PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 9, Pages 2697–2702
S 0002-9939(99)04922-9
Article electronically published on April 15, 1999

POLYNOMIALLY CONVEX HULLS OF GRAPHS ON THE SPHERE

TOSHIYA JIMBO AND AKIRA SAKAI

(Communicated by Theodore W. Gamelin)

ABSTRACT. Let Σ be the graph of a continuous map of the unit sphere of $\mathbb{C}^n$ into $\mathbb{C}^m$, and $h(\Sigma)$ the polynomially convex hull of Σ. Several examples of $h(\Sigma)$ for $n = m > 1$ are given, which have different properties from the known ones for $n > m$.

Let E be a compact subset of $\mathbb{C}^N$. The polynomially convex hull of E is, by definition, the set $h(E)$ of the points z of $\mathbb{C}^N$ for which

$$|p(z)| \leq \sup\{|p(\zeta)| : \zeta \in E\}$$

holds for all polynomials p. If $h(E) = E$, then E is said to be polynomially convex.

Let B be the open unit ball in $\mathbb{C}^n : \sum_{k=1}^{n} |z_k|^2 < 1$, and let S be the boundary of B. Consider a continuous map

$$f = (f_1, \dots, f_m) : S \to \mathbb{C}^m.$$

The graph of f is denoted by Σ. The polynomially convex hulls of Σ, especially in the case that $n = m = 1$ or $n > m$, have been studied by several authors (J. Wermer [7], H. Alexander [2], P. Ahern and W. Rudin [1], J. Anderson [3], and others). The following theorem for $n = m = 1$ is Wermer's maximality theorem [7].

Theorem W. *When $n = m = 1$, only two cases occur:*
(a) Σ is polynomially convex or;
(b) $h(\Sigma)$ is a graph of a function of $C(\overline{B})$ which is holomorphic in B.

We denote by π the projection of $\mathbb{C}^{n+m}$ onto $\mathbb{C}^n$. If E is a subset of $\mathbb{C}^{n+m}$ such that $\pi(E) = \overline{B}$, we say that E covers $\overline{B}$. When $n > 1$, Alexander [2] proved the following theorem.

Theorem A. *(1) If $n > m$, then $h(\Sigma)$ covers $\overline{B}$.*
(2) If $m = 1$ and if F is continuous on $\overline{B}$ and is pluriharmonic in B, then $h(\Sigma)$ is the graph of F.
(3) If $F = |g|$ for some holomorphic function g with no zeros, then $h(\Sigma)$ is the graph of F.

In the case that $n = m > 1$, the situation is somewhat different. For example, if $f_j(z) = \overline{z_j},\ j = 1, \dots, n$, Σ is polynomially convex, by Weierstrass' approximation theorem. In this paper, we give some examples in this case.

Received by the editors November 20, 1997.
1991 *Mathematics Subject Classification.* Primary 32E20.

©1999 American Mathematical Society

1.

In the first example, $h(\Sigma)$ is not polynomially convex and does not cover $\overline{B}$.

Example 1. We consider

$$f_k(z) = (z_1 - a)\overline{z_k},\ 0 < |a| < 1,\ k = 1, \dots , n.$$

Then $h(\Sigma)$ contains an $(n-1)$-ball and $\pi(h(\Sigma))$ does not contain the origin. That is, Σ is not polynomially convex and $h(\Sigma)$ does not cover $\overline{B}$.

Proof. We consider

$$\Sigma_0 = \{(a, z_2, \dots , z_n, 0, \dots , 0) \in \mathbb{C}^{2n} :\ \sum_{k=2}^{n} |z_k|^2 = 1 - |a|^2\}.$$

This set is the intersection of Σ and the complex subspace $z_1 = a, w_1 = \cdots = w_n = 0$. Hence $h(\Sigma)$ contains an $(n-1)$-ball

$$\{(a, z_2, \dots , z_n, 0, \dots , 0) : \sum |z_k|^2 < 1 - |a|^2\},$$

and hence $h(\Sigma) \neq \Sigma$.

