FURTHER EXTENSION OF THE HEINZ-KATO-FURUTA INEQUALITY

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(Communicated by David R. Larson)

ABSTRACT. Let T be a bounded operator on a Hilbert space $\mathfrak{H}$, and A, B positive definite operators. Kato has shown that if $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in \mathfrak{H}$, then $|(Tx, y)| \leq \|f(A)x\| \|g(B)y\|$, where $f(t), g(t)$ are operator monotone functions defined on $[0, \infty)$ such that $f(t)g(t) = t$. Furuta has shown that $|(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\| \|B^\beta y\|$, where $0 \leq \alpha, \beta \leq 1$, $1 \leq \alpha + \beta$. Let $f(t), g(t)$ be any continuous operator monotone functions, and set $h(t) = f(t)g(t)/t$ for $t > 0$. We will show that $Th(|T|)$ is well defined and $|(Th(|T|)x, y)| \leq \|f(A)x\| \|g(B)y\|$. Moreover, we will extend this result for unbounded closed operators densely defined on $\mathfrak{H}$.

1. INTRODUCTION

Let $B(\mathfrak{H})$ be the C^* -algebra of all bounded operators on a Hilbert space $\mathfrak{H}$. Let f be a real valued continuous function defined on $[0, \infty)$. If f satisfies $f(A) \leq f(B)$ whenever $0 \leq A \leq B$, $A, B \in B(\mathfrak{H})$, then f is called an *operator monotone function*. f is an operator monotone function if and only if f has an analytic extension that is holomorphic on the upper half plane and maps the upper half plane into it [4]. Therefore, if $0 \leq f(t)$ is an operator monotone function on $[0, \infty)$, then so is $f(\sqrt{t})^2$. This implies that for $A, B \geq 0$,

$$(1) \quad \|Ax\| \leq \|Bx\| \text{ for every } x \in \mathfrak{H} \rightarrow \|f(A)x\| \leq \|f(B)x\| \text{ for every } x \in \mathfrak{H}.$$

t^α ($0 < \alpha \leq 1$) and $\log(1+t)$ are operator monotone functions [2], [4]. Kato has shown the following.

Theorem A ([3]). *Let T be a bounded operator on a Hilbert space $\mathfrak{H}$, and A, B positive definite operators. If $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in \mathfrak{H}$, then*

$$(2) \quad |(Tx, y)| \leq \|A^\alpha x\| \|B^{1-\alpha} y\| \text{ for any } \alpha \in [0, 1].$$

Actually, he has shown that the above inequality holds for (unbounded) closed operators T, A, B , and furthermore that A^α and $B^{1-\alpha}$ in the above inequality can

Received by the editors June 6, 1997 and, in revised form, December 9, 1997.

1991 *Mathematics Subject Classification*. Primary 47A63, 47A30, 47B25.

Key words and phrases. Löwner theory, Heinz inequality, operator monotone function, closed operator, non-negative operator.

This research was partially supported by Grant-in-Aid for Scientific Research.

be replaced by $f(A)$ and $g(B)$, where $f(t)$, $g(t)$ are operator monotone functions defined on $[0, \infty)$ such that $f(t)g(t) = t$, that is,

$$(3) \quad |(Tx, y)| \leq \|f(A)x\| \|g(B)y\|.$$

Furuta has extended (2) as follows.

Theorem B ([1]). *Let T be a bounded operator on a Hilbert space $\mathfrak{H}$, and A, B positive definite operators. If $\|Tx\| \leq \|Ax\|$, $\|T^*y\| \leq \|By\|$ for all $x, y \in \mathfrak{H}$, then*

$$(4) \quad |(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\| \|B^\beta y\|,$$

where $0 \leq \alpha, \beta \leq 1$, $1 \leq \alpha + \beta$.

The aim of this paper is to extend (3) and (4). Actually, we will show that $T((fg)/t)(|T|)$ is well defined for every operator monotone function $f(t)$, $g(t)$ and

$$|(T((fg)/t)(|T|)x, y)| \leq \|f(A)x\| \|g(B)y\|.$$

We will treat bounded operators in the next section and unbounded closed operators in the last section.

