L^p ESTIMATES FOR OSCILLATORY INTEGRAL OPERATORS

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ABSTRACT. An endpoint boundedness result is established for a class of oscillatory integral operators.

1.

This paper deals with a problem left open in [6]. Let $\Delta = \{(x, x) : x \in \mathbf{R}\}$, $K \in C^\infty(\mathbf{R}^2 \setminus \Delta)$ and satisfy

$$(1) \quad \left| \frac{\partial^{j+l} K}{\partial x^j \partial y^l}(x, y) \right| \leq A_{jl} |x - y|^{-j-l}$$

for $(x, y) \in \mathbf{R}^2 \setminus \Delta$ and $j, l \geq 0$.

For $a, b \geq 1$, define the non-convolutional oscillatory integral operator $T_{a,b}$:

$$(2) \quad (T_{a,b}f)(x) = \int_{\mathbf{R}} e^{i|x|^a|y|^b} K(x, y) f(y) dy,$$

initially for $f \in \mathcal{S}(\mathbf{R})$.

Such operators often arise in harmonic analysis (see e.g. [7], [8], [9]). It should be noted that when $a = b = 1$ and $K \equiv 1$ the operator in (2) is essentially the Fourier transform.

The main problem under investigation concerns the L^p boundedness of the operators $\{T_{a,b}\}$. A quick examination of the operators $T_{a,b}$ with $K \equiv 1$ reveals that the L^p inequality

$$(3) \quad \|T_{a,b}f\|_p \leq C_{a,b,p} \|f\|_p$$

cannot hold (for all $f \in \mathcal{S}(\mathbf{R})$) unless $p = \frac{a+b}{a}$. Thus one is led to the problem of determining whether

$$(4) \quad \|T_{a,b}f\|_{\frac{a+b}{a}} \leq C_{a,b} \|f\|_{\frac{a+b}{a}}$$

holds for $f \in \mathcal{S}(\mathbf{R})$ and $a, b \geq 1$. The following is known:

Theorem A. *Suppose that K satisfies (1). Let $T_{a,b}$ be given as in (2). Then:*

- (i) *If $a > 1$ and $b > 1$, then $T_{a,b}$ is bounded from $L^{\frac{a+b}{a}}$ to itself.*
- (ii) *If $a = b = 1$ (hence $\frac{a+b}{a} = 2$), then $T_{a,b} = T_{1,1}$ is bounded from L^2 to itself.*

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Part (i) of Theorem A was established in [6] (Theorem 3.1 on page 212), while part (ii) can be found in [5] (see also [7]). Apparently the remaining issue is to answer the following question:

Question. *Is $T_{a,b}$ bounded from $L^{\frac{a+b}{a}}$ to itself when either $a = 1$ or $b = 1$?*

It would seem quite reasonable to believe that the answer is “yes”. In fact, this was shown to be true in the special case where $K(x, y) = |x - y|^{i\sigma}$ with $\sigma \in \mathbf{R}$ ([6]). However, for the operators with generic K 's that satisfy (1) the question has remained unanswered. The main purpose of this paper is to completely resolve this issue. Namely we have:

Theorem B. *Let $b > 1$, K satisfy (1), and $T_b = T_{1,b}$ be given as in (2). Then there exists a constant $C_b > 0$ such that*

$$(5) \quad \|T_b f\|_{b+1} \leq C_b \|f\|_{b+1}$$

holds for $f \in \mathcal{S}(\mathbf{R})$.

For $a > 1$ and $b = 1$ one obtains (4) from (5) by duality. It follows from Theorems A and B that $T_{a,b}$ is bounded on $L^{\frac{a+b}{a}}$ for all $a, b \geq 1$.

Part of our argument presented here is similar to the method used in [6], which goes back to [7]. The key new ingredients are reduction to pseudodifferential operators and the establishment of related L^2 estimates. This combination allows us to tackle the above mentioned problem and may have applications elsewhere.