We next consider the polynomial

$$p(z_1, \dots , z_n, w_1, \dots , w_n) = 1 + \overline{a}(\sum_{k=1}^{n} z_k w_k - z_1).$$

Then, we have $p(0, \dots , 0, w_1, \dots , w_n) = 1$ for all $(w_1, \dots , w_n) \in C^n$.

If $(z_1, \dots , z_n, w_1, \dots , w_n) \in \Sigma$, then $w_k = f_k(z)$, and hence, we have $p(z, w) = 1 - |a|^2$. This shows that the point $(0, \dots , 0, w_1, \dots , w_n)$ does not belong to $h(\Sigma)$ for every $(w_1, \dots , w_n)$. $\square$

2.

In this section, we deal with the case that f is the restriction on S of $F = (F_1, \dots, F_m)$ with $F_j \in C(\overline{B}), j = 1, \dots , m$. The graph of F on $\overline{B}$ will be denoted by $G(F)$.

We first prove the following

Lemma 1. *Let K be a compact polynomially convex subset of $\mathbb{C}^N$. If $F_j, j = 1, \dots , m$, are pluriharmonic in an open neighborhood U of K, then the graph $G(F)$ of $F = (F_1, \dots , F_m)$ on K is polynomially convex.*

Proof. Since K is polynomially convex, it is sufficient to show that, for any point (z^0, w^0) with $z^0 \in K$ and $w^0 \neq F(z^0)$, there exists a function $q(z, w)$ which is holomorphic in $U \times \mathbb{C}^m$ and satisfies $|q(z^0, w^0)| > \|q\|_{G(F)}$. We may assume that $\mathrm{Re}(F_k(z^0) - w_k^0) > 0$, for some k. We can write $F_k = g_k + \overline{h}_k$, where g_k and h_k are holomorphic in U. We set

$$q(z, w) = \exp(g_k(z) + h_k(z) - w_k).$$

Then q is holomorphic on B. Since $\mathrm{Re}F_k = \mathrm{Re}(g_k + \overline{h}_k) = \mathrm{Re}(g_k + h_k)$, we have $|q(z, w)| = 1$ if $w = F(z)$, and $|q(z^0, w^0)| = \exp(\mathrm{Re}(F_k(z^0) - w_k^0)) > 1$. $\square$

Corollary. *If F_j are pluriharmonic on $\overline{B}$, then $h(\Sigma) \subset G(F)$. If, in addition, $h(\Sigma)$ covers $\overline{B}$, then we have $h(\Sigma) = G(F)$.*

When $m < n$, (2) of Theorem A follows from this Corollary and (1) of Theorem A.

The following question naturally arises: Does $h(\Sigma) = G(F)$ hold when F_j are pluriharmonic and $h(\Sigma) \neq \Sigma$? The following example shows it is not true for $n = m$.

Example 2. For the pluriharmonic functions

$$f_1(z_1, z_2) = \overline{z}_1, \quad f_2(z_1, z_2) = \overline{z}_1\overline{z}_2,$$

we have $\Sigma \subsetneqq h(\Sigma) \subsetneqq G(F)$. In fact, we have

$$h(\Sigma) = \Sigma \cup \{(0, z_2, 0, 0) : |z_2| \leq 1\}.$$

Proof. $h(\Sigma)$ contains the disk

$$D = \{(0, z_2, 0, 0) : |z_2| \leq 1\}.$$

We show that $h(\Sigma) = \Sigma \cup D$. Let a and b be any complex numbers with $|a|^2+|b|^2 < 1$ and $a \neq 0$. We set

$$M = \max\{|z_1 - a|^2 + |z_1|^2|z_2 - b|^2 : (z_1, z_2) \in S\},$$

$$m = \min\{|z_1 - a|^2 + |z_1|^2|z_2 - b|^2 : (z_1, z_2) \in S\}.$$

Then we have $m > 0$. Consider the polynomial

$$P(z_1, z_2, w_1, w_2) = 1 - \alpha\{(z_1 - a)(w_1 - \overline{a}) + z_1(z_2 - b)(w_2 - \overline{b}w_1)\},$$

with $0 < \alpha < 2/M$.