2. BOUNDED OPERATORS

Lemma 1. *Let $A = \int_{-0}^{\infty} \lambda dE_\lambda$ be the spectral decomposition of $A \geq 0$. Then $A(A + \delta)^{-1}$ strongly converges to $I - E_0$ as $\delta \rightarrow +0$.*

Proof. $A(A + \delta)^{-1}E_0x = (A + \delta)^{-1}AE_0x = 0$ for $x \in \mathfrak{H}$. Thus we have

$$\begin{aligned} \|A(A + \delta)^{-1}x - (I - E_0)x\|^2 &= \|\{A(A + \delta)^{-1} - I\}(I - E_0)x\|^2 \\ &= \int_{-0}^{\infty} \left| \frac{\lambda}{\lambda + \delta} - 1 \right|^2 d\|E_\lambda(I - E_0)x\|^2 = \int_{+0}^{\infty} \left| \frac{\delta}{\lambda + \delta} \right|^2 d\|E_\lambda(I - E_0)x\|^2. \end{aligned}$$

From Lebesgue's dominated convergence theorem, this converges to 0 as $\delta \rightarrow +0$. $\square$

Lemma 2. *Let $T = V|T|$ be the polar decomposition of T . Then*

$$T(|T| + \delta)^{-1} \rightarrow V \text{ (strongly) as } \delta \rightarrow +0.$$

Proof. Let $\{E_\lambda\}$ be the spectral family for $|T|$. By Lemma 1, $|T|(|T| + \delta)^{-1} \rightarrow I - E_0$ as $\delta \rightarrow +0$. Thus we have

$$T(|T| + \delta)^{-1} \rightarrow V(I - E_0) \text{ (strongly) as } \delta \rightarrow +0.$$

Since E_0 is the projection onto the eigen-space $\{x : Tx = 0\}$, we get $VE_0 = 0$. Consequently,

$$T(|T| + \delta)^{-1} \rightarrow V \text{ (strongly) as } \delta \rightarrow +0. \quad \square$$

Definition 1. For a function $h(t)$ we set $h_\delta(t) = h(t + \delta)$.

Lemma 3. *Let $h(t)$ be a continuous function defined on $(0, \infty)$. If $th(t)$ has the continuous extension $f(t)$ on $[0, \infty)$, then*

$$Th_\delta(|T|) \rightarrow Vf(|T|) \text{ (strongly) as } \delta \rightarrow +0.$$

Proof. We note that $Th_\delta(|T|) = T(|T| + \delta)^{-1}f(|T| + \delta)$. Since $f(|T| + \delta)$ converges to $f(|T|)$ in the norm topology, from Lemma 2 it follows that $Th_\delta(|T|) \rightarrow Vf(|T|)$ (strongly) as $\delta \rightarrow +0$. $\square$

Definition 2. Under the same assumption as in Lemma 3, we denote the limit of $Th_\delta(|T|)$ as $\delta \rightarrow +0$ by $Th(|T|)$, which is equal to $Vf(|T|)$.

Let us remark that we defined not $h(|T|)$, but $Th(|T|)$. If $h(t)$ is continuous on $[0, \infty)$, then $h(|T|)$ is well defined, and the product of T and $h(|T|)$ is coincident with $Th(|T|)$ defined above. It is clear that if $h(t) = t^{-\alpha}$ ($0 < \alpha \leq 1$), then $Th(|T|)$ is well defined and $Th(|T|) = V|T|^{1-\alpha}$.

Theorem 4. Let f and g be any continuous operator monotone functions defined on $[0, \infty)$. Then $T((fg)/t)(|T|)$ is well defined for every T in $B(\mathfrak{H})$ in the sense of Definition 2. Moreover, if

$$\|Tx\| \leq \|Ax\|, \|T^*y\| \leq \|By\| \quad \text{for all } x, y \in \mathfrak{H},$$

where $A, B \geq 0$, then

$$(5) \quad |(T((fg)/t)(|T|)x, y)| \leq \|f(A)x\| \|g(B)y\| \quad \text{for all } x, y \in \mathfrak{H}.$$

Proof. Set $h(t) = f(t)g(t)/t$ for $t > 0$. It is clear that $th(t)$ is continuous on $[0, \infty)$ and hence $Th(|T|) = T((fg)/t)(|T|)$ is well defined. Let $T = V|T|$ be the polar decomposition of T . Then by Lemma 3 we obtain

$$(6) \quad \begin{aligned} Th(|T|) &= V(fg)(|T|) = Vf(|T|)g(|T|), \\ |(Th(|T|)x, y)| &= |(Vf(|T|)g(|T|)x, y)| = |(g(|T^*|)Vf(|T|)x, y)| \\ &\leq \|f(|T|)x\| \|g(|T^*|)y\|, \end{aligned}$$

and by (1)

$$\leq \|f(A)x\| \|g(B)y\|.$$

□

We remark that (5) is an extension of both (3) and (4): in fact, we removed the conditions $f(t)g(t) = t$ and $1 \leq \alpha + \beta$ from (3) and (4), respectively.

