EXISTENCE OF ERGODIC RETRACTION FOR NONCOMMUTATIVE SEMIGROUPS IN BANACH SPACES

OSAMU KADA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We prove the existence of ergodic retraction for a noncommutative semigroup which is right Eberlein-weakly almost periodic in a uniformly convex Banach space.

1. INTRODUCTION

Let S be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that for each $t \in S$, mappings $s \mapsto st$ and $s \mapsto ts$ from S to S are continuous. We denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all integers, nonnegative integers, positive integers, real numbers and nonnegative real numbers, respectively. Let C be a subset of a real Banach space E . Then a mapping $T : C \rightarrow C$ is called *nonexpansive* provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The following nonlinear ergodic theorem for nonexpansive mappings (in a Hilbert space) was established by Baillon [1, 2]:

Theorem 1. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself. If T has a fixed point, then the Cesàro means $(1/n) \sum_{k=0}^{n-1} T^k x$ converge weakly as $n \rightarrow \infty$ to a fixed point y of T , and if $-C = C$, $0 \in C$ and T is odd, then the Cesàro means converge strongly as $n \rightarrow \infty$ to a fixed point y of T .*

In these cases, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto the set $\text{Fix}(T)$ of fixed points of T such that $PT^n = T^n P = P$ for all $n \in \mathbb{Z}^+$, and $Px \in \text{cl co}\{T^n x; n \in \mathbb{Z}^+\}$ for each $x \in C$, where $\text{cl co } A$ is the closure of the convex hull of A . We call such a retraction “ergodic retraction”. Takahashi [29] proved the existence of ergodic retraction for a noncommutative semigroup in a Hilbert space, and Hirano, Kido and Takahashi [12] proved it for commutative semigroups in a uniformly convex Banach space. Recently, we studied the operator $T(\mu)$ on C when μ is an invariant mean on $C_b(S)$, which is the candidate of the ergodic retraction ([16]). But it remains open whether there exists ergodic retraction for noncommutative semigroups in Banach spaces. On the other hand, Ruess and Summers [24, 26, 27] proved strong mean ergodic theorems for Eberlein-weakly almost periodic functions over $S = \mathbb{R}^+$, which generalized Baillon [2] and Bruck

Received by the editors January 22, 1996 and, in revised form, July 18, 1997 and January 9, 1998.

1991 *Mathematics Subject Classification.* Primary 47A35, 47H09, 47H20.

[4, 5]. Further, we generalized this for commutative semigroups and obtained some results about ergodic projections and retractions ([13]).

In this paper, we prove the existence of ergodic retraction for a noncommutative semigroup in a uniformly convex Banach space, assuming the semigroup is right Eberlein-weakly almost periodic.

2. PRELIMINARIES

Throughout this paper we assume a Banach space E is real. Let S be a semitopological semigroup with identity, let $l^\infty(S)$ be the Banach space of all bounded real valued functions on S with the supremum norm, and let $l^1(S)$ be the Banach space of all functions $f : S \rightarrow \mathbb{R}$ such that $\|f\|_1 = \sum_{s \in S} |f(s)| < \infty$ with norm $\|\cdot\|_1$. Then $l^1(S)^* = l^\infty(S)$. We also denote by $C_b(S)$ the space of all bounded continuous real valued functions on S . Then for each $s \in S$ and $f \in l^\infty(S)$, we define elements $r(s)f$ and $l(s)f$ in $l^\infty(S)$ by $(r(s)f)(t) = f(ts)$, and $(l(s)f)(t) = f(st)$, $t \in S$. Let X be a subspace of $l^\infty(S)$ containing constant functions and translation invariant; i.e. for any $s \in S$, $r(s)X \subset X$ and $l(s)X \subset X$. Then an element μ of X^* , where X^* is the dual space of X , is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. As is known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$. A mean μ on X is invariant if $\mu(r(s)f) = \mu(f)$ for all $s \in S$ and $f \in X$. A finite mean is an element of $\text{co}\{\delta(s); s \in S\}$, where $\delta(s)(f) = f(s)$ for each $f \in l^\infty(S)$. Depending on time and circumstances, the values of mean μ at $f \in X$ will also be denoted by $\mu(f)$ or $\mu_s(f(s))$.

