LEVEL SETS OF A TYPICAL C^n FUNCTION

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ABSTRACT. We determine the level set structure of a typical C^n function.

INTRODUCTION

Assume that $\mathcal{F}$ is a function space complete with respect to some norm. We say that a *typical function in $\mathcal{F}$ satisfies property $\mathcal{P}$* if the subfamily of $\mathcal{F}$ consisting of those functions which satisfy $\mathcal{P}$ is residual (in the sense of Baire category) in $\mathcal{F}$. If y is in the range of f , we call $f^{-1}(\{y\})$ a *level set* of f . In [2] Bruckner and Garg gave a full description of level sets of a typical function from $C[0, 1]$, the space of real-valued, continuous functions defined on $[0, 1]$ and equipped with the sup norm. Namely, they proved:

Theorem 1 (Bruckner and Garg). *For a typical function $f \in C[0, 1]$ there exists a countable set $S_f \subseteq (\min f, \max f)$ such that the level set $f^{-1}(\{y\})$ is:*

1. *a nowhere dense perfect set if $y \notin S_f \cup \{\min f, \max f\}$,*
2. *a single point if $y = \min f$ or $\max f$, and*
3. *the union of a nowhere dense perfect set and an isolated point of $f^{-1}(\{y\})$ if $y \in S_f$.*

In this paper we give the description of level sets of typical functions in $C^n[0, 1]$. Namely, we prove that a typical $f \in C^1[0, 1]$ is either strictly monotone or f has uncountably many level sets having exactly one accumulation point and all other level sets of f are finite. For a typical function in $C^n[0, 1]$ with $n \geq 2$ the situation is simple. All level sets are finite.

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Let us now introduce some notation and definitions.

The symbols $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$ will denote the sets of all positive integers, rationals and reals, respectively. We use $|A|$ to denote the cardinality of set A . The restriction of a function f to a set A will be denoted by $f|_A$. We use $\lambda(A)$ to denote the Lebesgue measure of A for a Lebesgue measurable subset A of $\mathbf{R}$.

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Let X be a metric space with a norm $|\cdot|$. We use $B_{|\cdot|}(x, r)$ to denote the open ball in X with center x and radius r .

We shall consider the space $C^n[0, 1]$ ($n \geq 0$) endowed with Sobolev's norm, i.e. $\|f\|_n = \sum_{i=0}^n \|f^{(i)}\|$, where $\|\cdot\|$ stands for the sup norm and $f^{(i)}$ stands for i th derivative of f .

We prove a series of lemmas which lead us to our main results.

Lemma 1. *Fix $y_0 \in \mathbf{R}$. For a typical $f \in C[0, 1]$, $f^{-1}(\{y_0\})$ is either empty or perfect and nowhere dense.*

Proof. Let $p, q \in [0, 1]$ and $p < q$. Let $\mathcal{F}_{p,q} = \{f \in C[0, 1] : |f^{-1}(\{y_0\}) \cap [p, q]| = 1\}$. We first want to show that $\mathcal{F}_{p,q}$ is nowhere dense in $C[0, 1]$. We will accomplish this by showing that for each $h \in \mathcal{F}_{p,q}$ and $\epsilon > 0$, there exists a nonempty open set which misses $\mathcal{F}_{p,q}$ and is contained in $B_{\|\cdot\|}(h, \epsilon)$. To this end, let $h \in \mathcal{F}_{p,q}$ and $\epsilon > 0$. Let $(u, v) \subset [p, q]$ be such that $h((u, v)) \subset (y_0 - \epsilon/4, y_0 + \epsilon/4)$. Let $a < b < c$ be any points of (u, v) . Let us define a new function ϕ as follows: $\phi(x) = h(x)$ for $x \notin (u, v)$, $\phi(a) = \phi(c) = y_0 - \epsilon/4$, $\phi(b) = y_0 + \epsilon/4$ and make ϕ linear on intervals (u, a) , (a, b) , (b, c) and (c, v) . Every function from $B_{\|\cdot\|}(\phi, \epsilon/4)$ now takes the value y_0 in at least two points on the interval $(u, v) \subset [p, q]$ whence $B_{\|\cdot\|}(\phi, \epsilon/4) \cap \mathcal{F}_{p,q} = \emptyset$. As $\|h - \phi\| \leq \epsilon/2$, $B_{\|\cdot\|}(\phi, \epsilon/4) \subseteq B_{\|\cdot\|}(h, \epsilon)$. Thus, $\mathcal{F}_{p,q}$ is nowhere dense.

