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WEIGHTED CACCIOPPOLI-TYPE ESTIMATES AND WEAK REVERSE HÖLDER INEQUALITIES FOR A-HARMONIC TENSORS

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ABSTRACT. We obtain a local weighted Caccioppoli-type estimate and prove the weighted version of the weak reverse Hölder inequality for A-harmonic tensors.

1. INTRODUCTION

Harmonic functions have wide applications in many fields, such as potential theory, partial differential equations, harmonic analysis and the theory of H^p-spaces. A-harmonic tensors are interesting and important generalizations of p-harmonic tensors. In the meantime, p-harmonic tensors are extensions of conjugate harmonic functions and p-harmonic functions, $p > 1$. In recent years there have been remarkable advances made in the field of A-harmonic tensors. Many interesting results of A-harmonic tensors and their applications in fields such as potential theory, quasiregular mappings and the theory of elasticity have been found; see [1], [2], [3], [7], [8], [9], [10], [11], [12], [14]. For many purposes, we need to know the integrability of A-harmonic tensors and estimate the integrals for A-harmonic tensors. In this paper we first obtain the local weighted Caccioppoli-type estimate and the weighted version of the weak reverse Hölder inequality for A-harmonic tensors. These integral inequalities can be used to study the integrability of A-harmonic tensors and estimate the integrals for A-harmonic tensors.

We always assume Ω is a connected open subset of $\mathbf{R}^n$ throughout this paper. Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of $\mathbf{R}^n$. For $l = 0, 1, \cdots, n$, the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathbf{R}^n)$. The Grassmann algebra $\wedge = \bigoplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the Hodge star operator $\star$: $\wedge \to \wedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all α, $\beta \in \wedge$.

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Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}.$ The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star\,(-1)^{l(n-l)} : \wedge^l \to \wedge^l$. Let $0 < p < \infty$; we denote the weighted L^p-norm of a measurable function f over E by

$$||f||_{p,E,w} = \left(\int_E |f(x)|^p w(x) dx \right)^{1/p}.$$

A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbf{R}^n)$. We denote the space of differential l-forms by $D'(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$||\omega||_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega (\sum_I |\omega_I(x)|^2)^{p/2} dx \right)^{1/p}.$$

Similarly, $W_p^1(\Omega, \wedge^l)$ are those differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbf{R})$. The notations $W_{p,loc}^1(\Omega, \mathbf{R})$ and $W_{p,loc}^1(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$.

Recently there has been new interest developed in the study of the A-harmonic equation for differential forms, largely pertaining to applications in quasiconformal analysis and nonlinear elasticity, that is:

$$d^\star A(x, d\omega) = 0, \tag{1.1}$$

where $A : \Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1} \text{ and } \langle A(x, \xi), \xi \rangle \ge |\xi|^p \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W_{p,loc}^1(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi \rangle = 0$$

for all $\varphi \in W_p^1(\Omega, \wedge^{l-1})$ with compact support.

Definition 1.3. We call u an A-harmonic tensor in Ω if u satisfies the A-harmonic equation (1.1) in Ω.

Let us mention some basic terms for harmonic tensors as follows. A differential l-form $u \in D'(\Omega, \wedge^l)$ is called a closed form if $du = 0$ in Ω. A differential form u is called a p-harmonic tensor if

$$d^\star(|du|^{p-2} du) = 0 \text{ and } d^\star u = 0,$$

where $1 < p < \infty$. See [7] for more results about p-harmonic tensors. In order to formulate some estimates it is required first of all that the equation be written in the form of a first order differential system:

$$A(x, du) = d^\star v\ . \tag{1.4}$$

In this way we obtain a pair (u, v) of $(l-1)$-form u and $(l+1)$-form v, called the conjugate A-harmonic fields. Example: $du = d^*v$ is an analogue of a Cauchy-Riemann system in $\mathbf{R}^n$. Clearly, the A-harmonic equation is not affected by adding a closed form to u and coclosed form to v. Therefore, any type of estimates between u and v must be modulo such forms. Suppose that u is a solution to (1.1) in Ω. Then, at least locally in a ball B, there exists a form $v \in W_q^1(B, \wedge^{l+1})$, $\frac{1}{p} + \frac{1}{q} = 1$, such that (1.4) holds.