If $w_k = f_k(z), \quad z \in S, \quad k = 1, 2$, then we have

$$P(z_1, z_2, w_1, w_2) = 1 - \alpha(|z_1 - a|^2 + |z_1|^2|z_2 - b|^2),$$

and hence $|P| < 1$. Since $P(a, b, w_1, w_2) = 1$, the point (a, b, w_1, w_2) does not belong to $h(\Sigma)$ for any (w_1, w_2), that is, $(a, b) \notin \pi(h\Sigma))$. □

3. Joint spectrum

Let K be a compact subset of $\mathbb{C}^n$ and $u_1, \ldots, u_k$ functions of $C(K)$. We denote by $[u_1, \ldots, u_k; K]$ the algebra of uniform limits on K of polynomials of $u_1, \ldots, u_k$. In particular, $[z_1, \ldots, z_n; \overline{B}]$ is the uniform algebras of all continuous functions on $\overline{B}$ which are holomorphic in B and is denoted by $A(B)$.

To show further examples we need a lemma on the joint spectrum. Let A be a uniform algebra on K, and M_A the maximal ideal space of A. For functions $f_1, \ldots, f_k$ of A, the *joint spectrum* is, by definition,

$$\sigma(f_1, \ldots, f_k) = \{(\varphi(f_1), \ldots, \varphi(f_k)) \in \mathbb{C}^k : \varphi \in M_A\}.$$

It is known that if A is generated by $f_1, \ldots, f_k$, then $\sigma(f_1, \ldots, f_k)$ is polynomially convex. We set

$$K^* = \{(f_1(z), \ldots, f_k(z)) : z \in K\}.$$

If p is a polynomial in $\mathbb{C}^k$, then we have

$$p(\varphi(f_1), \ldots, \varphi(f_k)) = \varphi(p(f_1, \ldots, f_k)).$$

Therefore we have

Lemma 2. *If $f_1, \dots, f_k \in A$, then*

$$\sigma(f_1, \dots, f_k) \subset h(K^*).$$

If A is generated by $f_1, \dots, f_k$, then

$$\sigma(f_1, \dots, f_k) = h(K^*).$$

Proposition 1. *If $f_1, \dots, f_{n-1} \in C(S)$ and $f_n \in [f_1, \dots, f_{n-1}; S]$, then*

$$\pi(h(\Sigma)) = \overline{B}.$$

Let $F_1, \dots, F_{n-1}$ be functions of $C(\overline{B})$, and let $f_1, \dots, f_{n-1}$ be the restriction of $F_1, \dots, F_{n-1}$ on S, restrictively. We denote by Σ_1 the graph of $f' = (f_1, \dots, f_{n-1})$ on S.

Proposition 2. *If $f_n \in [f_1, \dots, f_{n-1}; S]$ and if $h(\Sigma_1) = G(F_1, \dots, F_{n-1})$, then there exists a function F_n of $C(\overline{B})$ such that $h(\Sigma)$ is the graph $G(F_1, \dots, F_n)$ of $(F_1, \dots, F_n)$ on $\overline{B}$.*

4.

Let $g_1(z_1, z_2)$ and $g_2(z_1, z_2)$ be holomorphic functions on $\overline{B}$ which have no zeros. Set $F = (|g_1|, |g_2|)$ and $f = F|_S$. We consider the problems asking if $h(\Sigma) = G(F)$. We give two counter-examples. In the first example, $h(\Sigma) = \Sigma$, in the second one, $\Sigma \subsetneqq h(\Sigma) \subsetneqq G(F)$. We also give an example in which g_1 and g_2 have zeros and $G(F) \subsetneqq h(\Sigma)$.

Example 3. Let $h_1(\zeta)$ and $h_2(\zeta)$ be holomorphic on the unit disk $\overline{D}$ such that h_1', h_2' have no zeros on $\overline{D}$. We set

$$g_1(z_1, z_2) = e^{h_1(z_1)}, \ g_2(z_1, z_2) = e^{h_2(z_2)}.$$

Then $h(\Sigma) = \Sigma$.