3. UNBOUNDED CLOSED OPERATORS

From now on, we consider the case where T, A, B are unbounded closed operators densely defined on $\mathfrak{H}$. Let us remember that for a self-adjoint operator A and for continuous functions f, g , $f(A)g(A) \subseteq (fg)(A)$ and that if $x \in \mathfrak{D}(g(A)) \cap \mathfrak{D}((fg)(A))$, then $x \in \mathfrak{D}(f(A)g(A))$ and $f(A)g(A)x = (fg)(A)x$.

It is known [4], [5] that an operator monotone function f on $[0, \infty)$ has an integral representation

$$f(t) = \alpha + \beta t - \int_0^\infty \left(\frac{1}{t+s} - \frac{s}{s^2+1} \right) dv(s) \quad \text{for } t > 0,$$

where α and β are real, $\beta \geq 0$, and v is a non-negative Borel measure such that

$$\int_0^\infty \frac{1}{1+s^2} dv(s) < \infty.$$

Since

$$\lim_{t \rightarrow +0} f(t) = f(0) \quad \text{and} \quad \lim_{t \rightarrow +0} \int_{\frac{1}{t}}^\infty \left(\frac{s}{s^2+1} - \frac{1}{t+s} \right) dv(s) = 0,$$

it follows that

$$\begin{aligned} f(0) &= \alpha - \lim_{t \rightarrow +0} \int_0^{\frac{1}{t}} \left(\frac{1}{t+s} - \frac{s}{s^2+1} \right) dv(s) \\ &= \alpha - \lim_{t \rightarrow +0} \int_0^\infty \left(\frac{1}{t+s} - \frac{s}{s^2+1} \right) \chi_{[0, \frac{1}{t}]} dv(s). \end{aligned}$$

From Fatou's theorem it follows that

$$\int_0^\infty \left(\frac{1}{s} - \frac{s}{s^2+1} \right) dv(s) < \infty, \text{ and hence } f(0) = \alpha - \int_0^\infty \left(\frac{1}{s} - \frac{s}{s^2+1} \right) dv(s).$$

Thus we have

$$(7) \quad f(t) = f(0) + \beta t + \int_0^\infty \left(\frac{1}{s} - \frac{1}{t+s} \right) dv(s).$$

Since $f(1) < \infty$, we get

$$(8) \quad \int_0^\infty \frac{1}{s(1+s)} dv(s) < \infty.$$

Lemma 5. *Let $0 \leq f(t)$, $g(t)$ be operator monotone functions, and let $A \geq 0$ be an unbounded self-adjoint operator. Then $\mathfrak{D}(A) \subseteq \mathfrak{D}(f(A))$, where $\mathfrak{D}(A)$ stands for the domain of A . Moreover, $\mathfrak{D}(A^2) \subseteq \mathfrak{D}(f(A)g(A)) \subseteq \mathfrak{D}((fg)(A))$.*

Proof. By (7) and (8), it is clear that $f(t)/t$ and $g(t)/t$ are bounded on $[1, \infty)$. Hence we can see

$$\mathfrak{D}(A) \subseteq \mathfrak{D}(f(A)) \quad \text{and} \quad \mathfrak{D}(A) \subseteq \mathfrak{D}(g(A)).$$

Similarly, from the boundedness of $f(t)g(t)/t^2$ on $[1, \infty)$ it follows that $\mathfrak{D}(A^2) \subseteq \mathfrak{D}((fg)(A))$. Thus for an arbitrary $x \in \mathfrak{D}(A^2)$, we have

$$x \in \mathfrak{D}(g(A)) \quad \text{and} \quad x \in \mathfrak{D}((fg)(A)),$$

and hence

$$x \in \mathfrak{D}(f(A)g(A)) \quad \text{and} \quad f(A)g(A)x = (fg)(A)x.$$

□

Proposition 6. *Let T be a densely defined closed operator, and let f and g be arbitrary operator monotone functions defined on $[0, \infty)$, such that $f \geq 0$, $g \geq 0$. Set $h(t) = ((fg)/t)$ for $t > 0$. Then*

$$\mathfrak{D}(Th_\delta(|T|)) = \mathfrak{D}((fg)(|T|)) \text{ for every } \delta > 0,$$

$$Th_\delta(|T|)x \rightarrow V(fg)(|T|x) \quad (s) \quad \text{as } \delta \rightarrow +0 \text{ for } x \in \mathfrak{D}((fg)(|T|)).$$

Proof. By (7),

$$0 \leq f(t+\delta) - f(t) \leq f(\delta) - f(0), \quad 0 \leq g(t+\delta) - g(t) \leq g(\delta) - g(0).$$