Definition 1.5. When u and v satisfy (1.4) in Ω, and A^{-1} exists in Ω, we call u and v conjugate A-harmonic tensors in Ω.

Definition 1.6. We call u a p-harmonic function if u satisfies the p-harmonic equation

$$\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$$

with $p > 1$. Its conjugate in the plane is a q-harmonic function v, $p^{-1} + q^{-1} = 1$, which satisfies

$$\nabla u|\nabla u|^{p-2} = (\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}).$$

Note that if $p = q = 2$, we get the usual conjugate harmonic functions.

We write $\mathbf{R} = \mathbf{R}^1$. Balls are denoted by B and σB is the ball with the same center as B and with $diam(\sigma B) = \sigma diam(B)$. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathbf{R}^n)$ and $w > 0$ a.e. Also in general $d\mu = w dx$ where w is a weight. The following result appears in [8]: Let $Q \subset \mathbf{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$ defined by

$$(K_y\omega)(x; \xi_1, \cdots, \xi_l) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \cdots, \xi_{l-1})dt$$

and the decomposition

$$\omega = d(K_y\omega) + K_y(d\omega).$$

We define another linear operator $T_Q : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$ by averaging K_y over all points y in Q

$$T_Q\omega = \int_Q \varphi(y)K_y\omega dy,$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)dy = 1$. We define the l-form $\omega_Q \in D'(Q, \wedge^l)$ by

$$\omega_Q = |Q|^{-1}\int_Q \omega(y)dy, \quad l = 0, \text{ and } \omega_Q = d(T_Q\omega), \quad l = 1, 2, \cdots, n,$$

for all $\omega \in L^p(Q, \wedge^l)$, $1 \le p < \infty$.

2. The local weighted Caccioppoli-type estimate

Definition 2.1. We say the weight $w(x)$ satisfies the A_r condition, $r > 1$, written $w \in A_r$, if $w(x) > 0$ a.e., and, for any ball $B \subset \mathbf{R}^n$,

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty.$$

See [5] and [6] for the basic properties of A_r-weights. We need the following lemma [5].

Lemma 2.2. *If $w \in A_r$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\| w \|_{\beta,B} \leq C|B|^{(1-\beta)/\beta} \| w \|_{1,B}$$

for all balls $B \subset \mathbf{R}^n$.

We will also need the following generalized Hölder's inequality.

Lemma 2.3. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\| fg \|_{s,\Omega} \leq \| f \|_{\alpha,\Omega} \cdot \| g \|_{\beta,\Omega} \tag{2.4}$$

for any $\Omega \subset \mathbf{R}^n$.

In [10], C. A. Nolder obtains the following local Caccioppoli-type estimate.

Theorem A. *Let u be an A-harmonic tensor in Ω and let $\sigma > 1$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B} \leq C|B|^{-1}\|u - c\|_{s,\sigma B}$$

for all balls or cubes B with $\sigma B \subset \Omega$ and all closed forms c. Here $1 < s < \infty$.

The following weak reverse Hölder inequality appears in [10].

Theorem B. *Let u be an A-harmonic tensor in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st}\|u\|_{t,\sigma B}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

We now generalize Theorem A into the following local weighted Caccioppoli-type estimate for A-harmonic tensors.

Theorem 2.5. *Let $u \in D'(\Omega, \wedge^l)$, $l = 0, 1, \cdots, n$, be an A-harmonic tensor in a domain $\Omega \subset \mathbf{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and $w \in A_r$ for some $r > 1$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w} \leq C|B|^{-1}\|u - c\|_{s,\rho B,w}, \tag{2.6}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

Note that (2.6) can be written as

$$\left(\int_B |du|^s w dx\right)^{1/s} \leq \frac{C}{|B|}\left(\int_{\rho B} |u - c|^s w dx\right)^{1/s}, \tag{2.6$'$}$$

or

$$\left(\int_B |du|^s d\mu\right)^{1/s} \leq \frac{C}{|B|}\left(\int_{\rho B} |u - c|^s d\mu\right)^{1/s}, \tag{2.6$''$}$$

where the measure μ is defined by $d\mu = w(x)dx$ and $w \in A_r$.

