INVOLUTIONS WITH $W(F) = 1$

ZHI LÜ

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ABSTRACT. Let (T, M^n) be a smooth involution on a closed n -dimensional manifold such that all Stiefel-Whitney classes of the tangent bundle to each component of the fixed point set F of (T, M^n) vanish in positive dimension. In this paper, we estimate the least possible lower bound of $\dim F$ if (T, M^n) does not bound.

1. INTRODUCTION

Let (T, M^n) be a smooth involution on a closed n -dimensional manifold. It is very well known that the fixed point set F of (T, M^n) is the finite disjoint union of closed submanifolds of M^n . Let F^{n-k} denote the union of those components of the fixed point set F having dimension $n - k$ and let ν^k denote the normal bundle of F^{n-k} in M^n . By $\dim F$ we mean the dimension of the highest dimensional non-empty component of the fixed point set F .

In [1] Conner and Floyd examined involutions (T, M^n) having the property $W(F^{n-k}) = 1$ for all k (here W denotes the total Stiefel-Whitney class), and proved that if all the components of F have same dimension, then (T, M^n) bounds equivariantly. At the same time, they also gave an example and showed that the result fails if the dimensions of the components of F are allowed to vary. Later on, Kosniowski and Stong in [4] discussed the problem again, and obtained that if (T, M^n) is an involution with $W(F^{n-k}) = 1$ for all k , then the equivariant bordism class of (T, M^n) is determined by the bordism class of M^n . In this paper, we prove that if $n > 2\dim F$, then (T, M^n) with $W(F^{n-k}) = 1$ for all k bounds equivariantly, and see from an example that $\frac{n}{2}$ is exactly the least possible lower bound of $\dim F$ if (T, M^n) with $W(F^{n-k}) = 1$ for all k does not bound. Our main result is stated as follows.

Theorem A. *Suppose (T, M^n) is a smooth involution such that $W(F^{n-k}) = 1$ for all k . If $n > 2\dim F$, then (T, M^n) bounds equivariantly.*

Remark. It should be pointed out that if we change the condition $n > 2\dim F$ into $n \leq 2\dim F$ in Theorem A, then the result fails. For example, an involution

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$(T, RP(2))$ defined by

$$T : [x_0, x_1, x_2] \longrightarrow [-x_0, x_1, x_2]$$

fixing $F = * \sqcup RP(1)$ ($*$ denotes a point) does not bound, but $2 \dim F = \dim RP(2)$.

This means that if (T, M^n) with $W(F^{n-k}) = 1$ for all k does not bound, then the least possible lower bound of $\dim F$ is exactly $n/2$, and thus the result of Theorem A is best possible.

In fact, the condition that $W(F^{n-k}) = 1$ for all k in Theorem A can also be changed into more general case. We state the result as follows.

Theorem B. *Suppose that (T, M^n) is a smooth involution such that $W(F^{n-k}) = 1$ for all $k \leq \frac{3}{5}n$, but not necessarily for $k > \frac{3}{5}n$. If $n > 2 \dim F$, then (T, M^n) bounds equivariantly.*

Throughout this paper, the coefficient group is $\mathbb{Z}_2$. Let $[N]$ denote the fundamental homology class of closed smooth manifold N , and $\{N\}$ the unoriented bordism class of N . $\sigma_i(x_1, \dots, x_n)$ denotes the i -th elementary symmetric function $\sum x_1 \cdots x_i$, and $S_\omega(x_1, \dots, x_n) = \sum x_1^{i_1} \cdots x_r^{i_r}$ denotes the usual smallest symmetric polynomial containing the given monomial, where $\omega = (i_1, \dots, i_r)$ is a partition of $|\omega| = i_1 + \cdots + i_r$.

2. PRELIMINARIES

First, we introduce an important formula given by Kosniowski and Stong [4] for the calculation of the Stiefel-Whitney numbers of M^n in terms of the fixed point data $\nu \rightarrow F = \bigsqcup_k \nu^k \rightarrow F^{n-k}$. That is the following

Proposition 1 (Kosniowski and Stong). *If $f(x_1, \dots, x_n)$ is any symmetric polynomial over $\mathbb{Z}_2$ in n variables of degree at most n , then*

$$f(x_1, \dots, x_n)[M^n] = \sum_k \frac{f(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k})}{\prod_{i=1}^k (1 + y_i)} [F^{n-k}]$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x)$, $\sigma_i(y)$, and $\sigma_i(z)$ by the Stiefel-Whitney classes $W_i(M^n)$, $W_i(\nu^k)$, and $W_i(F^{n-k})$ respectively, and taking the value of the resulting cohomology class on the fundamental homology class of M^n or F^{n-k} .