S is called *right reversible* if any two closed left ideals of S have nonvoid intersection. In this case, $(S, \leq)$ is a directed system when the binary relation “ $\leq$ ” on S is defined by $s \leq t$ if and only if $\{s\} \cup \text{cl}(Ss) \supset \{t\} \cup \text{cl}(St)$, $s, t \in S$.

Remark 2.1. Let S be a semitopological semigroup such that the topology is normal and $C_b(S)$ has a right (resp. left) invariant mean. Then S is right (resp. left) reversible; see Lau and Takahashi [21].

Let C be a closed subset of a Banach space E . A mapping V of C into itself is called *Lipschitzian* if there exists $K \geq 0$ such that $\|Vx - Vy\| \leq K\|x - y\|$ for every $x, y \in C$. We define the *Lipschitz norm* of V by

$$\|V\| := \sup \left\{ \frac{\|Vx - Vy\|}{\|x - y\|}; x, y \in C, x \neq y \right\}.$$

If $\|V\| \leq 1$, then V is called *nonexpansive*. We denote by $\text{Lip}(C)$, $\text{Cont}(C)$, $\text{Aff}(C)$ and $L(E)$, the semitopological semigroup of all Lipschitzian self-mappings of C , the semitopological semigroup of all nonexpansive self-mappings of C , the semitopological semigroup of all affine self-mappings of C and the semitopological semigroup of all bounded linear mappings of E , under composition and pointwise convergence topology, respectively. Let $T : S \rightarrow \text{Lip}(C)$, $\text{Cont}(C)$ or $L(E)$ be a *representation* (resp. *anti-representation*), i.e., $T(s+t) = T(s)T(t)$ (resp. $T(st) = T(t)T(s)$) for every $s, t \in S$, and $T(\cdot)x$ is continuous for every $x \in C$ or E . We denote by $\text{Fix}(T)$ the *set of common fixed points* of $T(s)$, $s \in S$, and by $O(x) = \{T(s)x; s \in S\}$ the orbit from $x \in C$. We also denote by $B(E) = \{x \in E; \|x\| \leq 1\}$ the unit ball of E . For two Banach spaces E and F , $L(E, F)$ is the set of all bounded linear mappings from E to F .

First we give a lemma.

Lemma 2.2. *Let S be a semitopological semigroup such that there exists a left invariant mean μ on $C_b(S)$. Let C be a closed convex subset of a Banach space E and let D be a bounded subset of C . Let $T : S \rightarrow \text{Cont}(C)$ be a representation such that $\{T(s)x_0; s \in S\}$ is bounded for some $x_0 \in C$. Then there exists a bounded closed convex subset M of C , which is $T(s)$ -invariant for every $s \in S$ and includes $K := \{T(s)x; s \in S, x \in D\}$.*

Proof. Put $a = \sup_{s \in S} \|T(s)x_0\|$, $b = \sup_{x \in D} \|x\|$, and define the set

$$M := \{z \in C; \mu_s \|T(s)x_0 - z\| \leq a + b\}.$$

Then we can easily see that M is as in the assertion. $\square$

Let S be a topological space, let E be a Banach space E and let $C_b(S, E)$ be the Banach space of all bounded continuous mappings from S to E with supremum norm, and let $C_C(S, E)$ be the set of all elements $f \in C_b(S, E)$ such that $f(S) := \{f(s); s \in S\}$ is relatively weakly compact. Then $C_C(S, E)$ is a linear subspace of $C_b(S, E)$. For any mean μ on $C_b(S)$, we define a “vector valued mean” $\tau(\mu)$, homomorphically; i.e., $\tau(\mu)$ is an element of $L(C_C(S, E), E)$ such that $\tau(\mu)x = x$ for all constant functions $x \in E$, and $\|\tau(\mu)\| = 1$. More generally, we give definitions as follows; see [18].

