THE CONNECTED STABLE RANK OF THE PURELY INFINITE SIMPLE C^* -ALGEBRAS

YIFENG XUE

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ABSTRACT. Suppose that $\mathcal{A}$ is a unital purely infinite simple C^* -algebra. If the class $[1]$ of the unit 1 in $K_0(\mathcal{A})$ has torsion, then $\text{csr}(\mathcal{A}) = \infty$; if $[1]$ is torsion-free in $K_0(\mathcal{A})$, then $\text{csr}(\mathcal{A}) = 2$. If $\mathcal{A}$ is a non-unital purely infinite simple C^* -algebra, then $\text{csr}(\mathcal{A}) = 2$.

Before considering the connected stable rank of the purely infinite simple C^* -algebra, we need to introduce some notation as follows. For the C^* -algebra $\mathcal{A}$ with unit 1, we denote by $\mathcal{U}_n(\mathcal{A})$ the group of unitary elements in the matrix algebra $M_n(\mathcal{A})$ and also denote by $\mathcal{U}_n^0(\mathcal{A})$ the connected component of 1_n in $\mathcal{U}_n(\mathcal{A})$. We view $\mathcal{A}^n = \{(a_1, \dots, a_n)^T \mid a_i \in \mathcal{A}\}$ as the set of all $n \times 1$ matrices over $\mathcal{A}$. For $a = (a_1, \dots, a_n)^T$, we set $a^* = (a_1^*, \dots, a_n^*)$ —the $1 \times n$ over $\mathcal{A}$. Put

$$S_n(\mathcal{A}) = \left\{ (a_1, \dots, a_n)^T \in \mathcal{A}^n \mid a^*a = \sum_{i=1}^n a_i^*a_i = 1 \right\}.$$

$S_n(\mathcal{A})$ has the base point $e_n = (1, 0, \dots, 0)^T \in \mathcal{A}^n$. Let $\pi_0(S_n(\mathcal{A}), e_n)$ denote the set of all path connected components of $S_n(\mathcal{A})$. We identify the path connected component containing e_n with zero element “0”. Thus from [Sr], we have

$$\text{csr}(\mathcal{A}) = \min\{n \mid \pi_0(S_m(\mathcal{A}), e_m) = 0, \forall m \geq n\}$$

where

$$\text{csr}(\mathcal{A}) = \min\{n \mid \mathcal{U}_m^0(\mathcal{A}) \text{ acts transitively on } S_m(\mathcal{A}), \forall m \geq n\}$$

is the connected stable rank of $\mathcal{A}$ defined in [Rf]. If no such integer exists, we set $\text{csr}(\mathcal{A}) = \infty$. If $\mathcal{A}$ is not unital, we set $\text{csr}(\mathcal{A}) = \text{csr}(\mathcal{A}^+)$, where $\mathcal{A}^+$ is the C^* -algebra obtained from $\mathcal{A}$ by adjoining the unit 1.

Let p, q be two projections in the C^* -algebra $\mathcal{A}$. We write $p \sim q$ if there is $u \in \mathcal{A}$ such that $p = u^*u, q = uu^*$ and denote by $[p]$ the equivalence class of p with respect to “ $\sim$ ” and we also put

$$D(\mathcal{A}) = \{[p] \mid p \text{ is a non-zero projection in } \mathcal{A}\}.$$

Recall that a projection p in $\mathcal{A}$ is called infinite if there is a projection q in $\mathcal{A}$ such that $p \sim q < p$. We say that $\mathcal{A}$ is purely infinite if the closure of $a\mathcal{A}a$ contains an infinite projection for any positive element a in $\mathcal{A}$ (cf. [Cu]).

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From [Cu] we have known that if $\mathcal{A}$ is the unital purely infinite simple C^* -algebra, then $D(\mathcal{A})$ becomes a group with the natural addition $[p] + [q] = [p' + q']$ and $D(\mathcal{A}) \cong K_0(\mathcal{A})$ where $p' \sim p, q' \sim q, p'q' = 0$ and $K_0(\mathcal{A})$ is the K_0 -group of $\mathcal{A}$ defined in [Bk].

Let p be a non-zero projection of the purely infinite simple C^* -algebra $\mathcal{A}$ with unit 1. Set

$$SU(\mathcal{A}) = \{u \in \mathcal{A} | u^*u = 1, uu^* < 1\}, \quad SU_p(\mathcal{A}) = \{u \in SU(\mathcal{A}) | uu^* \leq p\}.$$

If $p = 1$, then $SU_1(\mathcal{A}) = \mathcal{U}(\mathcal{A}) \cup SU(\mathcal{A})$. In this situation, $SU_1(\mathcal{A})$ is not connected for $\mathcal{U}(\mathcal{A}) \cap SU(\mathcal{A}) = \emptyset$ and $\mathcal{U}(\mathcal{A}), SU(\mathcal{A})$ are all closed in $SU_1(\mathcal{A})$.

