VANISHING OF A CERTAIN KIND OF VASSILIEV INVARIANTS OF 2-KNOTS

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(Communicated by Ronald A. Fintushel)

ABSTRACT. In knot theory, Vassiliev's 1-knot invariants are defined in a combinatorial way as finite type invariants. By a natural generalization of the combinatorial definition, one has a certain family of 2-knot invariants, which should be called finite type 2-knot invariants. They form a subspace of the whole space of "Vassiliev 2-knot invariants". In this paper we prove that it is 1-dimensional.

By a Σ -immersion, we mean an immersion $k : \Sigma \rightarrow R^4$ of an oriented, closed and connected 2-manifold Σ into 4-space. If there exist orientation-preserving diffeomorphisms of Σ and R^4 to themselves which make the following diagram commutative, then we say that k and k' are *equivalent*:

$$\begin{array}{ccc} \Sigma & \xrightarrow{k} & R^4 \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{k'} & R^4. \end{array}$$

A Σ -immersion is *generic* if the singularities are transverse double points. In this case, the normal Euler number $e(k)$ of k is equal to $2(d_+(k) - d_-(k))$, where $d_+(k)$ is the number of positive double points of k and $d_-(k)$ is that of negative ones.

By the Smale and Hirsch theorem [10], [11], [4], two Σ -immersions k_0 and k_1 are related by a regular homotopy if and only if $e(k_0) = e(k_1)$. Moreover, if k_0 and k_1 are generic, then by taking a regular homotopy generically, we have a generic one-parameter family of Σ -immersions k_t ($t \in [0, 1]$) being generic for all but a finite number of values t such that at each exceptional t a finger move or a Whitney trick occurs. (There exists an explicit procedure to construct such a one-parameter family, cf. [6].) A finger move (cf. [3], [7], [8], [9]) is a regular homotopy between Σ -immersions yielding a pair of positive and negative transverse double points, whose local image is illustrated in Figure 1 in the motion picture method. The inverse operation is a Whitney trick in dimension four. It is a difficult problem to determine whether or not a given pair of double points is a Whitney pair (cf. [3], [7], [9]).

Let $\mathcal{K}(\Sigma)_{(n)}$ be the set of equivalence classes of Σ -immersions whose singularities are (transverse or non-transverse) double points and the number of non-transverse

Received by the editors May 22, 1997 and, in revised form, February 1, 1998.

1991 *Mathematics Subject Classification.* Primary 57Q45, 57M25.

Key words and phrases. 2-knot, Vassiliev invariant, finite type invariant, finger move, regular homotopy.

FIGURE 1

ones is n . In particular, $\mathcal{K}(\Sigma)_{(0)}$ (denoted simply by $\mathcal{K}(\Sigma)$) is the set of equivalence classes of generic Σ -immersions. If a Σ -immersion k has a non-transverse double point, then there exists a finger move (regular homotopy) k_t ($t \in [0, 1]$) with $k_{0.5} = k$ as a bifurcation. We call the restricted homotopy k_t ($t \in [0.5, 1]$) a *positive desingularization*, and the inverse of the restricted homotopy k_t ($t \in [0, 0.5]$) a *negative desingularization*. The equivalence classes of k_0 and k_1 are uniquely determined from that of k .

Let $v : \mathcal{K}(\Sigma) \rightarrow F$ be a mapping of $\mathcal{K}(\Sigma)$ to a field F (which may be a commutative ring with unit or an abelian group). For a Σ -immersion $k_\times$ whose class $[k_\times]$ belongs to $\mathcal{K}(\Sigma)_{(1)}$, define $v([k_\times])$ by $v([k_+]) - v([k_-])$, where k_+ and k_- are Σ -immersions obtained from $k_\times$ by positive and negative desingularizations respectively. Inductively, the mapping $v : \mathcal{K}(\Sigma) \rightarrow F$ is extended to $v : \coprod_{n=0}^{\infty} \mathcal{K}(\Sigma)_{(n)} \rightarrow F$. (It seems reasonable that such a mapping is called Vassiliev's invariant of Σ -immersions, cf. [1]. The family of such mappings forms a linear space over F .) If there exists an integer n_0 such that v vanishes over all $\mathcal{K}(\Sigma)_{(n)}$ with $n \geq n_0$, we say that v is of *finite type*.

Theorem 1. *Let $v : \mathcal{K}(\Sigma) \rightarrow F$ be a mapping of finite type. Then $v([k]) = v([k'])$ for any generic Σ -immersions k and k' with $e(k) = e(k')$ and $d(k) = d(k')$.*

Here $d(k)$ stands for the number of (transverse) double points of k . The assumption of the theorem is equal to $d_+(k) = d_+(k')$ and $d_-(k) = d_-(k')$.