Notice that in this paper our interest mainly focuses on discussing the involution (M^n, T) having the property $W(F^{n-k}) = 1$ for all k . It is easy to see that if $W(F^{n-k}) = 1$ for all k , then we may choose $z_i = 0$ in Proposition 1 from the splitting principle. Thus we have (here we also give a simple algebraic proof)

Corollary 1. *Suppose (T, M^n) is an involution with $W(F^{n-k}) = 1$ for all k . Then*

$$f(x_1, \dots, x_n)[M^n] = \sum_k \frac{f(1 + y_1, \dots, 1 + y_k, \overbrace{0, \dots, 0}^{n-k})}{(1 + y_1) \cdots (1 + y_k)} [F^{n-k}]$$

for all the symmetric polynomials $f(x_1, \dots, x_n)$ in n variable of degree at most n .

Proof. From the fact [4, p.317, Lemma], that for each k ,

$$\begin{aligned} & \sigma_i(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k}) \\ &= \sum_{p+q \leq i} \binom{k-p}{i-p-q} \sigma_p(y_1, \dots, y_k) \sigma_q(z_1, \dots, z_{n-k}), \end{aligned}$$

it is easy to see that if $W(F^{n-k}) = 1$, i.e., $\sigma_q(z_1, \dots, z_{n-k}) = 0$ for all $q \geq 1$, then

$$\begin{aligned} (2.1) \quad & \sigma_i(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k}) \\ &= \sum_{p \leq i} \binom{k-p}{i-p} \sigma_p(y_1, \dots, y_k) \\ &= \sigma_i(1 + y_1, \dots, 1 + y_k, \underbrace{0, \dots, 0}_{n-k}). \end{aligned}$$

On the other hand, it is very well known that for the symmetric polynomial $f(x_1, \dots, x_n)$, there must exist a polynomial $\phi(t_1, \dots, t_n)$ in n variables such that

$$\begin{aligned} (2.2) \quad & f(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k}) \\ &= \phi(\sigma_1(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k}), \dots, \\ & \quad \sigma_n(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k})). \end{aligned}$$

Combining (2.1) and (2.2), it at once follows that if $\sigma_q(z_1, \dots, z_{n-k}) = 0$ for all $q \geq 1$, then

$$f(1 + y_1, \dots, 1 + y_k, z_1, \dots, z_{n-k}) = f(1 + y_1, \dots, 1 + y_k, \underbrace{0, \dots, 0}_{n-k}),$$

and the corollary thus holds by Proposition 1. This completes the proof. $\square$

Next, in order to prove our main theorem, we also need the following result due to Kosniowski and Stong (see [4, p. 326, Proposition]), and its proof is included here for local completeness.

Proposition 2. *Suppose that (T, M^n) is a smooth involution having the property $W(F^{n-k}) = 1$ for all k . Then (T, M^n) bounds equivariantly if and only if M^n bounds.*

Proof. First, it follows immediately that M^n bounds if (T, M^n) bounds equivariantly.

Conversely, for each k , since $W(F^{n-k}) = 1$, the unoriented bordism class of the normal bundle $\nu^k \rightarrow F^{n-k}$ is determined by the numbers

$$S_\omega(\nu^k)[F^{n-k}]$$

where $\omega = (i_1, \dots, i_r)$ is any a partition of $n - k$. In particular, if $r > k$, then $S_\omega(\nu^k)[F^{n-k}] = 0$ for $\nu^k \rightarrow F^{n-k}$ is a k -plane bundle. Let $\omega = (i_1, \dots, i_r)$ with $r \leq k$ be any partition of $n - k$, and take

$$f_\omega(x_1, \dots, x_n) = \sum (1 + x_1)^{i_1} x_1 \cdots (1 + x_r)^{i_r} x_r x_{r+1} \cdots x_k.$$

Then

$$\frac{f_\omega(1 + y_1, \dots, 1 + y_k, \underbrace{0, \dots, 0}_{n-k})}{(1 + y_1) \cdots (1 + y_k)} = \sum y_1^{i_1} \cdots y_r^{i_r} = S_\omega(y_1, \dots, y_k),$$

but for $h < k$,

$$f_\omega(1 + y_1, \dots, 1 + y_h, \underbrace{0, \dots, 0}_{n-h}) = 0$$

and for $h > k$,

$$(2.3) \quad \min\{\deg f_\omega(1 + y_1, \dots, 1 + y_h, \underbrace{0, \dots, 0}_{n-h})\} \geq n - k > n - h$$

(here $\min\{\deg f_\omega(1 + y_1, \dots, 1 + y_h, \underbrace{0, \dots, 0}_{n-h})\}$ denotes the degree of the least degree term in $f_\omega(1 + y_1, \dots, 1 + y_h, \underbrace{0, \dots, 0}_{n-h})$); hence by Corollary 1, we have

$$(2.4) \quad S_\omega(\nu^k)[F^{n-k}] = f_\omega(x)[M^n] = S_{(i_1+1, \dots, i_r+1, \underbrace{1, \dots, 1}_{k-r})}\{M^n\}.$$

