THERE ARE NO DENTING POINTS IN THE UNIT BALL OF $WC(K, X)$

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ABSTRACT. For an infinite compact set K and for any Banach space X we show that the unit ball of the space of X -valued functions that are continuous when X is equipped with the weak topology, has no denting points.

INTRODUCTION

For a compact Hausdorff space K and for any Banach space X , let $WC(K, X)$ denote the space of X -valued functions on K that are continuous when X has the weak topology, equipped with the supremum norm. A point x_0 in the unit ball X_1 of a Banach space X is said to be a denting point if x_0 is an extreme point and a point of continuity for the identity map $i : (X_1, \text{weak}) \rightarrow (X_1, \|\cdot\|)$. (That this is equivalent to the standard definition of a denting point (see [DU]) is a result from [LLT].) In this note we show that when K is infinite given any $f \in WC(K, X)$ of norm one, there exists a sequence $\{f_n\} \subset WC(K, X)_1$ such that $f_n \rightarrow f$ in the weak topology but not in the norm topology. As a consequence we conclude that $WC(K, X)_1$ has no denting points. Our results also extend some results of Grzaślewicz [G] on the non-availability of strongly exposed points in the unit ball of the space of operator valued continuous functions. For the space of operators we show that for any infinite compact set K and for any X there are no denting points in $\mathcal{L}(X, C(K))_1$. Turning to injective tensor products, we show that for any infinite dimensional space Y such that Y^* is isometric to an $L^1(\mu)$ space (the so-called L^1 predual spaces; see [L], Chapter 7) and for any X , there are no denting points in the unit ball of $Y \otimes_\epsilon X$.

MAIN RESULT

To establish our result, we need the description of the extreme points of $WC(K, X)_1^*$ given in [De]. Let us recall from [De] that a $\wedge \in WC(K, X)^*$ is a point functional if there is a $t_0 \in K$ so that $\wedge(fa) = f(t_0) \wedge(a)$ for each $f \in C(K)$ and $a \in WC(K, X)$.

Theorem 1 (De Reyna et al.). *If $\wedge$ is an extreme point of $WC(K, X)_1^*$, then $\wedge$ is a point functional (see [R1] for a simpler proof of this result).*

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Theorem 2. *Let K be an infinite compact set. Let $f \in WC(K, X)$, $\|f\| = 1$. There exists a sequence $\{f_n\}$ in $WC(K, X)_1$ such that $f_n \rightarrow f$ in the weak topology but not in the norm topology. Consequently there are no denting points in $WC(K, X)_1$.*

Proof. Throughout the proof to show the weak convergence of a sequence $\{f_n\}$, we invoke Rainwater's theorem ([D], page 155) and verify the convergence of $\{\wedge(f_n)\}$ for extreme points $\wedge$ of $WC(K, X)_1^*$. We now divide the proof into 3 cases.

Case (i). Suppose $f(K)$ is infinite and has a non-zero accumulation point.

Since $f(K)$ is a weakly compact set and since for such sets a result of Eberlein's guarantees that accumulation points can be approximated by sequences (see [D], page 22 and also [H], page 149), we may assume that there exists a sequence $\{t_n\} \subset K$ such that $f(t_n) \rightarrow x_0, x_0 \neq 0$ in the weak topology. Choose pairwise disjoint open sets U_n in K such that $t_n \in U_n$. Also choose $h_n \in C(K)$ with $0 \leq h_n \leq 1, h_n(t_n) = 1$ and $h_n = 0$ on $K \setminus U_n$. Put $g_n = 1 - h_n$. Clearly $\|g_n\| \leq 1$ and $g_n(t) \rightarrow 1$ for each $t \in K$.

Now $g_n f \in WC(K, X)_1$ and $g_n(t)f(t) \rightarrow f(t)$ for each $t \in K$. (For a future use in case (iii) we note here that until now the hypothesis of case (i) has not been used.)

Now if $\wedge$ is an extreme point of $WC(K, X)_1^*$, in view of the theorem quoted above, there exists a $t_0 \in K$ such that

$$\wedge(g_n f) = g_n(t_0) \wedge(f) \rightarrow \wedge(f).$$

Therefore $g_n f \rightarrow f$ in the weak topology.

However $\|g_n f - f\| = \|h_n f\| \geq \|f(t_n)\|$.

Since $\{f(t_n)\}$ converges weakly to a non-zero limit, we conclude that $g_n f \not\rightarrow f$ in the norm.

Case (ii). Suppose $f(K)$ is an infinite set with 0 as the only accumulation point.