Let us notice now that for every function f in the residual set

$$C[0, 1] \setminus \bigcup \{\mathcal{F}_{p,q} : p, q \in \mathbf{Q} \cap [0, 1], p < q\}$$

there is no isolated point in the level set $f^{-1}(\{y_0\})$. So $f^{-1}(\{y_0\})$ must be either empty or perfect.

It is a well-known fact that a typical function in $C[0, 1]$ is nowhere constant. (For example, this follows from the aforementioned theorem of Bruckner and Garg.) This together with what we just showed finishes the proof of the lemma.

The following is a C^1 counterpart of Lemma 1.

Lemma 2. *Fix $y_0 \in \mathbf{R}$. For a typical $f \in C^1[0, 1]$, $f^{-1}(\{y_0\})$ is finite.*

Proof. Of course, it is enough to show the lemma for $y_0 = 0$. It is easy to see that a function f that has infinite level set $f^{-1}(\{0\})$ must satisfy at some point the equation $f(x) = f'(x) = 0$. We shall show that the family of functions

$$\mathcal{F} = \{f \in C^1[0, 1] : \text{there exists } x \text{ such that } f(x) = f'(x) = 0\}$$

is of first category. One can readily observe that it is closed, so it is enough to show that its interior is empty.

Let $f \in \mathcal{F}$. From Sard's theorem (see for instance Theorem 6.3, p. 226, [3]) we know that $\lambda(f(\{x : f'(x) = 0\})) = 0$. Thus there exists as small a positive ϵ as we want such that $f(x) \neq \epsilon$ at any point x where $f'(x) = 0$. Let us now consider the function $h = f - \epsilon$. If $h(x) = h'(x) = 0$ at some $x \in [0, 1]$, then $\epsilon \in f(\{x : f'(x) = 0\})$ which is impossible. Obviously, $h \in B_{\|\cdot\|_1}(f, 2\epsilon)$. Thus the interior of $\mathcal{F}$ is empty. This finishes the proof.

Let us now recall the following definition. We say that a function $f : [0, 1] \rightarrow \mathbf{R}$ is *monotone at a point x* if there exists an open (relative to $[0, 1]$) neighborhood V

of x such that one of the following conditions is satisfied:

- $f(s) \leq f(x)$ whenever $s < x$ and $s \in V$, and $f(s) \geq f(x)$ whenever $s > x$ and $s \in V$; or
- $f(s) \geq f(x)$ whenever $s < x$ and $s \in V$, and $f(s) \leq f(x)$ whenever $s > x$ and $s \in V$.

Lemma 3. *A typical function f in $C^1[0, 1]$ has the property that if there is an x such that $f'(x) = 0$, then f is not monotone at x .*

Proof. Let

$$\mathcal{F} = \{f \in C^1[0, 1] : \text{there exists } x \text{ such that } f'(x) = 0 \text{ and } f \text{ is monotone at } x\}.$$