For a nonnegative locally integrable function w we shall write $\|f\|_{p,w}$ for $(\int_{\mathbf{R}} |f(x)|^p w(x) dx)^{1/p}$. When $w \equiv 1$, we shall simply write $\|f\|_p$ for $\|f\|_{p,w}$.

2.

Let $b > 1$, $\eta \in C^\infty(\mathbf{R})$ such that $\eta(x) \equiv 1$ when $|x| \leq 1/2$ and $\eta(x) \equiv 0$ when $|x| \geq 1$. For $K(x, y)$ that satisfies (1) and $m \in \mathbf{N}$ we define $K_m(\cdot, \cdot)$ and $\omega_m(\cdot, \cdot)$ by

$$(6) \quad K_m(x, y) = [1 - \eta(x - y)]K(x, y)\eta\left(\frac{x - y}{m}\right),$$

$$(7) \quad \omega_m(\tau, \xi) = \int_{\mathbf{R}} e^{ix(\tau - \xi)} K_m(x + |\tau|^{1/b}, |\tau|^{1/b}) dx.$$

We shall begin with two propositions.

Proposition 1. *There exist constants $\tilde{A}_{j,l}$ independent of m such that*

$$(8) \quad \left| \frac{\partial^{j+l} K_m}{\partial x^j \partial y^l}(x, y) \right| \leq \tilde{A}_{j,l} [1 + |x - y|]^{-j-l}$$

for $j, l \geq 0$ and $(x, y) \in \mathbf{R}^2 \setminus \Delta$.

Proposition 2. *There exists $A > 0$ which is independent of m such that*

$$(9) \quad |\omega_m(\tau, \xi)| \leq A \min\{|\tau - \xi|^{-1}, |\tau - \xi|^{-2}\}$$

for all $\tau, \xi \in \mathbf{R}$.

The proof of Proposition 1 is elementary and hence will be omitted.

Proof of Proposition 2. If $|\tau - \xi| \geq 1$, then by Integration by Parts and Proposition 1

$$\begin{aligned} |\omega_m(\tau, \xi)| &= \frac{1}{|\tau - \xi|^2} \left| \int_{\mathbf{R}} e^{ix(\tau - \xi)} \frac{d^2}{dx^2} K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}}) dx \right| \\ &\leq A |\tau - \xi|^{-2} \int_{|x| \geq \frac{1}{2}} \frac{dx}{|x|^2} \\ &= A \min\{|\tau - \xi|^{-1}, |\tau - \xi|^{-2}\}. \end{aligned}$$

Now suppose that $|\tau - \xi| < 1$ and write $\omega_m(\tau, \xi) = I_1 + I_2$ where

$$I_1 = \int_{\mathbf{R}} e^{ix(\tau - \xi)} K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}}) \eta(x(\tau - \xi)) dx$$

and

$$I_2 = \int_{\mathbf{R}} e^{ix(\tau - \xi)} K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}}) [1 - \eta(x(\tau - \xi))] dx.$$

Then

$$|I_1| \leq A_1 \int_{|x| \leq |\tau - \xi|^{-1}} dx = 2A_1 |\tau - \xi|^{-1}$$

and

$$\begin{aligned} |I_2| &\leq \frac{1}{|\tau - \xi|^2} \int_{\mathbf{R}} \left| \frac{d^2}{dx^2} \{K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}}) [1 - \eta(x(\tau - \xi))]\} \right| dx \\ &\leq \frac{A_2}{|\tau - \xi|^2} \left(\int_{|x| \geq (2|\tau - \xi|)^{-1}} \frac{dx}{|x|^2} + \int_{(2|\tau - \xi|)^{-1} \leq |x| \leq |\tau - \xi|^{-1}} \left[\frac{|\tau - \xi|}{|x|} + |\tau - \xi|^2 \right] dx \right) \\ &\leq 7A_2 |\tau - \xi|^{-1} = 7A_2 \min\{|\tau - \xi|^{-1}, |\tau - \xi|^{-2}\}. \end{aligned}$$

