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INTEGRABILITY OF SUPERHARMONIC FUNCTIONS IN A JOHN DOMAIN

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Abstract. The integrability of positive superharmonic functions on a bounded fat John domain is established. No exterior conditions are assumed. For a general bounded John domain the L^p-integrability is proved with the estimate of p in terms of the John constant.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. By $S^+(D)$ we denote the family of all positive superharmonic functions in D. Armitage [5], [6] proved that $S^+(D) \subset L^p(D)$ for $0 < p < n/(n-1)$, provided D is smooth. This result was extended by Maeda-Suzuki [11] to a Lipschitz domain. They gave an estimate of p in terms of Lipschitz constant. Their estimate has the correct asymptotic behavior: $p \to n/(n-1)$ as the Lipschitz constant tends to 0. As a result they showed that $S^+(D) \subset L^p(D)$ for $0 < p < n/(n-1)$, provided D is a C^1 domain. Masumoto [12], [13] succeeded in obtaining the sharp value of p for planar domains bounded by finitely many Jordan curves. For the higher dimensional case Aikawa [1] gave the sharp value of p for Lipschitz domains with the aid of the coarea formula and the boundary Harnack principle.

On the other hand, Stegenga-Ullrich [16] treated very non-smooth domains, such as John domains and domains satisfying the quasihyperbolic boundary condition [7, 3.6], which are called "Hölder domains" by Smith-Stegenga [15]. Let $\delta_D(x) = \operatorname{dist}(x, \partial D)$ and $x_0 \in D$. We say that D is a John domain with John constant $c_J > 0$ if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that

$$\delta_D(\xi) \geq c_J \ell(\gamma(x,\xi)) \quad \text{for all } \xi \in \gamma, \tag{1.1}$$

where $\gamma(x,\xi)$ is the subarc of γ from x to ξ and $\ell(\gamma(x,\xi))$ is the length of $\gamma(x,\xi)$. A John domain may be visualized as a domain satisfying a twisted cone condition. The quasi-hyperbolic metric $k_D(x_1,x_2)$ is defined by

$$k_D(x_1,x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta_D(x)},$$

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where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. We say that D satisfies a quasi-hyperbolic boundary condition if there are positive constants A_1 and A_2 such that

$$k_D(x, x_0) \le A_1 \log\left(\frac{1}{\delta_D(x)}\right) + A_2 \quad \text{for all } x \in D.$$

Smith-Stegenga [15] called a domain satisfying the quasihyperbolic boundary condition a Hölder domain. It is easy to see that a John domain satisfies the quasi-hyperbolic boundary condition (see [7, Lemma 3.11]). Stegenga-Ullrich [16] proved that $S^+(D) \subset L^p(D)$ with small $p > 0$ for a domain satisfying the quasihyperbolic boundary condition. Lindqvist [10] extends the result to positive supersolutions of certain nonlinear elliptic equations, such as the p-Laplace equation. Gotoh [8] also studies L^p-integrability. Unfortunately, their $p > 0$ is very small and it does not seem that $p \ge 1$ is obtained by their methods.

The main aim of the present paper is to show that $S^+(D) \subset L^1(D)$ for a "fat" John domain.

Theorem 1. *Let D be a bounded John domain with John constant $c_J \ge 1 - 2^{-n-1}$. Then $S^+(D) \subset L^1(D)$.*

The above bound $1 - 2^{-n-1}$ is not sharp. For more specific John domains we obtain the sharp bound. We say that D satisfies the interior cone condition with aperture ψ, $0 < \psi < \pi/2$, if for each point $x \in D$ there is a truncated cone with vertex at x, aperture ψ and a fixed radius lying in D. Obviously, a domain satisfying the interior cone condition with aperture ψ is a John domain with John constant $\sin\psi$.

Theorem 2. *Let D be a bounded domain satisfying the interior cone condition with aperture ψ with $\cos\psi > 1/\sqrt{n}$. Then $S^+(D) \subset L^1(D)$.*

For a "slim" John domain we will show $S^+(D) \subset L^p(D)$ for some $0 < p < 1$ with the estimate of p. This will give a larger p than that in Stegenga-Ullrich [16]. For details see Section 3.