Definition 2.3. Let S be a topological space and let E be a Banach space. For $\mu \in C_b(S)^*$ and $f \in C_C(S, E)$, let $x_{\mu, f}^{**} : x^* \mapsto \mu_s \langle f(s), x^* \rangle$ be an element of E^{**} . Then $x_{\mu, f}^{**} \in E$. We define an element $\tau = \tau^E \in L(C_b(S)^*, L(C_C(S, E), E))$ by $\tau(\mu)f = x_{\mu, f}^{**}$.

Remark 2.4. The following holds; see [18]. (i) If μ is a mean on $C_b(S)$, then $\tau(\mu)f \in \text{cl co}\{f(s); s \in S\}$; (ii) τ is injective and $\|\tau\| \leq 1$; (iii) $\tau(\mu)x = \mu(1)x$ for all $x \in E$, and if μ is a mean on $C_b(S)$, then $\|\tau(\mu)\| = 1$; (iv) τ maps the point evaluation $\delta(s)$ to the point evaluation $\varepsilon(s)$, $s \in S$, where $\varepsilon(s)f = f(s)$ for $f \in C_C(S, E)$; (v) $\tau(r(s)^*\mu) = r(s)^*\tau(\mu)$.

Let $T : S \rightarrow \text{Lip}(C)$ be a representation such that $T(\cdot)x \in C_C(S, E)$ for some $x \in C$. Then we shall denote $\tau(\mu)(T(\cdot)x)$ by $T(\mu)x$, which is an element of E .

Remark 2.5. From (ii) and (iii), for any $\{a_i\} \subset \mathbb{R}$, $\{s_i\} \subset S$, $f \in C_C(S, E)$, we get $\tau(\sum_{i=1}^n a_i \delta(s_i))f = \sum_{i=1}^n a_i \tau(\delta(s_i))f = \sum_{i=1}^n a_i \varepsilon(s_i)f = \sum_{i=1}^n a_i f(s_i)$.

We give two lemmas.

Lemma 2.6. *Let $f \in C_C(S, E)$ and let μ be a left (resp. right) invariant mean on $C_b(S)$. Then $\tau(\mu)f \in \bigcap_{s \in S} \text{cl co}\{f(st); t \in S\}$ (resp. $\tau(\mu)f \in \bigcap_{s \in S} \text{cl co}\{f(ts); t \in S\}$). If μ is an invariant mean on $C_b(S)$, $\tau(\mu)f \in \bigcap_{s, q \in S} \text{cl co}\{f(stq); t \in S\}$.*

Proof. For any $s \in S$, considering $l(s)f$ for f in Remark 2.4(i), we have $\tau(\mu)f = \tau(\mu)(l(s)f) \in \text{cl co}\{(l(s)f)(t); t \in S\} = \text{cl co}\{f(st); t \in S\}$. So, the first assertion follows. The other assertions are proved similarly. $\square$

Lemma 2.7. *Let μ be a right invariant mean on $C_b(S)$ and let λ be a finite mean. Putting $f(t) = T(l(t)^*\lambda)x$ for every $t \in S$ and $x \in E$, we have $T(\mu)x = \tau(\mu)f$.*

Proof. Let $\lambda = \sum_{i=1}^n a_i \delta(s_i)$. Then by Remark 2.5, $T(l(t)^* \lambda)x = \sum_{i=1}^n a_i T(ts_i)x$, hence

$$\begin{aligned} \tau(\mu)f &= \tau(\mu)(T(l(\cdot)^* \lambda)x) = \tau(\mu) \left(\sum_{i=1}^n a_i T(\cdot s_i)x \right) \\ &= \sum_{i=1}^n a_i \tau(\mu)(T(\cdot s_i)x) = \sum_{i=1}^n a_i \tau(\mu)(T(\cdot)x) \\ &= \tau(\mu)(T(\cdot)x) = T(\mu)x. \end{aligned}$$