Thus (9) always holds. Using the arguments given above, one easily obtains:

Proposition 3. *There exists $A > 0$ independent of m such that*

$$(10) \quad \left| \frac{\partial \omega_m}{\partial \xi}(\tau, \xi) \right| \leq A |\xi - \tau|^{-2}$$

and

$$(11) \quad \left| \frac{\partial \omega_m}{\partial \tau}(\tau, \xi) \right| \leq A(1 + |\tau|^{\frac{1}{b}-1}) |\xi - \tau|^{-2}$$

for all $\xi \in \mathbf{R}$ and $\tau \neq 0$.

Define the kernels Ω_m and operators H_m by

$$(12) \quad \Omega_m(\tau, \xi) = \omega_m(\tau, \xi) \eta(\xi - \tau) [1 - \eta(4^b |\tau|)]$$

and

$$(13) \quad H_m f(\xi) = \int_{\mathbf{R}} \Omega_m(\tau, \xi) f(\tau) d\tau.$$

Below is a uniform weighted norm estimate for H_m .

Theorem 4. *Let $p \in (1, \infty)$ and $w \in A_p$. Then there exists a constant $A_{p,w} > 0$ independent of m such that*

$$(14) \quad \int_{\mathbf{R}} |H_m f(x)|^p w(x) dx \leq A_{p,w} \int_{\mathbf{R}} |f(x)|^p w(x) dx$$

holds for all $f \in \mathcal{S}(\mathbf{R})$.

Here $A_p = A_p(\mathbf{R})$ represents the collection of Muckenhoupt's A_p -weights (see [4]).

Proof. Let $a(\tau, x) = K_m(x + |\tau|^{\frac{1}{b}}, |\tau|^{\frac{1}{b}})[1 - \eta(4^b|\tau|)]$. Then by Proposition 1 for $j, k \geq 0$ there exists $A_{j,k} > 0$ independent of m such that

$$\left| \frac{\partial^{j+k} a(\tau, x)}{\partial \tau^j \partial x^k} \right| \leq A_{j,k} (1 + |x|)^{-k}.$$

Thus $a(\cdot, \cdot)$ is a symbol in the class $S_{1,0}^0$. Let W_a be the pseudodifferential operator with symbol $a(\cdot, \cdot)$ i.e.

$$(15) \quad (W_a f)(\tau) = \int_{\mathbf{R}} e^{i\tau x} a(\tau, x) \hat{f}(x) dx.$$

Then there exists $C > 0$ independent of m such that

$$(16) \quad \|W_a f\|_2 \leq C \|f\|_2$$

for all $f \in \mathcal{S}(\mathbf{R})$ (see, for example, [10] on page 234).

For $u \in \mathbf{R}$ we shall let $f_u(\tau) = e^{i\tau u} f(-\tau)$. Then by (7), (12), (13) and (15)

$$(17) \quad H_m f(\xi) = \int_{\mathbf{R}} e^{i\xi \cdot u} \hat{\eta}(u) g(\xi, u) du$$

where $g(\xi, u) = \mathcal{F} \circ [W_a \circ \mathcal{F}^{-1}]^*(f_u)(\xi)$ and $\mathcal{F}$ represents the Fourier transform. By (16) and Plancherel's Theorem, one obtains:

$$(18) \quad \left(\int_{\mathbf{R}} |g(\xi, u)|^2 d\xi \right)^{\frac{1}{2}} \leq C \|f_u\|_2 = C \|f\|_2 \quad \forall \quad u \in \mathbf{R}.$$

It then follows from (17), (18) and Minkowski's Inequality that

$$(19) \quad \|H_m f\|_2 \leq C \|\hat{\eta}\|_1 \|f\|_2 \leq A \|f\|_2$$

for all $f \in \mathcal{S}(\mathbf{R})$.