In the previous paper [2], the above theorems are obtained with additional assumption: the capacity density condition (CDC). See [3] for more illustrations. For the 2-dimensional case CDC is equivalent to the uniform perfectness of the boundary; planar domains bounded by finitely many Jordan curves satisfy CDC. Thus all the known results for $S^+(D) \subset L^p(D)$ with $p \ge 1$ required CDC or some other stronger exterior condition. The above Theorems 1 and 2 first establish $S^+(D) \subset L^1(D)$ for a domain satisfying only an interior condition. Recently, Gustafsson, Sakai and Shapiro [9] considered the L^1-integrability in connection with quadrature domains. They showed that if D is a quadrature domain and the Green functions do not decay so fast near the boundary, then $S^+(D) \subset L^1(D)$ ([9, Corollary 5.4]).

2. Proof of the theorems

For an open set U we denote by G_U the Green function for U. Throughout this section D is a bounded John domain or a domain satisfying an interior condition. For simplicity we suppress the subscript D and write G for the Green function for D. Moreover, $x_0 \in D$ is a fixed point and let $g(x) = G(x, x_0)$. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_1, A_2, \dots,$ to specify them. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $A \geq 1$ such that $A^{-1} f_1 \leq f_2 \leq A f_1$. The constant A will be called the constant of comparison.

The proof of the theorems uses the following lower estimate of the Green function. For $0 < c_J < 1$ we let

$$\alpha_J = \log\left[\frac{1-c_J}{(1+c_J)^{n-1}}\right] \Big/ \log(1-c_J^2). \tag{2.1}$$

We observe that $\lim \alpha_J = 1$ as $c_J \to 1$. Let c_n be the solution of the equation $(1+t)^{n+1}(1-t) = 1$ for $0 < t < 1$. Then $\alpha_J = 2$ for $c_J = c_n$ and $1 < \alpha_J < 2$ for $c_n < c_J < 1$. We see that $n/(n+2) < c_n < 1 - 2^{-n-1}$.

Lemma 1 (see [2, Lemma 12])**.** (i) *If D is a John domain with John constant c_J, then $g(x) \geq A\delta_D(x)^{\alpha_J}$.*

(ii) *If D satisfies the interior cone condition with aperture ψ with $\cos\psi > 1/\sqrt{n}$, then there is $1 < \alpha(\psi) < 2$ such that $g(x) \geq A\delta_D(x)^{\alpha(\psi)}$.*

Theorems 1 and 2 readily follow from Lemma 1 and the following.

Theorem 3. *Let D be a John domain and suppose $g(x) \geq A\delta_D(x)^{\alpha}$ for $\alpha > 0$. For $\varepsilon > 0$ let $V = (\min\{g, 1\})^{\varepsilon - 2/\alpha}$. Then*

$$\int_D u(x)V(x)g(x)dx \leq Au(x_0) \quad \textit{for any } u \in S^+(D),$$

where A is independent of $u \in S^+(D)$. Moreover, if $0 < \alpha < 2$, then $S^+(D) \subset L^1(D)$.

We need one of the main results in [2]. Define the Green capacity $\mathrm{Cap}_U(E)$ for $E \subset U$ by

$$\mathrm{Cap}_U(E) = \sup\{\mu(E) : G_U\mu \leq 1 \text{ on } U,\ \mu \text{ is a Borel measure supported on } E\}.$$

By $B(x, r)$ we denote the open ball with center at x and radius r.

Lemma 2 ([2, Theorem 1])**.** *Let $0 < \eta < 1$. Then for an open set U with Green function G_U*

$$\sup_{x\in U}\int_U G_U(x, y)dy \leq Aw_\eta(U)^2,$$

where

$$w_\eta(U) = \inf\left\{\rho > 0 : \frac{\mathrm{Cap}_{B(x,2\rho)}(B(x,\rho) \setminus U)}{\mathrm{Cap}_{B(x,2\rho)}(B(x,\rho))} \geq \eta \quad \textit{for all } x \in U\right\}.$$

The above quantity $w_\eta(U)$ is called the *capacitary width* of U. The definition of John domain readily implies the following.