Proof. For any pair $u = (u_1, u_2)$ of real numbers, we set

$$l_k = \{\zeta \in \overline{D} : \mathrm{Re} h_k(\zeta) = u_k\}, \ k = 1, 2,$$

$$L_u = \{(z_1, z_2) \in \overline{D} \times \overline{D} : |g_1| = e^{u_1}, |g_2| = e^{u_2}\}.$$

For each k, $\mathrm{Re} h_k$ is harmonic. Hence l_k does not divide the plane and hence is polynomially convex. Since moreover $\partial h_k \neq 0$, it is a totally real set. Since $L_u = l_1 \times l_2$, L_u is a polynomially convex totally real set of $\mathbb{C}^2$. This implies $P(L_u) = C(L_u)$ (cf. [6]). By Theorem 3 of [6], which is a generalization of Merglyan's theorem [5], we have

$$[z_1, z_2, |g_1|, |g_2|; \overline{D} \times \overline{D}] = C(\overline{D} \times \overline{D}),$$

which implies $P(\Sigma) = C(\Sigma)$. Therefore Σ is polynomially convex. □

Example 4. We consider $g_1(z_1, z_2) = e^{2z_1}$ and $g_2(z_1, z_2) = e^{2z_1 z_2}$. Then $\Sigma \subsetneqq h(\Sigma) \subsetneqq G(F)$. In fact, we have

$$h(\Sigma) = \Sigma \cup \{(0, z_2, 1, 1) : |z_2| \leq 1\}.$$

Proof. We consider three algebras

$$A_1 = [z_1, z_2, \overline{z}_1, \overline{z_1 z_2}; S],$$

$$A_2 = [z_1, z_2, e^{\overline{z_1}}, e^{\overline{z_1 z_2}}; S],$$

$$A_3 = [z_1, z_2, e^{z_1+\overline{z_1}}, e^{z_1 z_2+\overline{z_1 z_2}}; S].$$

We denote by Σ_1, Σ_2 and Σ_3 the graphs on S of $(\overline{z_1}, \overline{z_1 z_2})$, $(e^{z_1}, e^{\overline{z_1 z_2}})$ and $(e^{z_1+\overline{z_1}}, e^{z_1 z_2+\overline{z_1 z_2}})$ respectively. By Example 2 and Lemma 2, we have

$$\sigma(z_1, z_2, \overline{z_1}, \overline{z_1 z_2}) = h(\Sigma_1) = \Sigma_1 \cup \{(0, z_2, 0, 0) : |z_2| \leq 1\}.$$

If we can prove that these algebras are the same, then so are the maximal ideal spaces of these algebras and hence, by Lemma 2, we have

$$h(\Sigma_3) = \sigma(z_1, z_2, e^{z_1+\overline{z_1}}, e^{z_1 z_2+\overline{z_1 z_2}}) = \Sigma_3 \cup \{(0, z_2, 1, 1) : |z_2| \leq 1\}.$$

We show $A_1 = A_2$. Evidently we have $A_2 \subset A_1$. Consider the algebra $[\zeta, e^{\overline{\zeta}}; \overline{D}]$ on the unit disk $\overline{D}$. Since $e^{\overline{\zeta}}$ is pluriharmonic, the graph G on $\overline{D}$ is polynomially convex, by Corollary of Lemma 1. G is a totally real set, since $\frac{\partial}{\partial \overline{\zeta}} e^{\overline{\zeta}} \neq 0$. Hence we have $P(G) = C(G)$ and $[\zeta, e^{\overline{\zeta}}; \overline{D}] = C(\overline{D})$. We have $\overline{\zeta} \in [\zeta, e^{\overline{\zeta}}; \overline{D}]$, which implies

$$\overline{z}_1 \in [z_1, e^{\overline{z}_1}; \overline{D} \times \overline{D}] \text{and } \overline{z_1 z_2} \in [z_1 z_2, e^{\overline{z_1 z_2}}; \overline{D} \times \overline{D}].$$

Therefore, we have

$$[z_1, z_2, \overline{z}_1, \overline{z_1 z_2}; \overline{D} \times \overline{D}] \subset [z_1, z_2, e^{\overline{z}_1}, e^{\overline{z_1 z_2}}; \overline{D} \times \overline{D}].$$

It follows that $A_1 \subset A_2$.

