequality (1.18).

The proof of Corollary 1.19 is similar. Choose in (1.15) the weight functions $\alpha(t) = \beta(t) \equiv 1$ and $p = q$. Then we have

$$\sup_{r \in (0, \varepsilon_0)} \int_r^{\varepsilon_0} S(t) dt \left(\int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{p-1} \leq \lambda^p.$$

Because

$$r^p = \left(\int_0^r dr \right)^p \leq \left(\int_0^r S(t) dt \right) \left(\int_0^r \left(\frac{1}{S(t)} \right)^{1/(p-1)} dt \right)^{p-1},$$

we have

$$\sup_{r \in (0, \varepsilon_0)} r^p \int_r^{\varepsilon_0} S(t) dt \bigg/ \int_0^r S(t) dt \leq \lambda^p.$$

Using the formulas

$$\int_0^r S(t) dt = \int_0^r dt \int_{\varepsilon(x)=t} \frac{dH^{n-1}}{|\nabla \varepsilon|} = \int_{\varepsilon < r} dv$$

and

$$\int_r^{\varepsilon_0} S(t) dt = \int_{\varepsilon(x) > r} dv,$$

we arrive at (1.20).

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