Since $\Omega_m(\tau, \xi) = 0$ when $|\tau| \leq 2^{-(2b+1)}$, (9), (10), (11), and (12) give us that

$$(20) \quad |\Omega_m(\tau, \xi)| \leq \frac{A}{|\tau - \xi|}$$

and

$$(21) \quad \left| \frac{\partial}{\partial \tau} \Omega_m(\tau, \xi) \right| + \left| \frac{\partial}{\partial \xi} \Omega_m(\tau, \xi) \right| \leq \frac{A}{|\tau - \xi|^2}$$

hold for $(\xi, \tau) \in \mathbf{R}^2 \setminus \Delta$ and some positive constant A independent of m . It follows from (19)-(21) that $\Omega_m(\tau, \xi)$ is a generalized Calderón-Zygmund kernel. By a minor modification of the arguments in [1] one obtains the weighted L^p inequality (14) with a constant $A_{p,w}$ independent of m (see also [10], page 221, and [3]).

Corollary 5. *If we let*

$$(22) \quad (J_m f)(\xi) = \int_{\mathbf{R}} \Omega_m(|\tau|^b, \xi) f(\tau) d\tau,$$

then there exists $A > 0$ independent of m such that

$$(23) \quad \int_{\mathbf{R}} |(J_m f)(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq A \int_{\mathbf{R}} |f(y)|^{b+1} dy$$

for all $f \in \mathcal{S}(\mathbf{R})$.

Proof. Let

$$f_1(t) = \frac{1}{b} f(t^{\frac{1}{b}}) t^{\frac{1}{b}-1} \chi_{[0,\infty)}(t)$$

and

$$f_2(t) = \frac{1}{b} f(-t^{\frac{1}{b}}) t^{\frac{1}{b}-1} \chi_{[0,\infty)}(t).$$

Then $J_m f = H_m f_1 + H_m f_2$. Let $w(\xi) = |\xi|^{b-1} \in A_{b+1}$. By Theorem 4 we get

$$\|J_m f\|_{b+1,w} \leq C(\|f_1\|_{b+1,w} + \|f_2\|_{b+1,w}) \leq A\|f\|_{b+1}.$$

This proves (23).

Lemma 6. Define L_m by:

$$(24) \quad (L_m f)(\xi) = \int_{\mathbf{R}} e^{iy(|y|^b - \xi)} \Omega_m(|y|^b, \xi) f(y) dy$$

for $\xi \in \mathbf{R}$. Then there exists $A > 0$ independent of m such that

$$(25) \quad \int_{\mathbf{R}} |(L_m f)(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq A \int_{\mathbf{R}} |f(y)|^{b+1} dy$$

for $f \in \mathcal{S}(\mathbf{R})$.

This result follows from Corollary 5 by using the method employed in [6] on pages 217 and 218, which can be traced all the way back to [7] and [9]. Below we shall give a sketch of the proof.

Without loss of generality we may assume that $\text{supp}(f) \subset [0, \infty)$. For $j \geq 0$ let $I_j = [j^{1/b}, (j+1)^{1/b})$, $f_j(y) = f(y) \chi_{I_j}(y)$, and $g_j(y) = e^{ij^{1/b} y^b} f_j(y)$. Then

$$(26) \quad (L_m f)(\xi) = \sum_{j=0}^{\infty} (L_m f_j)(\xi).$$

By (12) and (9)

$$\begin{aligned} & |(L_m f_j)(\xi) - e^{-ij^{1/b} \xi} (J_m g_j)(\xi)| \\ & \leq A \int_{|y^b - \xi| \leq 1} |e^{i(y-j^{1/b})(y^b - \xi)} - 1| |\xi - y^b|^{-1} |f_j(y)| dy \leq A(\tilde{M} f_j)(\xi), \end{aligned}$$