Evidently, we have $A_3 \subset A_1$. Since e^{-z_1} and $e^{-(z_1+z_2)}$ are approximated by polynomials of z_1, z_2, we have $A_2 \subset A_3$. Therefore, we have $A_1 = A_2 = A_3$. □

In the following example, g_1 and g_2 have zeros and $h(\Sigma)$ covers $\overline{B}$.

Example 5. If

$$g_1(z_1, z_2) = z_1^2, \quad g_2(z_1, z_2) = z_2^2,$$

then $h(\Sigma)$ covers $\overline{B}$. But $h(\Sigma)$ is not a graph of the function $F = (|g_1|, |g_2|)$. In fact, $h(\Sigma)$ coincides with a closed manifold

$$M = \{(z_1, z_2, w_1, w_2) \colon \mathrm{Im} w_1 = \mathrm{Im} w_2 = 0, |z_1|^2 \leq \mathrm{Re} w_1 \leq 1 - |z_2|^2, \mathrm{Re} w_2 = 1 - \mathrm{Re} w_1\}.$$

The real dimension of M is 5.

Proof. Set $\Sigma_1 = \{(z_1, z_2, |z_1|^2) \in \mathbb{C}^3 : (z_1, z_2) \in S\}$. For any point (a, b, t^2) with $|a|^2 + |b|^2 < 1, |a|^2 < t^2 \leq 1 - |b|^2, t > 0$, we consider the map

$$F = (F_1, F_2, F_3) : \lambda \mapsto \left(\frac{t(t\lambda + a)}{t + \overline{a}\lambda}, \frac{\sqrt{1-t^2}(\sqrt{1-t^2}\lambda + b)}{\sqrt{1-t^2} + \overline{b}\lambda}, t^2 \right)$$

of the disk $|\lambda| \leq 1$ into $\mathbb{C}^3$. If $|\lambda| = 1$, then $|F_1(\lambda)|^2 + |F_2(\lambda)|^2 = 1$. Hence $(a, b, t^2) \in h(\Sigma_1)$.

By taking appropriate polynomials, it follows that other points do not belong to $h(\Sigma_1)$. Hence we have $h(\Sigma_1) = \{(z_1, z_2, w_1) : |z_1|^2 \leq \mathrm{Re} w_1 \leq 1 - |z_2|^2, \mathrm{Im} w_1 = 0\}$.

Since $h(\Sigma) = \sigma(z_1, z_2, |z_1|^2, 1 - |z_1|^2)$, we have $h(\Sigma) = M$. □

References

[1] P. Ahern and W.Rudin: *Hulls of 3-spheres in* $\mathbb{C}^3$, Contemporary Math. 137 (1992), 1 - 27. MR **93k:**32020

[2] H. Alexander: *Polynomial hulls of graphs*, Pacific J. Math. 147 (1991), 201 - 212. MR **91k:**32013

[3] J. T. Anderson: *On an example of Ahern and Rudin*, Proc. Amer. Math. Soc. 116 (1992), 695 - 699. MR **93c:**32021

[4] H. Alexander and J. Wermer: *Polynomial hulls with convex fibers*, Math. Ann. 271 (1985), 99-109. MR **86i:**32025

[5] S. N. Mergelyan: *Uniform approximation to functions of a complex variable*, Uspehi Mat. Nauk. 7 (1952), No.2, 31-122; Amer. Math. Soc Transl. 3 (1962), Ser.1, 294-391. MR **14:**547e

[6] A. Sakai: *Uniform approximation in several complex variables*, Osaka J. Math. 15 (1978), 586 - 611. MR **81e:**32019

[7] J. Wermer: *On algebras of continuous functions*, Proc. Amer. Math. Soc. 4 (1953), 866-869. MR **15:**440g

Department of Mathematics, Nara University of Education, Takabatake, Nara 630-8528, Japan
E-mail address: `jimbo@nara-edu.ac.jp`

Yamadanishi 2-9,A3-612 Suita, Osaka 565-0824, Japan
E-mail address: `CXH02215@niftyserve.or.jp`