It is known from [Rf, Proposition 6.5] that the topological stable rank of $\mathcal{A}$ is ∞ if $\mathcal{A}$ is a purely infinite simple C^* -algebra. But what is the $\text{csr}(\mathcal{A})$? In this paper, we will show that $\text{csr}(\mathcal{A}) \in \{2, \infty\}$. First we have the following known lemma which could be deduced directly from [Cu, Lemma 1.8]:

Lemma 1. *Suppose that $\mathcal{A}$ is a purely infinite simple C^* -algebra with unit 1. Then for $k \geq 2$ there are isometries $s_1, \dots, s_k$ in $\mathcal{A}$ such that $\sum_{i=1}^k s_i s_i^* = p$ is a projection in $\mathcal{A}$. Furthermore, if $[1]$ is torsion-free in $K_0(\mathcal{A})$ or if $k \not\equiv 1 \pmod n$ when $[1]$ has order n ($1 \leq n < \infty$) in $K_0(\mathcal{A})$, then $[p] \neq [1]$ and if $k \equiv 1 \pmod n$ for above n , then p can be chosen as $p = 1$.*

From Lemma 1, we can choose k isometries $s_1, \dots, s_k$ in the unital purely infinite simple C^* -algebra $\mathcal{A}$ such that $\sum_{i=1}^k s_i s_i^* = p$ is a projection in $\mathcal{A}$. Now define a map ϕ_k of $M_k(\mathcal{A})$ to $\mathcal{A}$ by

$$\phi_k((a_{ij})_{k \times k}) = \sum_{i,j=1}^k s_i a_{ij} s_j^* + 1 - p.$$

It is easy to check that ϕ_k is a $*$ -homomorphism with $\phi_k(1_k) = 1$.

The following corollary somewhat enhances a partial result of [Cu, Lemma 1.8].

Corollary. *Let $\mathcal{A}$ be a purely infinite simple C^* -algebra with unit 1. Then for any $u \in \mathcal{U}_k(\mathcal{A})$ ($k \geq 2$), there is $u_0 \in \mathcal{U}_k^0(\mathcal{A})$ such that $u = u_0 \text{diag}(\phi_k(u), 1_{k-1})$.*

Proof. Let $s_1, \dots, s_k$ be as above and put

$$X = \begin{pmatrix} s_1 & s_2 & \dots & s_k \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} X & 1_k - XX^* \\ 0 & X^* \end{pmatrix}.$$

Then it is clear that $X^*X = 1_k$, $Y \in \mathcal{U}_{2k}(\mathcal{A})$ and moreover, by the definitions of ϕ_k and Y , we have the following identity:

$$(1) \quad Y \text{diag}(u, 1_k) Y^* = \text{diag}(\phi_k(u), 1_{2k-1}).$$

Noting that Y can be decomposed as the form

$$Y = \begin{pmatrix} 1_k & X \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} 0 & 1_k - 2XX^* \\ 1_k & 0 \end{pmatrix} \begin{pmatrix} 1_k & X^* \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} 1_k & 0 \\ -X & 1_k \end{pmatrix},$$

we obtain that $Y \in \mathcal{U}_{2k}^0(\mathcal{A})$. Therefore applying [LZ, Condition (ii)] and [Lin, Lemma 2.2] to (1), we get the assertion. $\square$

Let $s_1, \dots, s_k$ be as above. Then $(s_1^*u, \dots, s_k^*u) \in S_k(\mathcal{A})$ for any $u \in SU_p(\mathcal{A})$ because $uu^* \leq p$ $pu = u$ for $u \in SU_p(\mathcal{A})$. This leads to the following lemma.

Lemma 2. *The map $\beta: (SU_p(\mathcal{A}), s_1) \rightarrow (S_k(\mathcal{A}), e_k)$ given by $\beta(u) = (s_1^*u, \dots, s_k^*u)$ is homeomorphic.*

Proof. Obviously, β is continuous and $\beta(s_1) = e_k$. Now for any $(a_1, \dots, a_k) \in S_k(\mathcal{A})$, put $u = \sum_{i=1}^k s_i a_i$. Since $s_i^* s_j = \delta_{ij} 1$ and $ps_i = s_i$, $i, j = 1, \dots, k$, it follows that $pu = u$ and

$$u^*u = \sum_{i,j} a_i^* s_i^* s_j a_j = \sum_{i=1}^k a_i^* a_i = 1,$$

i.e., $u \in SU_p(\mathcal{A})$. Therefore we can define a continuous map $\gamma: (S_k(\mathcal{A}), e_k) \rightarrow (SU_p(\mathcal{A}), s_1)$ by $\gamma((a_1, \dots, a_k)) = \sum_{i=1}^k s_i a_i$. By the definitions of β and γ , the assertion follows. $\square$

Using Lemma 1 and Lemma 2, we then establish the theorem in the following.