This implies

$$\begin{aligned} 1 \leq \frac{f(t+\delta)}{f(t)} \frac{g(t+\delta)}{g(t)} &\leq \left(1 + \frac{f(\delta) - f(0)}{f(t)}\right) \left(1 + \frac{g(\delta) - g(0)}{g(t)}\right) \\ &\leq \left(1 + \frac{f(\delta) - f(0)}{f(1)}\right) \left(1 + \frac{g(\delta) - g(0)}{g(1)}\right) \quad (t > 1). \end{aligned}$$

Thus we can see that

$$\mathfrak{D}((fg)(|T| + \delta)) = \mathfrak{D}((fg)(|T|)),$$

$$(9) \quad (fg)(|T| + \delta)x \rightarrow (fg)(|T|x) \quad (s) \quad (\delta \rightarrow +0) \text{ for } x \in \mathfrak{D}((fg)(|T|)).$$

By $(fg)(t + \delta) \geq \delta h(t + \delta)$, we obtain

$$\mathfrak{D}((fg)(|T| + \delta)) \subseteq \mathfrak{D}(h(|T| + \delta)).$$

Therefore, using $(|T| + \delta)(h(|T| + \delta)) \subseteq (fg)(|T| + \delta)$ we get

$$(|T| + \delta)(h(|T| + \delta)) = (fg)(|T| + \delta).$$

Thus we obtain

$$Th_\delta(|T|) = \{V|T|(|T| + \delta)^{-1}(|T| + \delta)\}h_\delta(|T|) = \{V|T|(|T| + \delta)^{-1}\}(fg)(|T| + \delta);$$

here we used $\mathfrak{D}(|T|h(|T| + \delta)) = \mathfrak{D}((|T| + \delta)(h(|T| + \delta)))$.

In a fashion similar to Lemma 1, we can see

$$V|T|(|T| + \delta)^{-1} \rightarrow V(1 - E_0) = V \text{ (strongly) as } \delta \rightarrow +0.$$

Since $V|T|(|T| + \delta)^{-1}$ is contractive, this and (9) imply

$$Th_\delta(|T|x) \rightarrow V(fg)(|T|x) \quad (\delta \rightarrow +0) \text{ for } x \in \mathfrak{D}((fg)(|T|)).$$

□

Definition 3. Under the same condition as in Proposition 6, we denote the limit of $Th_\delta(|T|)$ by $Th(|T|)$. This is equal to $V(fg)(|T|)$, where $(fg)(t) = th(t)$.

For non-negative self-adjoint operators A and B , $A \ll B$ means $\mathfrak{D}(B) \subseteq \mathfrak{D}(A)$ and $\|Ax\| \leq \|Bx\|$ for $x \in \mathfrak{D}(B)$.

If $0 \leq f$ is an operator monotone function defined on $[0, \infty)$, then $f(A) \ll f(B)$ whenever $A \ll B$ (see [3]).

Theorem 7. Let T be a densely defined closed operator, and let A and B be non-negative self-adjoint operators. Let f and g be arbitrary operator monotone functions defined on $[0, \infty)$, such that $f \geq 0$, $g \geq 0$. Then $T((fg)/t)(|T|)$ is well defined in the sense of Definition 3. Moreover, if $|T| \ll A$, $|T^*| \ll B$, then

$$|(T((fg)/t)(|T|x), y)| \leq \|f(A)x\| \|g(B)y\|$$

for every $x \in \mathfrak{D}(A) \cap \mathfrak{D}(fg(|T|))$ and for every $y \in \mathfrak{D}(B)$.

Proof. By Proposition 6 and Definition 3, $T((fg)/t)(|T|)$ is well defined and equal to $V(fg)(|T|)$.

The assumption $|T| \ll A$, $|T^*| \ll B$ implies that $f(|T|) \ll f(A)$, $g(|T^*|) \ll g(B)$. Suppose

$$x \in \mathfrak{D}(A) \cap \mathfrak{D}(fg(|T|)) \quad \text{and} \quad y \in \mathfrak{D}(B).$$

By Lemma 5, we get

$$x \in \mathfrak{D}(f(|T|)) \quad \text{and} \quad y \in \mathfrak{D}(g(|T^*|)).$$

Therefore, $x \in \mathfrak{D}(f(|T|)) \cap \mathfrak{D}(fg(|T|))$ and hence $(fg)(|T|x) = g(|T|)f(|T|x)$. Consequently we get

$$\begin{aligned} |(Th(|T|x), y)| &= |(V(gf)(|T|x), y)| = |(Vg(|T|)f(|T|x), y)| \\ &= |(g(|T^*|)Vf(|T|x), y)| \leq \|f(|T|x)\| \|g(|T^*|)y\| \leq \|f(A)x\| \|g(B)y\|. \quad \square \end{aligned}$$

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