Therefore $T(\mu)x = \tau(\mu)f$. $\square$

3. EXISTENCE OF ERGODIC RETRACTION

In this section, we prove that there exists the ergodic retraction for a noncommutative semigroup which is right Eberlein-weakly almost periodic. First, we give the definition of vector valued weakly almost periodic functions; see Ruess and Summers [24].

Definition 3.1. Let S be a semitopological semigroup and let E be a Banach space. We say a function $f \in C_b(S, E)$ is *right (resp. left) Eberlein-weakly almost periodic* if $\{r(s)f; s \in S\}$ (resp. $\{l(s)f; s \in S\}$) is a relatively weakly compact subset of $C_b(S, E)$. If $f \in C_b(S, E)$ is right and left Eberlein-weakly almost periodic, then we say f is *Eberlein-weakly almost periodic*. We denote by $WR(S, E)$ (resp. $WL(S, E), W(S, E)$) the space of all of right Eberlein- (resp. left Eberlein-, Eberlein-) weakly almost periodic functions. Let $RUC(S, E)$ be the set of all *right bounded uniformly continuous* functions from S to E , i.e., the set of all $f \in C_b(S, E)$ such that the map $s \mapsto r(s)f$ from S to $C_b(S, E)$ is continuous. Let C be a closed subset of E . We say a representation $T : S \rightarrow \text{Cont}(C)$ is *right Eberlein- (resp. left Eberlein-, Eberlein-) weakly almost periodic* on a subset D of C if for any $x \in D, T(\cdot)x \in WR(S, E)$ (resp. $WL(S, E), W(S, E)$).

Proposition 3.2. (a) $WR(S, E), WL(S, E)$ and $W(S, E)$ are closed translation invariant linear subspaces of $C_b(S, E)$, and (b) $RUC(S, E)$ is a closed left translation invariant linear subspace of $C_b(S, E)$.

Proof. (a) (i) Closeness of $WR(S, E), WL(S, E)$ and $W(S, E)$: this is proved just as in the proof of Berglund, Junghenn and Milnes [3, Theorem 4.2.5].

(ii) Translation invariance of $WR(S, E), WL(S, E)$ and $W(S, E)$: For $f \in WR(S, E)$ and $t \in S$, since $\{r(s)r(t)f; s \in S\} \subset \{r(s)f; s \in S\}$, we see that $r(t)f \in WR(S, E)$. So, $WR(S, E)$ is right translation invariant.

Let $f \in WR(S, E)$ and $t \in S$. Since $\text{cl}\{r(s)l(t)f; s \in S\} = \text{cl}\{l(t)r(s)f; s \in S\} = l(t)\text{cl}\{r(s)f; s \in S\}$, and $l(t)$ is weak to weak continuous, $\text{cl}\{r(s)l(t)f; s \in S\}$ is weakly compact. So, we have $l(t)f \in WR(S, E)$, and $WR(S, E)$ is left translation invariant. Similarly, we see $WL(S, E)$ and $W(S, E)$ are translation invariant.

(b) Closeness of $RUC(S, E)$: Let $f \in \text{cl } RUC(S, E)$, and let $\{f_\alpha\} \subset RUC(S, E)$ be such that $f_\alpha \rightarrow f$. We show $f \in RUC(S, E)$. For this it is sufficient to show the following: let $\{s_\beta\}$ be a net of S such that $s_\beta \rightarrow s$. Then $r(s_\beta)f \rightarrow r(s)f$. For any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that

$$\|f - f_\alpha\| \leq \varepsilon.$$

Then there exists $\beta_0 = b_0(\alpha)$ such that for any $\beta \geq \beta_0$,

$$\|r(s_\beta)f_\alpha - r(s)f_\alpha\| \leq \varepsilon.$$

Then we have

$$\begin{aligned} \|r(s_\beta)f - r(s)f\| &\leq \|r(s_\beta)f - r(s_\beta)f_\alpha\| + \|r(s_\beta)f_\alpha - r(s)f_\alpha\| + \|r(s)f_\alpha - r(s)f\| \\ &\leq \varepsilon + 2\|f - f_\alpha\| \leq 3\varepsilon. \end{aligned}$$