As before let $t_n \in K$ and $f(t_n) \rightarrow 0$ weakly. If $\|f(t_n)\| \not\rightarrow 0$ the arguments as given in case (i) yield the required result.

So we assume w.l.o.g that $\|f(t_n)\| \rightarrow 0$.

Let $0 < \delta < 1$ and N be such that $\|f(t_n)\| < \delta$ for all $n \geq N$.

Let $U_n = f^{-1}(f(t_n))$. Then U_n 's are clopen and pairwise disjoint.

Let $x_0 \in X, \|x_0\| = 1 - \delta$. By $\tilde{x}_0$ we denote the constant function x_0 in $WC(K, X)$.

Let $f_n = \chi_{U_n} \tilde{x}_0$ for $n \geq N$. Clearly $f_n(t) \rightarrow 0 \forall t \in K$ and $\|f_n\| = \|x_0\| = 1 - \delta$.

Therefore $(f_n + f)(t) \rightarrow f(t)$ for all $t \in K$. Thus $f_n + f \rightarrow f$ weakly but not in norm. Note that for $t \in K \setminus U_n, f_n(t) = 0$ and for $t \in U_n, \|f_n(t) + f(t)\| = \|x_0 + f(t_n)\| \leq (1 - \delta) + \delta = 1$.

Hence the claim.

Case (iii). Suppose $f(K)$ is a finite set. Let $t_0 \in K$ be such that $f(t_0) \neq 0$. Let $U = f^{-1}(f(t_0))$. Suppose the clopen set U is infinite. Get a sequence $\{g_n\}$ in $C(K)$ as in case (i) such that $g_n = 0$ on $K \setminus U$ and $g_n \rightarrow \chi_U$ pointwise. Now $g_n f + \chi_{K \setminus U} f \rightarrow f$ pointwise and thus weakly.

Also,

$$\begin{aligned} \|g_n f + \chi_{K \setminus U} f\| &= \max\{\|f\|_U, \|f\|_{K \setminus U}\} \\ &= \|f\| = 1 \end{aligned}$$

and

$$\begin{aligned}
 \|g_n f + \chi_{K \setminus U} f - f\| &= \|g_n f + \chi_U f\| \\
 &= \|h_n f\|_U \text{ (recall from case (i) that } g_n = 1 - h_n \text{ on } U) \\
 &= \|f(t_0)\|.
 \end{aligned}$$

Therefore $g_n f + \chi_{K \setminus U} f \not\rightarrow f$ in the norm. The other situations are similarly handled.

Therefore f is not a point of weak-norm continuity in $WC(K, X)_1$.

For any Banach spaces X, Y let $\mathcal{L}(X, Y)$ denote the space of bounded operators.

Remark 1. For a dual space X^* , the space $WC(K, X^*)$ can be identified with $\mathcal{F}(X, C(K))$ (space of weakly compact operators; see [DS], page 490). When X is reflexive, clearly $\mathcal{F}(X, C(K)) = \mathcal{L}(X, C(K))$. Hence, when K is infinite these spaces do not have denting points in their unit ball. If $(\Omega, \mathcal{A}, \mu)$ is an infinite measure space, this author has remarked in [R2] that the space $\mathcal{F}(L^1(\mu), X)$ is isometric to $WC(K, X)$ for an infinite compact space K . Therefore there are no denting points in the unit ball of $\mathcal{F}(L^1(\mu), X)$.

Corollary 1. *For an infinite compact set K and for any separable Banach space X , there are no points of weak-norm continuity in $\mathcal{L}(X, C(K))_1$.*

Proof. It is well-known that the space $\mathcal{L}(X, C(K))$ can be identified with the space of X^* valued functions on K that are continuous when X^* has the weak*-topology (denoted as $W^*C(K, X^*)$; see [DS] page 490). The arguments given during the proof of Theorem 1 in [R1] can be used to show that any extreme point of $\mathcal{L}(X, C(K))_1^*$ is a point functional. Now the conclusion is obtained by proceeding along the same lines as in the proof of the above theorem. While in case (i), since X is separable, note that $f(K)$ is a weak*-compact metric space.

If one merely wants to show that there are no denting points, then a shorter argument can be given using Theorem 1 of [S].

Corollary 2. *For an infinite compact set K and for any Banach space X , there are no denting points in $\mathcal{L}(X, C(K))_1$.*

Proof. Let T be a denting point in $\mathcal{L}(X, C(K))_1$. Since T is in particular an extreme point, we have from Theorem 1 of [S], that T^* attains its norm on an infinite set. Thus, as in case (i) of the above proof, a sequence can be constructed to converge weakly but not in the norm to T .