Lemma 3. *Let D be a John domain. Then $w_\eta(\{x \in D : \delta_D(x) \leq r\}) \leq Ar$.*

The following estimate of the Green potential is called the basic estimate [4, Theorem 3].

Lemma 4. *Let u be a positive continuous superharmonic function on D. For an integer j we put $D_j = \{x \in D : 2^{j-1} < u(x) < 2^{j+2}\}$ and let G_j be the Green function for D_j. If f is a nonnegative measurable function on D, then*

$$\sup_{x\in D} \frac{1}{u(x)} \int_D G(x,y)f(y)dy \le 4 \sum_{j=-\infty}^{\infty} \sup_{x\in D_j} \frac{1}{u(x)} \int_{D_j} G_j(x,y)f(y)dy.$$

Proof of Theorem 3. Apply Lemma 4 to $u = g$ and $f = Vg$ to obtain

$$\sup_{x\in D} \frac{1}{g(x)} \int_D G(x,y)V(y)g(y)dy \le 32 \sum_{j=-\infty}^{\infty} \sup_{x\in D_j} \int_{D_j} G_j(x,y)V(y)dy,$$

where $D_j = \{x \in D : 2^{j-1} < g(x) < 2^{j+2}\}$. Since $g(x) \ge A\delta_D(x)^\alpha$, it follows that $D_j \subset \{x \in D : \delta_D(x) \le A2^{j/\alpha}\}$ and hence from Lemmas 2 and 3 that

$$\sum_{j=-\infty}^{0} \sup_{x\in D_j} \int_{D_j} G_j(x,y)V(y)dy \le A \sum_{j=-\infty}^{0} (2^j)^{\varepsilon-2/\alpha}(2^{j/\alpha})^2 \le A \sum_{j=-\infty}^{0} 2^{\varepsilon j} < \infty.$$

On the other hand, if $j \ge 1$, then $D_j \subset B(x_0, A2^{j/(2-n)})$ if $n \ge 3$ and $D_j \subset B(x_0, \exp(-A2^j))$ if $n = 2$. Hence Lemma 2 implies

$$\begin{aligned}\sum_{j=1}^{\infty} \sup_{x\in D_j} \int_{D_j} G_j(x,y)V(y)dy &= \sum_{j=1}^{\infty} \sup_{x\in D_j} \int_{D_j} G_j(x,y)dy \\ &\le \begin{cases} A\sum_{j=1}^{\infty} 2^{2j/(2-n)} < \infty & \text{if } n \ge 3, \\ A\sum_{j=1}^{\infty} \exp(-A2^j) < \infty & \text{if } n = 2. \end{cases}\end{aligned}$$

Thus

$$\int_D G(x,y)V(y)g(y)dy \le Ag(x) = AG(x,x_0).$$

Integrate the above inequality with respect to $d\mu(x)$ and use Fubini's theorem. Then we have

$$\int_D u(y)V(y)g(y)dy \le Au(x_0)$$

with $u = G\mu$. Every $u \in S^+(D)$ can be approximated from below by a Green potential, so that the monotone convergence theorem proves the first assertion.

Finally, suppose $0 < \alpha < 2$. Let $\varepsilon = -1 + 2/\alpha > 0$ and observe that $Vg \ge 1$ and $\int_D u dx \le Au(x_0)$ for $u \in S^+(D)$. If $u(x_0) < \infty$, then $u \in L^1(D)$ obviously. If $u(x_0) = \infty$, then replace u by its Poisson integral over a small ball with center at x_0. The replaced function belongs to $S^+(D)$ and its value at x_0 is finite, so that it belongs to $L^1(D)$ by the previous observation. This, together with the local integrability of u, proves $u \in L^1(D)$. □

3. L^p-integrability

For a bounded John domain with John constant smaller than that in Theorem 1, we shall obtain L^p-integrability of positive superharmonic functions with $0 < p < 1$. The exponent p will be estimated in terms of John constant. To this end we show the following lemma, which is inspired by [14, Theorem 4].