Proof. Since $w \in A_r$ for some $r > 1$, by Lemma 2.2, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\| w \|_{\beta,B} \le C_1 |B|^{(1-\beta)/\beta} \| w \|_{1,B} \tag{2.7}$$

for any cube or any ball $B \subset \mathbf{R}^n$. Choose $t = s\beta/(\beta-1)$; then $1 < s < t$ and $\beta = t/(t-s)$. Since $1/s = 1/t + (t-s)/st$, by Hölder's inequality, Theorem A and (2.7), we have

$$\begin{aligned} \|du\|_{s,B,w} &= \left(\int_B \left(|du| w^{1/s} \right)^s dx \right)^{1/s} \\ &\le \left(\int_B |du|^t dx \right)^{1/t} \left(\int_B \left(w^{1/s} \right)^{st/(t-s)} dx \right)^{(t-s)/st} \\ &\le C_2 \|du\|_{t,B} \cdot \|w\|_{\beta,B}^{1/s} \\ &\le C_3 |B|^{-1} \|u-c\|_{t,\sigma B} \cdot \|w\|_{\beta,B}^{1/s} \\ &\le C_4 |B|^{-1} |B|^{(1-\beta)/\beta s} \|w\|_{1,B}^{1/s} \cdot \|u-c\|_{t,\sigma B} \\ &= C_4 |B|^{-1} |B|^{-1/t} \cdot \|w\|_{1,B}^{1/s} \cdot \|u-c\|_{t,\sigma B} \end{aligned} \tag{2.8}$$

for all balls B with $\sigma B \subset \Omega$ and all closed forms c. Since c is a closed form and u is an A-harmonic tensor, then $u - c$ is still an A-harmonic tensor. Taking $m = s/r$, we find that $m < s < t$. Applying Theorem B yields

$$\begin{aligned} \|u-c\|_{t,\sigma B} &\le C_5 |B|^{(m-t)/mt} \|u-c\|_{m,\sigma^2 B} \\ &\le C_5 |B|^{(m-t)/mt} \|u-c\|_{m,\rho B} \end{aligned} \tag{2.9}$$

where $\rho = \sigma^2$. Substituting (2.9) in (2.8), we have

$$\|du\|_{s,B,w} \le C_6 |B|^{-1} |B|^{-1/m} \cdot \|w\|_{1,B}^{1/s} \cdot \|u-c\|_{m,\rho B}. \tag{2.10}$$

Now $1/m = 1/s + (s-m)/sm$; by Hölder's inequality again, we obtain

$$\begin{aligned} \|u-c\|_{m,\rho B} &= \left(\int_{\rho B} |u-c|^m dx \right)^{1/m} \\ &= \left(\int_{\rho B} \left(|u-c| w^{1/s} w^{-1/s} \right)^m dx \right)^{1/m} \\ &\le \left(\int_{\rho B} |u-c|^s w dx \right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w} \right)^{m/(s-m)} dx \right)^{(s-m)/sm} \\ &\le \|u-c\|_{s,\rho B,w} \cdot \|1/w\|_{m/(s-m),\rho B}^{1/s} \end{aligned} \tag{2.11}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. Combining (2.10) and (2.11), we obtain

$$\|du\|_{s,B,w} \le C_6 |B|^{-1} |B|^{-1/m} \cdot \|w\|_{1,B}^{1/s} \cdot \|1/w\|_{m/(s-m),\rho B}^{1/s} \cdot \|u-c\|_{s,\rho B,w}. \tag{2.12}$$

Since $w \in A_r$, we then have

$$\begin{aligned}
\|w\|_{1,B}^{1/s} \cdot \|1/w\|_{m/(s-m),\rho B}^{1/s} &= \left(\int_B w dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{m/(s-m)} dx\right)^{(s-m)/sm} \\
&\le \left(\left(\int_{\rho B} w dx\right)\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(s/m-1)} dx\right)^{s/m-1}\right)^{1/s} \\
&= \left(|\rho B|^{s/m}\left(\frac{1}{|\rho B|}\int_{\rho B} w dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{1/s} \\
&\le C_7 |B|^{1/m}.
\end{aligned} \tag{2.13}$$

Substituting (2.13) in (2.12), we find that

$$\|du\|_{s,B,w} \le C|B|^{-1}\|u-c\|_{s,\rho B,w}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c. This ends the proof of Theorem 2.5.

3. The weighted version of the weak reverse Hölder inequality

We now generalize Theorem B into the following weighted form.