Remark 2. The identification of $\mathcal{L}(\ell^1)$ as $\mathcal{L}(c_0, C(\beta(N)))$ and that of $\mathcal{L}(\ell^\infty)$ as $\mathcal{L}(\ell^\infty, C(\beta(N)))$ shows that there are no denting points in their respective unit balls.

Remark 3. If W is a closed subspace of $\mathcal{L}(X, C(K))$ containing $C(K, X^*)$ and is a $C(K)$ module, then using arguments similar to the ones given above one can show that there are no denting points in the unit ball of W . An application of this idea gives that there are no denting points in $\mathcal{L}(c_0)_1$ and in $\mathcal{L}(L^1(\mu))_1$.

Remark 4. Recall that a point x_0 of X_1 is said to be strongly exposed if there exists an $x_0^* \in X^*$ such that $x_0^*(x_0) > x_0^*(x)$ for all other $x \in X_1$ and for any sequence $\{x_n\}$ in X_1 , $x_0^*(x_n) \rightarrow x_0^*(x_0)$ implies $\|x_n - x_0\| \rightarrow 0$. Thus a strongly exposed point

is in particular a denting point. Therefore our results give easy proofs of parts of Theorem 2 and Theorem 3 from [G] where it is shown that for an infinite K and for any Hilbert space H there are no strongly exposed points in $C(K, \mathcal{L}(H))_1$.

Even though we do not yet know the nature of extreme points of $W^*C(K, \mathcal{L}(H))_1$, the next proposition exhibits exposed points.

Proposition. *Every exposed point of $C(K, \mathcal{L}(H))_1$ is an exposed point of $W^*C(K, \mathcal{L}(H))_1$.*

Proof. Let T be an exposed point of the unit ball of $C(K, \mathcal{L}(H))_1$. It follows from Theorem 2 in [G] that H is separable and K carries a strictly positive measure, say μ . Now define a linear functional ψ on $W^*C(K, \mathcal{L}(H))$ as in the proof of first part of Theorem 2 in [G]. Then the arguments given in [G] allow us to conclude that ψ exposes T .

We conclude this note by applying these ideas in deciding the denting point question for the unit ball of injective tensor product spaces.

Theorem 3. *Let X be any Banach space and Y an infinite dimensional L^1 predual space. There are no denting points in $(Y \otimes_\epsilon X)_1$.*

Proof. Since Y is an infinite dimensional space, clearly Y^{**} is isometric to an infinite dimensional $C(K)$ space. Suppose x_0 is a denting point of $(Y \otimes_\epsilon X)_1$. It is well known that x_0 is a denting point of $(Y \otimes_\epsilon X)_1^{**}$. It follows from an observation in [E] that $Y^{**} \otimes_\epsilon X$ is, under the canonical embedding, a subspace of $(Y \otimes_\epsilon X)^{**}$. It is clear that x_0 continues to be a point of weak-norm continuity for the identity mapping on $(Y^{**} \otimes_\epsilon X)_1$. Since $Y^{**} \otimes_\epsilon X$ is isometric to $C(K, X)$, we get a contradiction. Hence the result.

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The results of Benyamini, Rudin and Wage (Continuous images of weakly compact subsets of Banach spaces, Pacific J. Math. 70 (1977), 309-324) show that if K is an infinite Eberlein compact space or X is a subspace of a weakly compactly generated Banach space, then again accumulation points of $f(K)$ can be approximated by sequences and the arguments of this note work to show that there are no points of weak-norm continuity in $\mathcal{L}(X, C(K))_1$. In a recent work (Denting and strongly extreme points in the unit ball of spaces of operators) this author has shown that $\mathcal{L}(L^1(\mu), X)_1$ has a point of weak-norm continuity iff $L^1(\mu)$ is finite dimensional and X_1 has a point of weak-norm continuity. Also for any infinite, totally disconnected compact set K , there is no point of weak-norm continuity in $\mathcal{L}(X, C(K))_1$. Here however we could only exhibit nets that converge weakly but not in the norm. The general question is still open.

Added in proof. Theorem 2 was also proved using different methods by Z. Hu and M. A. Smith in their paper *On the extremal structure of the unit balls of Banach spaces of weakly continuous functions and their duals*, that has appeared in Trans. Amer. Math. Soc. **349** (1997), 1901–1918. $\square$

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