This implies $r(s_\beta)f \rightarrow r(s)f$.

(ii) Left translation invariance of $RUC(S, E)$: Let $f \in RUC(S, E)$ and $q \in S$. Then we have

$$\begin{aligned} \|r(s)l(q)f - r(t)l(q)f\| &= \|l(q)r(s)f - l(q)r(t)f\| \\ &\leq \|r(s)f - r(t)f\|, \end{aligned}$$

so that $l(q)f \in RUC(S, E)$. $\square$

Remark 3.3. Let C be a closed convex subset of a uniformly convex Banach space E and let $T : S \rightarrow \text{Cont}(C)$ be a representation such that $\text{Fix}(T) \neq \emptyset$. Then we know that when $S = (\mathbb{Z}^+)^d$ or $(\mathbb{R}^+)^d$, $d \in \mathbb{N}$, if the representation T is strongly asymptotically isometric on a subset D of C , then T is Eberlein-weakly almost periodic on D . Here T is called strongly asymptotically isometric on D if the following hold: Let $\|\cdot\|$ be the restriction to S of any norm on $(\mathbb{R})^d$. For any $x, y \in D$, $\lim_{\|s\| \rightarrow \infty} \|T(s+h)x - T(s)y\|$ exists uniformly over $h \in S$ (see [13, 24]).

As in the proof of Ruess and Summers [27, Proposition 2.1], we have the following lemma.

Lemma 3.4. $f \in WR(S, E)$ (resp. $WL(S, E)$) if and only if $\{\langle f(t \cdot), x^* \rangle; t \in S, x^* \in B(E^*)\}$ (resp. $\{\langle f(\cdot t), x^* \rangle; t \in S, x^* \in B(E^*)\}$) is relatively weakly compact in $C_b(S)$.

Proof. By Milnes [22], $f \in WR(S, E)$ if and only if for any sequences $\{(t_m, x_m^*)\}$ in $S \times B(E^*)$ and $\{s_n\}$ in S ,

$$\lim_n \lim_m \langle f(t_m s_n), x_m^* \rangle = \lim_m \lim_n \langle f(t_m s_n), x_m^* \rangle,$$

whenever both iterated limits exists. From this, the assertion follows. The second assertion is proved similarly. $\square$

Let C be a closed convex subset of a Banach space E , and let T be a nonexpansive mapping on C and $\varepsilon > 0$. Then we define

$$F_\varepsilon(T) = \{x \in C; \|x - Tx\| \leq \varepsilon\}.$$

The following is Bruck's result [5].

Lemma 3.5. Let C be a bounded closed convex subset of a uniformly convex Banach space E . Then for any $\varepsilon > 0$ and any nonexpansive mapping T on C , there exists $\delta > 0$ such that

$$\text{cl co } F_\delta(T) \subset F_\varepsilon(T).$$

The following lemma is a noncommutative version of a result of Hirano, Kido and Takahashi [11]; see [16].