Lemma 5. *Let D be a bounded John domain with John constant c_J. Then there is a positive constant τ_J depending only on c_J and the dimension n such that*

$$\int_D \delta_D(x)^{-\tau} dx < \infty \quad \text{for } 0 < \tau < \tau_J.$$

Here τ_J can be estimated as $\tau_J \geq \dfrac{\log(1 + (c_J/20)^n)}{\log 2}$.

Proof. Let $\widetilde{D}_j = \{x \in D : 2^{-j-1} \leq \delta_D(x) < 2^{-j}\}$. Then $\bigcup_{j=j_0}^{\infty} \widetilde{D}_j$ is a disjoint decomposition of D with some j_0. Observe that $\sum_{j=j_0}^{\infty} |\widetilde{D}_j| = |D| < \infty$, where $|\widetilde{D}_j|$ denotes the volume of $\widetilde{D}_j$. Without loss of generality we may assume that $j_0 = 0$ and $x_0 \in D_0$. Suppose $x \in \bigcup_{i=j+1}^{\infty} \widetilde{D}_i$ with $j \geq 1$, i.e., $\delta_D(x) < 2^{-j-1}$. By definition there is a rectifiable curve γ connecting x and x_0 with (1.1). We find a point $\xi \in \gamma$ such that $\delta_D(\xi) = 2^{-j}$. By (1.1)

$$2^{-j} = \delta_D(\xi) \geq c_J \ell(\gamma(x, \xi)) \geq c_J |x - \xi|,$$

so that $|x - \xi| \leq c_J^{-1} 2^{-j}$. Hence

$$\bigcup_{i=j+1}^{\infty} \widetilde{D}_i \subset \bigcup_{\delta_D(\xi) = 2^{-j}} C(\xi, c_J^{-1} 2^{-j}),$$

where $C(\xi, c_J^{-1} 2^{-j})$ is the closed ball with center at ξ and radius $c_J^{-1} 2^{-j}$. Suppose for a moment $\delta_D(\xi) = 2^{-j}$. Then, by definition, there is a point $x_\xi \in \partial D$ such that $|x_\xi - \xi| = 2^{-j}$. Let ξ' be the point on the line segment $\overline{x_\xi \xi}$ with $|\xi - \xi'| = 2^{-j-1}$. Then an elementary geometrical observation shows that $\delta_D(\xi') = \frac{1}{2}(2^{-j} + 2^{-j-1})$ and $B(\xi', 2^{-j-2}) \subset \widetilde{D}_j$, so that

$$|\widetilde{D}_j \cap C(\xi, c_J^{-1} 2^{-j})| \geq A_0 (2^{-j-2})^n = \left(\frac{c_J}{20}\right)^n |C(\xi, 5c_J^{-1} 2^{-j})|, \tag{3.1}$$

where A_0 is the volume of a unit ball. By the covering lemma (see e.g. [17, Theorem 1.3.1]) we can find ξ_k such that $\delta_D(\xi_k) = 2^{-j}$, $\{C(\xi_k, c_J^{-1} 2^{-j})\}_k$ is disjoint and

$$\bigcup_{i=j+1}^{\infty} \widetilde{D}_i \subset \bigcup_k C(\xi_k, 5c_J^{-1} 2^{-j}).$$

In view of (3.1) we have

$$\sum_{i=j+1}^{\infty} |\widetilde{D}_i| \leq \sum_k |C(\xi_k, 5c_J^{-1} 2^{-j})| \leq \left(\frac{20}{c_J}\right)^n \sum_k |\widetilde{D}_j \cap C(\xi_k, c_J^{-1} 2^{-j})| \leq \left(\frac{20}{c_J}\right)^n |\widetilde{D}_j|.$$