We first show that the complement of $\mathcal{F}$ is dense in $C^1[0, 1]$. Let $f \in C^1[0, 1]$ and $\epsilon > 0$. Let us define $g \in B_{\|\cdot\|_1}(f, \epsilon/3)$. If $f'(0) = 0$ or $f'(1) = 0$, let us put $g = f + \alpha x$ with $\alpha < \epsilon/6$ such that $g'(0) \neq 0$ and $g'(1) \neq 0$. If $f'(0) \neq 0$ and $f'(1) \neq 0$, then we put $g = f$. One can easily see that $g \in B_{\|\cdot\|_1}(f, \epsilon/3)$. As the set $M = \{x \in [0, 1] : g'(x) = 0\}$ is compact and the set $\{x \in [0, 1] : |g'(x)| < \epsilon/6\}$ is open in $[0, 1]$ and contains M , there exists a finite sequence of points $0 \leq a_1 < b_1 < \dots < a_n < b_n \leq 1$ such that $|g'|(|(a_i, b_i) < \epsilon/6$, for each $i \leq n$, and $M \subset \bigcup \{(a_i, b_i) : i \leq n\}$. Let $\phi \in C[0, 1]$ be defined as follows: $\phi(x) = g'(x)$ if $x \notin \bigcup \{(a_i, b_i) : i \leq n\}$ and ϕ is linear on every interval (a_i, b_i) , $i \leq n$. Finally, let $h \in C^1[0, 1]$ be defined as

$$h(x) = g(0) + \int_0^x \phi(t) dt.$$

It is easy to observe that h has strict extremum at every point x where $h'(x) = 0$ and thus is not monotone at any such point. We also have $\|f - h\|_1 \leq \|f - g\|_1 + \|g - h\|_1 < \epsilon$. Thus, the complement of $\mathcal{F}$ is dense in $C^1[0, 1]$.

To complete the proof of the lemma, we now show that $\mathcal{F}$ is F_σ . Let us first observe that

$$\mathcal{F} = \bigcup_{n=1}^{\infty} (\mathcal{F}_n^+ \cup \mathcal{F}_n^-),$$

where

$$\mathcal{F}_n^+ = \{f \in C^1[0, 1] : \text{there exists } p \in [0, 1] \text{ such that } f'(p) = 0 \text{ and}$$

$$f|(p - 1/n, p] \cap [0, 1] \leq f(p) \text{ and } f|[p, p + 1/n) \cap [0, 1] \geq f(p)\}$$

and

$$\mathcal{F}_n^- = \{f \in C^1[0, 1] : \text{there exists } p \in [0, 1] \text{ such that } f'(p) = 0 \text{ and}$$

$$f|(p - 1/n, p] \cap [0, 1] \geq f(p) \text{ and } f|[p, p + 1/n) \cap [0, 1] \leq f(p)\}.$$

It is easy to see that $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are closed in $C^1[0, 1]$. Thus $\mathcal{F}$ is of first category.

Proposition 1. *The family*

$$\mathcal{S} = \{f \in C^1[0, 1] : \text{there is } 0 \leq u < v \leq 1 \text{ such that } f(u) = f(v)$$

$$\text{and } (u = 0 \text{ or } f'(u) = 0) \text{ and } (v = 1 \text{ or } f'(v) = 0)\}$$

is of first category in $C^1[0, 1]$.

Proof. We shall use the fact that the space $C^1[0, 1]$ is homeomorphic with the product space $\mathbf{R} \times C[0, 1]$ where the homeomorphism $\Psi : \mathbf{R} \times C[0, 1] \rightarrow C^1[0, 1]$ is given by the formula

$$(1) \quad \Psi(a, f)(x) = a + \int_0^x f(t) dt.$$

We have $\Psi^{-1}(\mathcal{S}) = \mathbf{R} \times \mathcal{G}$, where

$$\mathcal{G} = \left\{ \varphi \in C[0, 1] : \text{there exist } 0 \leq u < v \leq 1 \text{ such that} \right.$$

$$\left. (u = 0 \text{ or } \varphi(u) = 0) \text{ and } (v = 1 \text{ or } \varphi(v) = 0) \text{ and } \int_u^v \varphi(t) dt = 0 \right\}.$$

Thus to show that $\mathcal{S}$ is of first category in $C^1[0, 1]$ or, equivalently, that $\Psi^{-1}(\mathcal{S})$ is of first category in $\mathbf{R} \times C[0, 1]$ it is enough to show that $\mathcal{G}$ is of first category in $C[0, 1]$.