Lemma 3.6. *Let C be a closed subset of a Banach space E , let S be a right reversible semitopological semigroup, and let $T : S \rightarrow \text{Cont}(C)$ be a representation. Assume $\text{Fix}(T) \neq \emptyset$. Then for any $x \in C$ and any finite mean $\lambda \in \text{co}\{\delta(s); s \in S\}$,*

$$\lim_t \|T(s)T(l(t)^*\lambda)x - T(l(st)^*\lambda)x\| = 0 \quad \text{uniformly over } s \in S.$$

Let $\langle F, G \rangle$ be a duality of vector spaces. We do not assume it is separated. Let $\sigma(F, G)$ be the weak topology on F generated by G , and let $\tau(F, G)$ be the Mackey topology on F , i.e., the topology of uniform convergence on all $\sigma(G, F)$ -compact convex circled subsets of G .

Lemma 3.7. *Let X be a translation invariant closed subspace of $l^\infty(S)$ containing constants, and assume that X has a left invariant mean μ . Then there exist nets $\{\phi_\alpha\}$ and $\{\varphi_\alpha\}$ of finite means such that for any $s \in S$, $l(s)^*\phi_\alpha - \phi_\alpha$ and $l(s)^*\varphi_\alpha - \mu$ converge to 0 in the $\tau(l^1(S), X)$ topology and $\tau(X^*, X)$ topology, respectively.*

Proof. The first assertion is proved in [16]. The second assertion is proved similarly. Indeed, by the Mackey-Arens theorem, the continuous dual of $(X^*, \tau(X^*, X))$ is also X . Let E be the product $(X^*, \tau(X^*, X))^S$. Let Φ be the set of finite means and define a map $T : \Phi \rightarrow E$ given by

$$T(\phi)(s) = l(s)^*\phi - \mu, \quad \phi \in \Phi, \quad s \in S.$$

Then $T(\Phi)$ is a convex subset of E . There exists a net $\{\lambda_\alpha\} \subset \Phi$ such that $\langle f, l(s)^*\lambda_\alpha - \mu \rangle \rightarrow 0$ for each $s \in S$ and $f \in X$. Hence $T(\lambda_\alpha) \rightarrow 0$ in the weak topology of E , i.e., $0 \in$ the weak closure of $T(\Phi)$. Hence by Mazur's theorem, $0 \in$ the strong closure of $T(\Phi)$, that is, there exists a net $\{\varphi_\alpha\}$ of finite means such that $l(s)^*\varphi_\alpha - \mu$ converges to 0 in the $\tau(X^*, X)$ topology for each $s \in S$. $\square$

Now we can prove our main theorem on the existence of ergodic retraction for a noncommutative semigroup which is right Eberlein-weakly almost periodic in a uniformly convex Banach space.

Theorem 3.8. *Let S be a normal semitopological semigroup such that there exists a left invariant mean on $C_b(S)$. Let C be a closed convex subset of a uniformly convex Banach space E , and let $T : S \rightarrow \text{Cont}(C)$ be a right Eberlein-weakly almost periodic representation on a subset D of C . Assume $\text{Fix}(T) \neq \emptyset$. Then for each right invariant mean μ on $C_b(S)$, if it exists, $T(\mu)$ is a nonexpansive retraction from D onto $\text{Fix}(T)$ such that $T(\mu)T(s) = T(s)T(\mu) = T(\mu)$ for each $s \in S$ and $T(\mu)x \in \text{clco } O(x)$ for each $x \in C$.*

Proof. By Lemma 2.2, we may assume C is bounded. Fix any $s \in S$ and $\varepsilon > 0$. By Lemma 3.5, there exists $\delta > 0$ such that

$$(3.1) \quad \text{clco } F_\delta(T(s)) \subset F_\varepsilon(T(s)).$$

Let λ be a left invariant mean. Since $T(\cdot)x \in WR(S, E)$, by Lemma 3.4 and Krein's theorem we have that the convex, circled hull of $\{\langle T(t)x, x^* \rangle; t \in S, x^* \in B(E^*)\}$ is relatively weakly compact. So, by Lemma 3.7 there exists a finite mean λ_0 such that the following holds:

$$\sup\{|\langle \lambda_0 - \lambda, T(t)x, x^* \rangle|; t \in S, x^* \in B(E^*)\} \leq \delta/4.$$