Consider a special case that Σ is a 2-sphere S^2 . We call a 2-knot invariant valued over F a *finite type* invariant if it is extended to a mapping $v : \mathcal{K}(S^2) \rightarrow F$ of finite type. This is a natural analogue of finite type 1-knot invariants and Vassiliev's invariants in knot theory. Theorem 1 implies the following theorem.

Theorem 2. *The set of finite type 2-knot invariants is 1-dimensional, i.e., they are constant maps.*

1. THE VASSILIEV MODULE OF Σ -IMMERSIONS

Let k be a Σ -immersion whose class $[k]$ belongs to $\mathcal{K}(\Sigma)_{(n)}$ and $v_1, \dots, v_n$ the non-transverse double points of k . For each n -tuple of signs $(\epsilon_1, \dots, \epsilon_n)$, let $k_{\epsilon_1, \dots, \epsilon_n}$ denote a generic Σ -immersion obtained by desingularizations on $v_1, \dots, v_n$ in directions according to $(\epsilon_1, \dots, \epsilon_n)$. Then an element

$$\sum_{(\epsilon_1, \dots, \epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1, \dots, \epsilon_n}] \in \mathbb{Z}\mathcal{K}(\Sigma)$$

is determined from $[k]$ up to sign. Denote by L_n the subspace of $\mathbb{Z}\mathcal{K}(\Sigma)$ spanned by such elements for all $[k] \in \mathcal{K}(\Sigma)_{(n)}$. Evidently, we have $L_1 \supset L_2 \dots$. The Smale and Hirsch theorem implies that $[k] - [k'] \in L_1$ if and only if $e(k) = e(k')$. The n -th Vassiliev module is the $(\mathbb{Z})$ -module $\mathbb{Z}\mathcal{K}(\Sigma)/L_n$. (By tensor product with F , one may consider it a vector space or a module over F .) A mapping $v : \mathcal{K}(\Sigma) \rightarrow F$ is of finite type if and only if its extension $v : \mathbb{Z}\mathcal{K}(\Sigma) \rightarrow F$ is factored through this module for some n .

The following is our main theorem, which implies Theorem 1.

Theorem 3. *Let k and k' be generic Σ -immersions with $e(k) = e(k')$ and $d(k) = d(k')$. Then $[k] - [k'] \in L_n$ for every $n \in \mathbb{N}$.*

 2. UNKNOTTED Σ -IMMERSIONS

A generic Σ -immersion is called an *unknotted* one if it is equivalent to a standard embedding of Σ with some (or no) “trivial kinks” (see Figure 2). We denote by u_d^e an unknotted Σ -immersion with $e(u_d^e) = e$ and $d(u_d^e) = d$. Notice that the equivalence class of u_d^e is unique.

Let $k : \Sigma \rightarrow R^4$ be a Σ -immersion. A *chord* γ attached to $k(\Sigma)$ means a smooth simple arc in R^4 whose endpoints are distinct points of $k(\Sigma)$ except the singularities and the interior of γ is disjoint from it. By a standard argument on general position, the ambient isotopy class of γ by isotopies of R^4 keeping $k(\Sigma)$ setwise fixed may be treated as a homotopy class in the sense of [2], [5]. If k' is a Σ -immersion obtained from k by a finger move along γ (see Figure 1), then $[k']$ is uniquely determined from $[k]$ and the class of γ . (The Σ -immersion appearing as a bifurcation in the finger move is obtained from k by shrinking γ .)

Lemma 1. *Let u be an unknotted Σ -immersion. Any Σ -immersion obtained from u by a finger move is also unknotted.*

FIGURE 2

FIGURE 3

Proof. Let γ be a chord attached to $u(\Sigma)$ along which the finger move is applied. Since the fundamental group $\pi_1(R^4 \setminus u(\Sigma), *)$ is an infinite cyclic group generated by the meridian of $u(\Sigma)$, by arguments of [5], [2], we may assume that γ is a chord attached to $u(\Sigma)$ trivially. The finger move brings the Σ -immersion u a pair of positive and negative trivial kinks, see Figure 3. $\square$

Lemma 2. *If a Σ -immersion k' is obtained from another k by a Whitney trick followed by a finger move, then it is also obtained, up to equivalence, from k by a finger move followed by a Whitney trick.*

Proof. Let V be a 4-ball in R^4 such that a Whitney trick performed in V as in Figure 1 changes k into a Σ -immersion k'' and a finger move along a chord γ attached to $k''(\Sigma)$ changes k'' into k' . If γ is disjoint from V , the assertion is obvious. Since the inclusion-induced homomorphism $\pi_1(\partial V \setminus k''(\Sigma), *) \rightarrow \pi_1(V \setminus k''(\Sigma), *)$ is surjective (in fact, isomorphic), we may assume that γ is disjoint from V . $\square$

Lemma 3 (Unknotting Lemma). *For any generic Σ -immersion k , there exists a family of mutually disjoint chords attached to $k(\Sigma)$ such that finger moves along them change k into an unknotted one*