where $(\tilde{M} f)(\xi)$ represents the Hardy-Littlewood maximal function of $g(t) = f(t^{\frac{1}{b}}) t^{\frac{1}{b}-1}$. By Corollary 5 and the $L^{b+1}(\mathbf{R}, |x|^{b-1} dx) \rightarrow L^{b+1}(\mathbf{R}, |x|^{b-1} dx)$ boundedness of the Hardy-Littlewood maximal function ([4]), we obtain:

$$\begin{aligned} (27) \quad & \int_{\mathbf{R}} |L_m f_j(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq C \left(\int_0^{\infty} |g_j(y)|^{b+1} + \int_0^{\infty} [|f_j(y^{\frac{1}{b}})| y^{\frac{1}{b}-1}]^{b+1} y^{b-1} dy \right) \\ & = A \int_{\mathbf{R}} |f_j(y)|^{b+1} dy. \end{aligned}$$

By $\text{supp}(L_m f_j) \subseteq [j-1, j+2]$, (26), and (27), we get

$$\int_{\mathbf{R}} |L_m f(\xi)|^{b+1} |\xi|^{b-1} d\xi \leq \tilde{A} \int_{\mathbf{R}} |f(y)|^{b+1} dy.$$

Lemma 7. *Let*

$$(Pf)(x) = \int_{\mathbf{R}} e^{ix|y|^b} [1 - \eta(x - y)] K(x, y) [1 - \eta(4^b |y|^b)] f(y) dy.$$

Then there exists $A > 0$ such that

$$(28) \quad \|Pf\|_{b+1} \leq A\|f\|_{b+1}$$

holds for all $f \in \mathcal{S}(\mathbf{R})$.

Proof. Let $f \in \mathcal{S}(\mathbf{R})$. For $m \in N$ and $x \in \mathbf{R}$ let

$$(29) \quad P_m(x) = \int_0^\infty e^{ixy^b} [1 - \eta(x - y)] K(x, y) \eta\left(\frac{x - y}{m}\right) [1 - \eta(4^b y^b)] f(y) dy.$$

Then by (29), (24) and (12) we obtain

$$\hat{P}_m(\xi) = \int_0^\infty e^{iy(y^b - \xi)} [1 - \eta(4^b y^b)] \omega_m(y^b, \xi) f(y) dy$$

and

$$L_m(\chi_{(0, \infty)} f)(\xi) = \int_0^\infty e^{iy(y^b - \xi)} [1 - \eta(4^b y^b)] \omega_m(y^b, \xi) \eta(\xi - y^b) f(y) dy.$$

By (9) we get that for $\xi \in \mathbf{R}$

$$(30) \quad \begin{aligned} & |\hat{P}_m(\xi) - L_m(\chi_{(0, \infty)} f)(\xi)| \\ & \leq A \int_0^\infty \frac{|1 - \eta(\xi - y^b)|}{|\xi - y^b|^2} |f(y)| dy \leq A(\tilde{M}f)(\xi). \end{aligned}$$

By Pitt's Inequality ([2]), (30), and Lemma 6,

$$(31) \quad \begin{aligned} \|P_m\|_{b+1} & \leq \left(\int_{\mathbf{R}} |\hat{P}_m(\xi)|^{b+1} |\xi|^{b-1} d\xi \right)^{\frac{1}{b+1}} \\ & \leq C[\|L_m(\chi_{(0, \infty)} f)\|_{b+1, |x|^{b-1}} + \|\tilde{M}f\|_{b+1, |x|^{b-1}}] \\ & \leq C\|f\|_{b+1} + C \left[\int_0^\infty |f(y^{\frac{1}{b}}) y^{\frac{1}{b}-1}|^{b+1} y^{b-1} dy \right]^{\frac{1}{b+1}} = A\|f\|_{b+1}. \end{aligned}$$

By letting $m \rightarrow \infty$ in (31), we obtain

$$\left\| \int_0^\infty e^{ix|y|^b} [1 - \eta(x - y)] K(x, y) [1 - \eta(4^b |y|^b)] f(y) dy \right\|_{b+1} \leq A\|f\|_{b+1}.$$

Similarly,

$$\left\| \int_{-\infty}^0 e^{ix|y|^b} [1 - \eta(x - y)] K(x, y) [1 - \eta(4^b |y|^b)] f(y) dy \right\|_{b+1} \leq A\|f\|_{b+1}.$$

This proves (28).