Then, for any $t \in S$,

$$\begin{aligned} & \|T(l(st)^*\lambda_0)x - T(l(t)^*\lambda_0)x\| \\ &= \|\tau(l(st)^*\lambda_0 - l(t)^*\lambda_0)(T(\cdot)x)\| \\ &\leq \|\tau(l(st)^*\lambda_0 - \lambda)(T(\cdot)x)\| + \|\tau(l(t)^*\lambda_0 - \lambda)(T(\cdot)x)\| \\ &= \sup\{(\lambda_0 - \lambda)\langle T(st\cdot)x, x^* \rangle; x^* \in B(E^*)\} \\ &\quad + \sup\{(\lambda_0 - \lambda)\langle T(t\cdot)x, x^* \rangle; x^* \in B(E^*)\} \\ &\leq \delta/4 + \delta/4 = \delta/2. \end{aligned}$$

By Remark 2.1 and Lemma 3.6, there exists $t_0 \in S$ such that

$$\|T(s)T(l(t)^*\lambda_0)x - T(l(st)^*\lambda_0)x\| \leq \delta/2 \quad \text{for all } t \geq t_0.$$

Put $f(t) = T(l(t)^*\lambda_0)x$. Then for any $t \geq t_0$,

$$\begin{aligned} & \|T(s)f(t) - f(t)\| \\ &= \|T(s)T(l(t)^*\lambda_0)x - T(l(t)^*\lambda_0)x\| \\ &\leq \|T(s)T(l(t)^*\lambda_0)x - T(l(st)^*\lambda_0)x\| + \|T(l(st)^*\lambda_0)x - T(l(t)^*\lambda_0)x\| \\ &\leq \delta/2 + \delta/2 = \delta. \end{aligned}$$

So, we have

$$(3.2) \quad f(t) \in F_\delta(T(s)) \quad \text{for all } t \geq t_0.$$

By Lemma 2.7, Lemma 2.6, (3.2) and (3.1), we have $T(\mu)x = \tau(\mu)f \in \text{cl co}\{f(t); t \geq t_0\} \subset \text{cl co } F_\delta(T(s)) \subset F_\varepsilon(T(s))$. Since $\varepsilon > 0$ and $s \in S$ are arbitrary, we obtain $T(\mu)x \in \text{Fix}(T)$. So, for any $s \in S$, $T(s)T(\mu) = T(\mu)$. Since μ is right invariant, we have, using Remark 2.4,

$$\begin{aligned} T(\mu)T(s)x &= \tau(\mu)(T(\cdot)T(s)x) = \tau(\mu)(T(\cdot s)x) \\ &= \tau(r(s)^*\mu)(T(\cdot)x) = \tau(\mu)(T(\cdot)x) \\ &= T(\mu)x, \end{aligned}$$

and hence $T(\mu)T(s) = T(\mu)$ for every $s \in S$. Since $T(\mu)^2x = \tau(\mu)(T(\cdot)T(\mu)x) = \tau(\mu)T(\mu)x = T(\mu)x$, we obtain $T(\mu)^2 = T(\mu)$, which means $T(\mu)$ is a retraction. From Lemma 2.6, we have $T(\mu)x \in \text{cl co } O(x)$.

$$\begin{aligned} \|T(\mu)x - T(\mu)y\| &= \|\tau(\mu)(T(\cdot)x - T(\cdot)y)\| \leq \|T(\cdot)x - T(\cdot)y\| \\ &\leq \|x - y\|, \end{aligned}$$

so that $T(\mu)$ is nonexpansive. This completes the proof. $\square$

REFERENCES

- [1] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), 1511–1514. MR **51**:11205
- [2] ———, *Quelques propriétés de convergence asymptotique pour les semigroupes de contractions impaires*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), 75–78. MR **55**:1153
- [3] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on semigroups*, John Wiley & Sons. MR **91b**:43001
- [4] R. E. Bruck, *On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω -limit set*, Israel J. Math. **29** (1978), 1–16. MR **58**:2474
- [5] ———, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1979), 304–314. MR **82h**:47051
- [6] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. 1, American Mathematical Society, 1961. MR **24**:A2627