Theorem 1. *Suppose that $\mathcal{A}$ is a unital purely infinite simple C^* -algebra. If the class $[1]$ of the unit 1 in $K_0(\mathcal{A})$ has torsion in $K_0(\mathcal{A})$, then $\text{csr}(\mathcal{A}) = \infty$; if $[1]$ is torsion-free in $K_0(\mathcal{A})$, then $\text{csr}(\mathcal{A}) = 2$.*

Proof. We assume that n ($1 \leq n < \infty$) is the order of $[1]$ in $K_0(\mathcal{A})$, i.e., n is the least positive integer such that $n[1] = 0$ in $K_0(\mathcal{A})$.

If $k \equiv 1 \pmod{n}$, then $SU_1(\mathcal{A})$ is not connected by Lemma 1 and consequently $\pi_0(S_k(\mathcal{A}), e_k) \neq 0$ by Lemma 2. If $k \not\equiv 1 \pmod{n}$, then there are k isometries $s_1, \dots, s_k$ in $\mathcal{A}$ such that $\sum_{i=1}^k s_i s_i^* = p$ is a projection in $\mathcal{A}$ with $[p] \neq [1]$ by Lemma 1. In this case, we have $uu^* \neq p$ for all $u \in SU_p(\mathcal{A})$.

Since $D(\mathcal{A})$ is a group and

$$[p] = [p - uu^*] + [uu^*] = [p - s_1 s_1^*] + [s_1 s_1^*]$$

in $D(\mathcal{A})$ for any $u \in SU_p(\mathcal{A})$, it follows that $[p - uu^*] = [p - s_1 s_1^*]$ in $D(\mathcal{A})$. Noting that $uu^* \leq p$, $s_1 s_1^* \leq p$, we have $p - uu^*, p - s_1 s_1^* \in p\mathcal{A}p$ and $p - uu^* \sim p - s_1 s_1^*$ in $p\mathcal{A}p$. Since $us_1^* \in p\mathcal{A}p$ and $uu^* = (us_1^*)(us_1^*)^*$, $s_1 s_1^* = (us_1^*)^*(us_1^*)$ in $p\mathcal{A}p$, it follows that there is $w_0 \in \mathcal{U}_1(p\mathcal{A}p)$ such that $uu^* = w_0^* s_1 s_1^* w_0$. Put $a = s_1^* w_0^* u$ and $w = (s_i^* w_0 s_j)_{k \times k}$. Then

$$(2) \quad u = w_0 s_1 a, \quad a \in \mathcal{U}_1(\mathcal{A}) \text{ and } w \in \mathcal{U}_k(\mathcal{A}).$$

Since $(s_1^* u, \dots, s_k^* u)^T = w(a, 0, \dots, 0)^T$ by (2), it follows from the Corollary that there exists $u_0 \in \mathcal{U}_k^0(\mathcal{A})$ such that

$$\begin{aligned} (s_1^* u, \dots, s_k^* u)^T &= u_0 \text{diag}(\phi_k(w), 1_{k-1})(a, 0, \dots, 0)^T \\ &= u_0 \text{diag}(w_0 + 1 - p, 1_{k-1})(a, 0, \dots, 0)^T \\ &= u_0 \text{diag}(b, b^*, 1_{k-2})e_k, \end{aligned}$$

where $b = (w_0 + 1 - p)a \in \mathcal{U}_1(\mathcal{A})$, that is, $(s_1^* u, \dots, s_k^* u)^T$ is in the component of e_k for $u_0 \text{diag}(b, b^*, 1_{k-2}) \in \mathcal{U}_k^0(\mathcal{A})$. Therefore $\pi_0(S_k(\mathcal{A}), e_k) = 0$ by Lemma 2 when $k \not\equiv 1 \pmod{n}$.

The above shows that $\text{csr}(\mathcal{A}) = \infty$ when $[1]$ has torsion.