Proof. By the Smale and Hirsch theorem, there exists a generic regular homotopy between k and an unknotted Σ -immersion u . Using the previous lemma, we see that there exists a Σ -immersion k' such that (i) it is obtained from k by a sequence of finger moves and (ii) it is obtained also from u by a sequence of finger moves. By (ii) and Lemma 1, k' is unknotted. By a similar argument as in the proof of Lemma 2, it is easily seen that a sequence of finger moves may be replaced with the same number of simultaneous finger moves. Thus, from (i), we have the result. $\square$

The *unknotting number* of a generic Σ -immersion k is defined by the minimum number of chords as in Lemma 3, which we denote by $u(k)$. By the proof of

Lemma 3, we see that the unknotting number $u(k)$ equals the minimum number of finger moves appearing in generic regular homotopies between k and unknotted Σ -immersions.

3. PROOF OF THEOREM 3

Let k be a generic Σ -immersion and $\gamma_1, \dots, \gamma_n$ mutually disjoint chords attached to $k(\Sigma)$. For each n -tuple of signs $(\epsilon_1, \dots, \epsilon_n)$, let $k_{\epsilon_1, \dots, \epsilon_n}$ denote a generic Σ -immersion obtained by finger moves along γ_i for i with $\epsilon_i = +1$ and by eliminating γ_i for i with $\epsilon_i = -1$. When one considers a Σ -immersion with n non-transverse double points obtained from k by shrinking the chords, $k_{\epsilon_1, \dots, \epsilon_n}$ is the same as before. Thus, in this situation, we have

$$\sum_{(\epsilon_1, \dots, \epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1, \dots, \epsilon_n}] \in L_n.$$

Proof of Theorem 3. It is sufficient to prove the following assertion for every $m \in \mathbb{N}$.

Assertion (m). Let k and k' be generic Σ -immersions with $e(k) = e(k')$ and $d(k) = d(k')$. If $u(k) \leq m$ and $u(k') \leq m$, then $[k] - [k'] \in L_n$ for any $n \geq m + 1$.

Since $u(k) \leq m$, there exist m mutually disjoint chords attached to $k(\Sigma)$, say $\gamma_1, \dots, \gamma_m$, such that the finger moves along them change k into an unknotted one, which is u_{d+2m}^e where $e = e(k)$ and $d = d(k)$. Let n be an integer with $n \geq m + 1$, and take parallel copies $\gamma_{m+1}, \dots, \gamma_n$ of γ_1 such that $\gamma_1, \dots, \gamma_n$ are mutually disjoint chords attached to $k(\Sigma)$. For each n -tuple of signs $(\epsilon_1, \dots, \epsilon_n)$, let $k_{\epsilon_1, \dots, \epsilon_n}$ denote a generic Σ -immersion obtained by finger moves along γ_i for i with $\epsilon_i = +1$ as before. Let $p = p(\epsilon_1, \dots, \epsilon_n)$ be the number of positive signs. If p is not zero, then by Lemma 1 we have $u(k_{\epsilon_1, \dots, \epsilon_n}) \leq m - 1$, $e(k_{\epsilon_1, \dots, \epsilon_n}) = e$ and $d(k_{\epsilon_1, \dots, \epsilon_n}) = d + 2p$. We use the induction on m . If $m = 1$, then $k_{\epsilon_1, \dots, \epsilon_n} = u_{d+2p}^e$ for any $(\epsilon_1, \dots, \epsilon_n)$ with $p = p(\epsilon_1, \dots, \epsilon_n) \neq 0$. Thus,

$$\sum_{(\epsilon_1, \dots, \epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1, \dots, \epsilon_n}] = \sum_{p=1}^n (-1)^{n-p} \binom{n}{p} [u_{d+2p}^e] + (-1)^n [k].$$

Since the left-hand side belongs to L_n and a similar equation holds for k' , we have $[k] - [k'] \in L_n$. If $m \geq 2$, then by the induction hypothesis we have $[k_{\epsilon_1, \dots, \epsilon_n}] - [u_{d+2p}^e] \in L_n$ for any $(\epsilon_1, \dots, \epsilon_n)$ with $p = p(\epsilon_1, \dots, \epsilon_n) \neq 0$. Thus,

$$\sum_{(\epsilon_1, \dots, \epsilon_n)} \epsilon_1 \dots \epsilon_n [k_{\epsilon_1, \dots, \epsilon_n}] \equiv \sum_{p=1}^n (-1)^{n-p} \binom{n}{p} [u_{d+2p}^e] + (-1)^n [k] \pmod{L_n}.$$

Therefore, we see that $[k] - [k'] \in L_n$. $\square$

The author would like to thank the referee for informing him that Tobias Ekholm has considered a similar problem under the direction of Oleg Viro.

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