Let us first notice that $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$, where

$$\mathcal{G}_n = \left\{ \varphi \in C[0, 1] : \text{there exist } 0 \leq u < v \leq 1 \text{ such that } v - u \geq 1/n, \right.$$

$$\left. (u = 0 \text{ or } \varphi(u) = 0) \text{ and } (v = 1 \text{ or } \varphi(v) = 0) \text{ and } \int_u^v \varphi(t) dt = 0 \right\}.$$

It is easy to check that $\mathcal{G}_n$ is closed in $C[0, 1]$. Now to show that $\mathcal{G}$ is of first category it is enough to show that the complement of $\mathcal{G}$ is dense in $C[0, 1]$.

Let $f \in C[0, 1]$. Let $\epsilon > 0$. By the Weierstrass approximation theorem there exists a nonzero polynomial w such that $\|f - w\| < \epsilon/2$ (of course, we consider only the restriction of w to $[0, 1]$). Let $0 = x_0 \leq x_1 < \dots < x_{n-1} \leq x_n = 1$ and the set $\{x_1, \dots, x_{n-1}\}$ be the set of all roots of w which belong to $[0, 1]$. Let

$$\delta = \min \left\{ \left| \int_{x_i}^{x_j} w(t) dt \right| : 0 \leq i < j \leq n \text{ and } \int_{x_i}^{x_j} w(t) dt \neq 0 \right\}.$$

Let

$$0 < \alpha < \min \left\{ \frac{\epsilon}{2\|w^2\|}, \frac{1}{\|w\|}, \frac{\delta}{\|w^2\|} \right\}.$$

We define a new polynomial $v(t) = w(t) + \alpha w^2(t)$. Let us notice that

- v has the same roots in the interval $[0, 1]$ as w ,
- $\|v(t) - w(t)\| < \epsilon/2$, and
- for $x_i < x_j$ we have that

$$\left| \int_{x_i}^{x_j} v(t) dt \right| \geq \min \left\{ \delta - \alpha\|w^2\|, \alpha \cdot \int_{x_i}^{x_j} w^2(t) dt \right\} > 0.$$

Thus, we have found a $v \in C[0, 1]$ such that $\|v - f\| < \epsilon$ and $v \notin \mathcal{G}$. Therefore, the complement of $\mathcal{G}$ is dense in $C[0, 1]$ and the proof of the proposition is complete.

The next three lemmas are immediate corollaries to Proposition 1.

Lemma 4. *No level set of a typical function in $C^1[0, 1]$ contains two points at which f attains a local extremum.*

Let us notice that if an extremum is attained at 0 or at 1 the derivative need not be equal to zero, and this was the reason for considering $u = 0$ and $v = 1$ in Proposition 1.

Lemma 5. *A typical function f in $C^1[0, 1]$ has the property that if f attains a local extremum at x , then $f^{-1}(f(x))$ is finite.*

Lemma 6. *No level set of a typical function in $C^1[0, 1]$ contains two accumulation points.*

Lemma 7. *A typical function $f \in C^1[0, 1]$ has the property that $\{x : f'(x) = 0\}$ is either empty or perfect.*

Proof. Let us consider the family $\mathcal{T}$ of functions $f \in C^1[0, 1]$ for which the set $\{x : f'(x) = 0\}$ is non-empty and is not perfect. We want to show that $\mathcal{T}$ is of first category. We again use the fact that the space $C^1[0, 1]$ is homeomorphic with the product space $\mathbf{R} \times C[0, 1]$ where the homeomorphism $\Psi : \mathbf{R} \times C[0, 1] \rightarrow C^1[0, 1]$ is given by the formula (1). Let us notice that $\Psi^{-1}(\mathcal{T}) = \mathbf{R} \times \mathcal{H}$, where

$$\mathcal{H} = \{\phi \in C[0, 1] : \phi^{-1}(\{0\}) \neq \emptyset \text{ and } \phi^{-1}(\{0\}) \text{ is not perfect}\}.$$

By Lemma 1 $\mathcal{H}$ is of first category in $C[0, 1]$ and thus $\mathcal{T}$ is of first category in $C^1[0, 1]$.