3.

We shall finish the paper with a proof of our main result.

Proof of Theorem B. For $f \in \mathcal{S}(\mathbf{R})$ write

$$(32) \quad (T_b f)(x) = F_1(x) + F_2(x) + F_3(x)$$

where

$$F_1(x) = \int_{\mathbf{R}} e^{i|x||y|^b} [1 - \eta(x-y)] K(x, y) [1 - \eta(4^b|y|^b)] f(y) dy,$$

$$F_2(x) = \int_{\mathbf{R}} e^{i|x||y|^b} [1 - \eta(x-y)] K(x, y) \eta(4^b|y|^b) f(y) dy,$$

and

$$F_3(x) = \int_{\mathbf{R}} e^{i|x||y|^b} \eta(x-y) K(x, y) f(y) dy.$$

By considering $x \geq 0$ and $x < 0$ separately and applying Lemma 7, it follows that

$$(33) \quad \|F_1\|_{b+1} \leq A\|f\|_{b+1}.$$

Since $\|F_3\|_{b+1} \leq A(|\eta| * |f|)(x)$,

$$(34) \quad \|F_3\|_{b+1} \leq A\|f\|_{b+1}.$$

For $(x, y) \in \text{supp}[(1 - \eta(x-y))\eta(4^b|y|^b)]$ we have $|x-y| \geq \frac{1}{2}$ and $|y| \leq \frac{1}{4}$. Let

$$I(x) = K(x, 0) \int_{\mathbf{R}} e^{i|x||y|^b} [1 - \eta(x-y)] \eta(4^b|y|^b) f(y) dy$$

and

$$\Pi(x) = \frac{\partial K}{\partial y}(x, 0) \int_{\mathbf{R}} e^{i|x||y|^b} [1 - \eta(x-y)] \eta(4^b|y|^b) y f(y) dy.$$

Then by (1)

$$(35) \quad |F_2(x) - [I(x) + \Pi(x)]| \leq A \int_{\mathbf{R}} \frac{|f(y)|}{1 + |x-y|^2} dy.$$

Clearly

$$|I(x)| \leq A[|\check{g}(|x|)| + (|\eta| * |f|)(x)]$$

where

$$g(t) = \frac{1}{b} [f(t^{\frac{1}{b}}) + f(-t^{\frac{1}{b}})] t^{\frac{1}{b}-1} \eta(4^b t) \chi_{(0, \infty)}(t).$$

Hence by Pitt's inequality

$$\begin{aligned} \int_{\mathbf{R}} |I(x)|^{b+1} dx &\leq A \left[\int_0^\infty |\check{g}(x)|^{b+1} dx + \|\eta\|_1^{b+1} \|f\|_{b+1}^{b+1} \right] \\ &\leq A \left[\int_{\mathbf{R}} |g(t)|^{b+1} |t|^{b-1} dt + \|\eta\|_1^{b+1} \|f\|_{b+1}^{b+1} \right] \leq A \|f\|_{b+1}^{b+1}. \end{aligned}$$

Similarly,

$$\int_{\mathbf{R}} |\Pi(x)|^{b+1} dx \leq A \|f\|_{b+1}^{b+1}.$$

Using (35), we obtain:

$$(36) \quad \|F_2\|_{b+1} \leq A\|f\|_{b+1}.$$

It now follows from (32)-(34) and (36) that

$$\|T_b f\|_{b+1} \leq A\|f\|_{b+1}$$

for all $f \in \mathcal{S}(\mathbf{R})$. This completes the proof of Theorem B.

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