Theorem 3.1. *Let $u \in D'(\Omega, \wedge^l)$, $l = 0, 1, \cdots, n$, be an A-harmonic tensor in a domain $\Omega \subset \mathbf{R}^n$, $\sigma > 1$. Assume that $0 < s, t < \infty$ and $w \in A_r$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |u|^s w dx\right)^{1/s} \le C|B|^{(t-s)/st}\left(\int_{\sigma B} |u|^t w^{t/s} dx\right)^{1/t} \tag{3.2}$$

for all balls B with $\sigma B \subset \Omega$.

The proof of Theorem 3.1 is similar to that of Theorem 2.5. For completion of the paper, we prove Theorem 3.1 as follows.

Proof. Since $w \in A_r$ for some $r > 1$, by Lemma 2.2, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\| w \|_{\beta,B} \le C_1 |B|^{(1-\beta)/\beta} \| w \|_{1,B} \tag{3.3}$$

for any cube or any ball $B \subset \mathbf{R}^n$. Choose $k = s\beta/(\beta-1)$; then $s < k$ and $\beta = k/(k-s)$. By (3.3) and Hölder's inequality, we have

$$\begin{aligned}
\|u\|_{s,B,w} &\le \left(\int_B |u|^k dx\right)^{1/k}\left(\int_B \left(w^{1/s}\right)^{sk/(k-s)} dx\right)^{(k-s)/sk} \\
&= \|u\|_{k,B} \cdot \|w\|_{\beta,B}^{1/s} \\
&\le C_2 |B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s} \cdot \|u\|_{k,B} \\
&= C_2 |B|^{-1/k}\|w\|_{1,B}^{1/s} \cdot \|u\|_{k,B}
\end{aligned} \tag{3.4}$$

for all balls B with $\sigma B \subset \Omega$. Choosing $m = st/(s + t(r-1))$, by Theorem B we obtain

$$\|u\|_{k,B} \le C_3 |B|^{(m-k)/km} \|u\|_{m,\sigma B}. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\|u\|_{s,B,w} \le C_4 |B|^{-1/m} \|w\|_{1,B}^{1/s} \cdot \|u\|_{m,\sigma B}. \tag{3.6}$$

Since $m < t$, by Hölder's inequality, we have

$$\begin{aligned} \|u\|_{m,\sigma B} &= \left(\int_{\sigma B} \left(|u| w^{1/s} w^{-1/s} \right)^m dx \right)^{1/m} \\ &\le \left(\int_{\sigma B} |u|^t w^{t/s} dx \right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w} \right)^{mt/(s(t-m))} dx \right)^{(t-m)/mt} \\ &\le \|1/w\|_{mt/(s(t-m)),\sigma B}^{1/s} \left(\int_{\sigma B} |u|^t w^{t/s} dx \right)^{1/t}. \end{aligned} \tag{3.7}$$

By the choice of m, we find that $r-1 = s(t-m)/mt$. Since $w \in A_r$, we then obtain

$$\begin{aligned} &\|w\|_{1,B}^{1/s} \cdot \|1/w\|_{mt/(s(t-m)),\sigma B}^{1/s} \\ &\quad = \left(\left(\int_B w dx \right) \left(\int_{\sigma B} \left(\frac{1}{w} \right)^{mt/(s(t-m))} dx \right)^{s(t-m)/mt} \right)^{1/s} \\ &\quad \le \left(|\sigma B|^{1+s(t-m)/tm} \left(\frac{1}{|\sigma B|} \int_{\sigma B} w dx \right) \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} \right)^{1/s} \\ &\quad \le C_5 |B|^{1/s + 1/m - 1/t}. \end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$\begin{aligned} \|u\|_{s,B,w} &\le C_4 |B|^{-1/m} \|w\|_{1,B}^{1/s} \cdot \|1/w\|_{mt/(s(t-m)),\sigma B}^{1/s} \left(\int_{\sigma B} |u|^t w^{t/s} dx \right)^{1/t} \\ &\le C_6 |B|^{1/s - 1/t} \left(\int_{\sigma B} |u|^t w^{t/s} dx \right)^{1/t}. \end{aligned} \tag{3.9}$$

It is easy to see that (3.9) is equivalent to (3.2). This ends the proof of Theorem 3.1.

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