We are now in a position to prove our main theorem.

Theorem 2. *A typical function $f \in C^1[0, 1]$ is either strictly monotone or there exist a perfect nowhere dense subset P_f of $(\min f, \max f)$ and a countable dense subset D_f of P_f such that the level set $f^{-1}(\{y\})$ is:*

1. *a set with exactly one accumulation point if $y \in P_f \setminus D_f$,*
2. *a finite set if $y \in D_f \cup ((\min f, \max f) \setminus P_f)$; and*
3. *a single point if $y \in \{\min f, \max f\}$.*

Proof. As the intersection of countably many residual families is residual in a complete space, we have that a typical function satisfies the properties of Lemmas 3 - 7. Let f be such a function.

Assume that f is not monotone. Then, there exists a point $x \in (0, 1)$ where $f'(x) = 0$. As f satisfies the property of Lemma 7, $M_f = \{x : f'(x) = 0\}$ is perfect. As f satisfies the property of Lemma 6, the set $P_f = f(M_f)$ is also perfect. Let

$$D_f = \{y \in P_f : \text{there is } x \in f^{-1}(\{y\}) \text{ such that } f \text{ has a local extremum at } x\}.$$

It is well-known that D_f has to be countable [1]. Of course, D_f is dense in P_f .

Let $y \in P_f \setminus D_f$. As f has no extremum at any point of the level set $f^{-1}(\{y\})$ and at some point $x \in f^{-1}(\{y\})$ we have $f'(x) = 0$, then, as f satisfies the property of Lemma 3, this must be an accumulation point of the level set $f^{-1}(\{y\})$. But, as f satisfies the property of Lemma 6, there is no other accumulation point of the level set $f^{-1}(\{y\})$. Thus 1. is satisfied.

As f satisfies the property of Lemma 5 and the fact that for any y for which $f^{-1}(\{y\})$ is infinite there must be a point $x \in f^{-1}(\{y\})$ such that $f'(x) = 0$, we have that 2. is satisfied.

That 3. is satisfied follows from the fact that f satisfies the property of Lemma 4.

Let us finally state the following theorem on the level sets of a typical function in $C^n[0, 1]$ for $n \geq 2$.

Theorem 3. *A typical function in $C^n[0, 1]$, $n \geq 2$, has all level sets finite.*

Proof. Let us first prove the theorem for $n = 2$. Notice that the space $C^2[0, 1]$ is homeomorphic with the product space $\mathbf{R} \times C^1[0, 1]$ where the homeomorphism $\Psi : \mathbf{R} \times C^1[0, 1] \rightarrow C^2[0, 1]$ is again given by the formula (1). Let

$$\mathcal{W} = \{f \in C^2[0, 1] : \text{there exists } y \text{ such that the set } f^{-1}(\{y\}) \text{ is infinite}\}.$$

Let us notice that

$$\Psi^{-1}(\mathcal{W}) \subseteq \mathbf{R} \times \mathcal{J},$$

where

$$\mathcal{J} = \{\phi \in C^1[0, 1] : \text{the set } \phi^{-1}(\{0\}) \text{ is infinite}\}.$$

By Lemma 2 $\mathcal{J}$ is of first category in $C^1[0, 1]$ whence $\mathcal{W}$ is of first category in $C^2[0, 1]$. Thus the conclusion of our theorem is proved for $n = 2$.

For $n > 2$, the theorem can be easily proved in a similar fashion with induction.

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