As to the case that $[1]$ is torsion-free in $K_0(\mathcal{A})$, we can use Lemma 1 and the same method as in the proof of $\pi_0(S_k(\mathcal{A}), e_k) = 0$ when $k \not\equiv 1 \pmod{n}$ to deduce that $\pi_0(S_k(\mathcal{A}), e_k) = 0 \forall k \geq 2$, i.e., $\text{csr}(\mathcal{A}) \leq 2$. Since $\mathcal{A}$ contains an isometry s_1 , $\text{csr}(\mathcal{A})$ must be equal to two. $\square$

Let $\mathcal{O}_n$ be the Cuntz algebra. Then by Theorem 1, we have $\text{csr}(\mathcal{O}_n) = \infty$ ($2 \leq n < \infty$), and $\text{csr}(\mathcal{O}_\infty) = 2$ since $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ ($n < \infty$), $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and the class [1] of unit 1 is the generator of $K_0(\mathcal{O}_n)$ (cf. [Cu]).

As an end of this paper, we will consider the $\text{csr}(\mathcal{A})$ if $\mathcal{A}$ is a non-unital purely infinite simple C^* -algebra. Our result is the following:

Theorem 2. *Let $\mathcal{A}$ be a non-unital purely infinite simple C^* -algebra. Then $\text{csr}(\mathcal{A}) = 2$.*

In order to prove this theorem, we need the following lemmas.

Lemma 3. *Let $\mathcal{A}$ be a C^* -algebra with unit 1 and let $a = (a_1, \dots, a_n)^T$, $b = (b_1, \dots, b_n)^T \in S_n(\mathcal{A})$ such that $\|a_i - b_i\| \leq (\sum_{i=1}^n \|a_i\|)^{-1}$. Then a, b are in the same component of $S_n(\mathcal{A})$.*

Proof. Put $c_t = ((1-t)a + tb)^*((1-t)a + tb) \forall t \in [0, 1]$. Since $a^*a = b^*b = 1$, it follows that

$$\|1 - c_t\| = \|(1-t)t(a^*(b-a) + (b-a)^*a)\| \leq \frac{1}{2} \quad \forall t \in [0, 1].$$

Set $G_t = ((1-t)a + tb)(c_t)^{-1/2}$. Then G_t is the path from a to b in $S_n(\mathcal{A})$. $\square$

Lemma 4. *Suppose that $\mathcal{A}$ is a purely infinite simple C^* -algebra. Then there exists a in $\mathcal{A}$ with $0 \leq a < 1$ such that $a\overline{\mathcal{A}}a$ is non-unital.*

Proof. Choose a non-trivial projection q in $\mathcal{A}$. Since $\mathcal{A}$ is purely infinite simple, there is a projection q_0 in $\mathcal{A}$ such that $q_0 < q$ and $q_0 \sim q$. Using the same method as in the proof of [Cu, Lemma 1.8], we can find a sequence of pairwise orthogonal projections $r_i < q - r_0$ in $q\mathcal{A}q$ such that $r_i \sim q$, $i \geq 0$. Let $\{\beta_n\}_0^\infty$ be a sequence in $(0, 1)$ with $\lim_{n \rightarrow \infty} \beta_n = 0$. Set $a = \sum_{n=0}^\infty \beta_n r_n$. Then $a \in \mathcal{A}$ and $0 \leq a < 1$.

We now claim that $\overline{a\mathcal{A}a}$ is non-unital for such a . If the statement is not true, then there is a projection R in $\overline{a\mathcal{A}a}$ such that R is the unit of $\overline{a\mathcal{A}a}$. Thus we have $Ra^3 = a^3$ and $\|R = axa\| < \frac{1}{2}$ for some $x \in \mathcal{A}$. Consequently, $r_n \leq R$ and

$$\|r_n - \beta_n^2 r_n x r_n\| \leq \|r_n(R - axa)r_n\| < \frac{1}{2}.$$

This implies that $\beta_n^2 \|x\| \geq \beta_n^2 \|r_n x r_n\| > \|r_n\| - \frac{1}{2} = \frac{1}{2}$ which is contrary to the assumption that $\lim_{n \rightarrow \infty} \beta_n = 0$. $\square$

Proof of Theorem 2. If $\mathcal{A}$ is a σ -unital, then by [Zh, Theorem 1.2] $\mathcal{A}$ is stable, i.e., $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of all compact operators on a separable, infinite dimensional Hilbert space over $\mathbb{C}$. In this case, $\text{csr}(\mathcal{A}) \leq 2$ by [Ni, Corollary 2.5].