Multiply the above inequalities by r^j and take the summation for $j = 1, \ldots, N-1$, where $r > 1$ is a constant to be determined. Then

$$\begin{aligned}
\left(\frac{20}{c_J}\right)^n \sum_{j=1}^{N-1} r^j |\widetilde{D}_j| &\geq \sum_{j=1}^{N-1} \sum_{i=j+1}^{\infty} r^j |\widetilde{D}_i| = \sum_{i=2}^{\infty} \sum_{j=1}^{\min\{N,i\}-1} r^j |\widetilde{D}_i| \\
&\geq \sum_{i=2}^{N-1} \sum_{j=1}^{i-1} r^j |\widetilde{D}_i| = \sum_{i=2}^{N-1} \frac{r^i - r}{r-1} |\widetilde{D}_i| \\
&= \frac{1}{r-1} \sum_{i=1}^{N-1} r^i |\widetilde{D}_i| - \frac{r}{r-1} \sum_{i=1}^{N-1} |\widetilde{D}_i|,
\end{aligned}$$

so that

$$\frac{r}{r-1}\sum_{i=1}^{N-1}|\widetilde{D}_i| \ge \left(\frac{1}{r-1} - \left(\frac{20}{c_J}\right)^n\right)\sum_{i=1}^{N-1} r^i|\widetilde{D}_i|.$$

Letting $N \to \infty$, we obtain

$$\infty > \frac{r}{r-1}|D| = \frac{r}{r-1}\sum_{i=1}^{\infty}|\widetilde{D}_i| \ge \left(\frac{1}{r-1} - \left(\frac{20}{c_J}\right)^n\right)\sum_{i=1}^{\infty} r^i|\widetilde{D}_i|.$$

Let $1 < r < 1 + (c_J/20)^n$. Then

$$\left(\frac{1}{r-1} - \left(\frac{20}{c_J}\right)^n\right) > 0$$

and the above inequality implies

$$\sum_{i=1}^{\infty} r^i|\widetilde{D}_i| < \infty.$$

We observe that

$$r^i \approx \delta_D(x)^{-\log r/\log 2} \qquad \text{for } x \in \widetilde{D}_i,$$

whence

$$\int_D \delta_D(x)^{-\log r/\log 2} dx < \infty.$$

This proves the lemma. ☐

Theorem 4. *Let D be a bounded John domain with John constant c_J. Suppose $g(x) \ge A\delta_D(x)^\alpha$ for $x \in D$ with $\alpha \ge 2$ and*

$$\int_D \delta_D(x)^{-\tau} dx < \infty \tag{3.2}$$

with $\tau > 0$. Then $S^+(D) \subset L^p(D)$ for $0 < p < \tau/(\alpha - 2 + \tau)$.

Remark. Let α_J be as in (2.1) and let τ_J be as in Lemma 5. If $\alpha_J \ge 2$, then Lemmas 1 and 5 show that $S^+(D) \subset L^p(D)$ with $0 < p < p_J = \tau_J/(\alpha_J - 2 + \tau_J)$. Observe that $p_J \approx c_J^{n+1}$ as $c_J \to 0$; $p_J \to 1$ as $c_J \to c_n$.

Proof. Let $0 < p < \tau/(\alpha - 2 + \tau)$. Put

$$\varepsilon = \frac{1}{\alpha}\left(\frac{(1-p)\tau}{p} - \alpha + 2\right).$$

Then $\varepsilon > 0$ and $\alpha(\varepsilon - 2/\alpha + 1)p/(1-p) = \tau$. Let $D' = \{x \in D : g(x) \le 1\}$. Take $u \in S^+(D)$. Then Hölder's inequality and Theorem 3 yield

$$\begin{aligned}\int_{D'} u^p dx &\le \left(\int_{D'} u g^{\varepsilon - 2/\alpha + 1} dx\right)^p \left(\int_{D'} g^{-(\varepsilon - 2/\alpha + 1)p/(1-p)} dx\right)^{1-p} \\ &\le A u(x_0)^p \left(\int_{D'} \delta_D^{-\tau} dx\right)^{1-p} \le A u(x_0)^p,\end{aligned}$$

where (3.2) is used in the last inequality. By the same reasoning as in the proof of Theorem 3, we have $\int_{D'} u^p dx < \infty$. This, together with the local integrability of a superharmonic function, proves the theorem. ☐

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