- [7] M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544. MR **19**:1067c
- [8] ———, *Fixed point theorem for compact convex sets*, Illinois J. Math. **5** (1961), 585–590. MR **25**:1547
- [9] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240. MR **12**:112a
- [10] S. Goldberg and P. Irwin, *Weakly almost periodic vector-valued functions*, Diss. Math. **157** (1979), 46 pp. MR **80e**:43010
- [11] N. Hirano, K. Kido and W. Takahashi, *Asymptotic behavior of commutative semigroups of nonexpansive mappings in Banach spaces*, Nonlinear Analysis **10** (1986), 229–249. MR **87d**:47078
- [12] ———, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Analysis **12** (1988), 1269–1281. MR **90a**:47138
- [13] O. Kada, *Strong ergodic theorems for commutative semigroups of operators*, (submitted).
- [14] ———, *Ergodic theorems for noncommutative semigroups of operators*, preprint.
- [15] ———, *Strong ergodic theorems for noncommutative semigroups in Banach spaces*, preprint.
- [16] O. Kada, A. Lau and W. Takahashi, *Asymptotically invariant net and fixed point set for semigroup of nonexpansive mappings*, Nonlinear Anal. **29** (1997), 539–550. MR **98d**:47156
- [17] O. Kada and W. Takahashi, *Nonlinear ergodic theorems for almost nonexpansive curves over commutative semigroups*, Topological Methods in Nonlinear Analysis **5** (1995), 305–324. MR **97f**:47053
- [18] ———, *Strong convergence and nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings*, Nonlinear Analysis **28** (1997), 495–511. MR **97j**:47075
- [19] J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, 1963. MR **29**:3851
- [20] K. Kido and W. Takahashi, *Mean ergodic theorems for semigroups of linear operators*, J. Math. Anal. Appl. **103** (1984), 387–394. MR **86g**:47052
- [21] A. T. Lau and W. Takahashi, *Invariant means and fixed point properties for nonexpansive representations of topological semigroups*, Topological Methods in Nonlinear Analysis **5** (1995), 39–57. MR **96i**:47101
- [22] P. Milnes, *On vector-valued weakly almost periodic functions*, J. London Math. Soc. (2) **22** (1980), 467–472. MR **82i**:43007
- [23] I. Namioka, *Følner's condition for amenable semigroups*, Math. Scand. **15** (1964), 18–28. MR **31**:5062
- [24] W. M. Ruess and W. H. Summers, *Weak almost periodicity and the strong ergodic limit theorem for contraction semigroups*, Israel J. Math. **64** (1988), 139–157. MR **90c**:47118
- [25] ———, *Integration of asymptotically almost periodic functions and weak asymptotic almost periodicity*, Diss. Math. **279** (1989). MR **90d**:46056
- [26] ———, *Weakly almost periodic semigroups of operators*, Pacific J. Math. **143** (1990), 175–193. MR **91b**:47142
- [27] ———, *Ergodic theorems for semigroups of operators*, Proc. Amer. Math. Soc. **114** (1992), 423–432. MR **92e**:47016
- [28] H. H. Schaefer, *Topological vector spaces*, Springer-Verlag, 1971. MR **49**:7722
- [29] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256. MR **82f**:47079
- [30] ———, *Nonlinear Functional Analysis (Japanese)*, Kindai-kagakusha, Japan, 1988.

DEPARTMENT OF INFORMATION SCIENCES, TOKYO INSTITUTE OF TECHNOLOGY, OH-OKAYAMA,
 MEGURO-KU, TOKYO 152, JAPAN
E-mail address: kada@math.sci.ynu.ac.jp