Consider the general case. For $k \geq 2$, let $\tilde{a} = (a_1 + \lambda_1, \dots, a_k + \lambda_k)^T \in S_k(\mathcal{A}^+)$ where $a_i \in \mathcal{A}$, $\lambda_i \in \mathbb{C}$. Set

$$x = \sum_{i=1}^k (a_i^* a_i + a_i a_i^*).$$

Then for $\varepsilon = \frac{1}{2} \min((6k)^{-1}, (1+12k)^{-1}(\sum_{i=1}^n \|a_i + \lambda_i\|)^{-1})$, there is a projection r in $\mathcal{A}$ such that $\|x(1-r)\| < \varepsilon^2$ by [LZ, Condition (ii)] and [BP, Theorem 2.6]. Since $(1-r)\mathcal{A}(1-r)$ is purely infinite simple ([Zh, Theorem 1.3]), there is by Lemma 4 a c in $(1-r)\mathcal{A}(1-r)$ with $0 \leq c < 1-r$ such that $c\overline{\mathcal{A}}c$ is non-unital. Put $d = r + c$.

Then $0 \leq d \leq 1$ and $\mathcal{B} = \overline{d\mathcal{A}d}$ is non-unital (if Q is the unit of $\mathcal{B}$, $Q - r$ must be the unit of $\overline{c\mathcal{A}c}$) and furthermore, we have

$$(3) \quad x^{1/2}(1-d)^2x^{1/2} \leq x^{1/2}(1-d)x^{1/2} \leq x^{1/2}(1-r)x^{1/2} \leq \varepsilon^2.$$

Now (3) indicates that $\|a_i(1-d)\| < \varepsilon$ and $\|(1-d)a_i\| < \varepsilon$, $1 \leq i \leq k$. Set $b_i = da_id + \lambda_i$, $b = \sum_{i=1}^k b_i^*b_i$. Then $b_i, b \in \mathcal{B}^+$, $\|a_i + \lambda_i - b_i\| < 2\varepsilon$, $1 \leq i \leq k$, and

$$\begin{aligned} \|1-b\| &= \left\| \sum_{i=1}^k (a_i + \lambda_i - b_i)^*(a_i + \lambda_i) + \sum_{i=1}^k b_i^*(a_i + \lambda_i - b_i) \right\| \\ &\leq 3 \sum_{i=1}^k \|a_i + \lambda_i - b_i\| < 6k\varepsilon < \frac{1}{2} \end{aligned}$$

(for $\|a_i + \lambda_i\| \leq 1$, $\|\lambda_i\| \leq 1$ and $\|b_i\| = \|d(a_i + \lambda_i)d + \lambda_i(1-d^2)\| \leq 2$). Thus b is invertible in $\mathcal{B}^+$ with $\|b^{-1}\| < 2$. Since $b > 0$, we get that $\|b^{-1/2}\| < 2^{1/2}$ and

$$\|1 - b^{-1/2}\| = \|b^{-1/2}(1 - b^{1/2})\| < 2^{1/2}\|1 - b\| < 12k\varepsilon.$$

Now set $c_i = b_i b^{-1/2} \in \mathcal{B}^+$. Then $(c_1, \dots, c_k)^T \in S_k(\mathcal{B}^+)$ and

$$(4) \quad \begin{aligned} \|a_i + \lambda_i - c_i\| &= \|a_i + \lambda_i - b_i + b_i(1 - b^{-1/2})\| \\ &< 2\varepsilon + 24k\varepsilon \leq \left(\sum_{i=1}^k \|a_i + \lambda_i\| \right)^{-1}. \end{aligned}$$

Since $\mathcal{B}$ is a non-unital, σ -unital, hereditary C^* -subalgebra of $\mathcal{A}$, it follows from [Zh, Theorem 1.2] that $\mathcal{B}$ is stable and consequently, by means of the above argument, there exists $w \in \mathcal{U}_k^0(\mathcal{B}^+) \subset \mathcal{U}_k^0(\mathcal{A}^+)$ such that $(c_1, \dots, c_k)^T = we_k$. Applying Lemma 3 to (4), we obtain that $\tilde{a}$ is in the component containing e_k . Thus $\text{csr}(\mathcal{A}^+) \leq 2$. Now choose an isometry s in $r\mathcal{A}r$ with $s^*s = r$, $ss^* < r$. Set $T = s + 1 - r$. Then T is an isometry in $\mathcal{A}^+$. Therefore $\text{csr}(\mathcal{A}) = 2$. $\square$

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INSTITUTE OF FUNDAMENTAL EDUCATION, EAST CHINA UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHANGHAI 200237, PEOPLE'S REPUBLIC OF CHINA

Current address: Department of Mathematics, East China University of Science and Technology, Shanghai 200237, People's Republic of China