Note that, for each component F^{n-h} with $h > k$,

$$\frac{f_\omega(1 + y_1, \dots, 1 + y_h, \underbrace{0, \dots, 0}_{n-h})}{(1 + y_1) \cdots (1 + y_h)}[F^{n-h}] = 0$$

by (2.3). From (2.4), it follows at once that if M^n bounds, then $\nu^k \rightarrow F^{n-k}$ bounds for each k . Again by [3, Theorem (25.2)], we obtain that (T, M^n) bounds equivariantly. This completes the proof of the proposition. $\square$

From the proof of Proposition 2, we easily see

Corollary 2. *Let (T, M^n) be an involution with $W(F^{n-k}) = 1$ for all k . Then M^n is decomposable if and only if the normal bundle of the $(n-1)$ -dimensional fixed point set component bounds.*

3. PROOFS OF THEOREM A AND THEOREM B

As in defined in [5], we say that $\omega = (i_1, \dots, i_s)$ is a non-dyadic partition of the positive integer l if $|\omega| = i_1 + \dots + i_s = l$ and the none of $i_\alpha \in \omega$ is of the form $2^p - 1$.

Now we begin with the proof of Theorem A.

Proof of Theorem A. Let $\omega = (2i_1+1, \dots, 2i_s+1, 2j_1, \dots, 2j_t)$ be any a non-dyadic partition of n . Take

$$\begin{aligned} & f_\omega(x_1, \dots, x_n) \\ &= \sum (1 + x_1)^{i_1+1} x_1^{i_1} \cdots (1 + x_s)^{i_s+1} x_s^{i_s} (1 + x_{s+1})^{j_1} x_{s+1}^{j_1} \cdots (1 + x_{s+t})^{j_t} x_{s+t}^{j_t}, \end{aligned}$$

since $W(F^{n-k}) = 1$ for all k ; then

$$(3.1) \quad S_\omega\{M^n\} = \sum_k \frac{f_\omega(1 + y_1, \dots, 1 + y_k, \underbrace{0, \dots, 0}_{n-k})}{(1 + y_1) \cdots (1 + y_k)}[F^{n-k}]$$

by Corollary 1. When $k < s + t$, it is obvious that

$$f_\omega(1 + y_1, \dots, 1 + y_k, \underbrace{0, \dots, 0}_{n-k}) = 0,$$

and thus

$$(3.2) \quad \sum_{k < s+t} \frac{f_\omega(1+y_1, \dots, 1+y_k, \overbrace{0, \dots, 0}^{n-k})}{(1+y_1) \cdots (1+y_k)} [F^{n-k}] = 0.$$

When $k \geq s+t$, we have

$$\begin{aligned} & f_\omega(1+y_1, \dots, 1+y_k, \underbrace{0, \dots, 0}_{n-k}) \\ &= \sum y_1^{i_1+1} (1+y_1)^{i_1} \cdots y_s^{i_s+1} (1+y_s)^{i_s} y_{s+1}^{j_1} (1+y_{s+1})^{j_1} \cdots y_{s+t}^{j_t} (1+y_{s+t})^{j_t}, \end{aligned}$$

and thus

$$\begin{aligned} & \min\{\deg f_\omega(1+y_1, \dots, 1+y_k, \underbrace{0, \dots, 0}_{n-k})\} \\ & \geq (i_1+1) + \cdots + (i_s+1) + j_1 + \cdots + j_t \\ & = \frac{n+s}{2} \\ & \geq \frac{n}{2}. \end{aligned}$$

Since $n > 2\dim F$, i.e., $2k > n$ for all k , we have

$$(3.3) \quad \sum_{k \geq s+t} \frac{f_\omega(1+y_1, \dots, 1+y_k, \overbrace{0, \dots, 0}^{n-k})}{(1+y_1) \cdots (1+y_k)} [F^{n-k}] = 0.$$

Hence it follows from (3.1), (3.2) and (3.3) that $S_\omega\{M^n\} = 0$, i.e., M^n bounds. Again by Proposition 2, we obtain that (T, M^n) bounds equivariantly. This completes the proof of Theorem A. $\square$

The proof of Theorem B is mainly based on the following lemma and the Boardman 5/2-Theorem (see [6]). It should be pointed out that Kosniowski and Stong [4] obtained the stronger Boardman 5/2-Theorem, which here we will use.

Lemma 3.1. *Suppose (T, M^n) is a smooth involution such that $W(F^{n-k}) = 1$ for all $k \leq \frac{3}{5}n$, but not necessarily for $k > \frac{3}{5}n$. If $n > 2\dim F$, then M^n bounds.*

Proof. Let $\omega = (2i_1+1, \dots, 2i_s+1, 2j_1, \dots, 2j_t)$ be any a non-dyadic partition of n . Choose

$$\begin{aligned} & f_\omega(x_1, \dots, x_n) \\ &= \sum (1+x_1)^{i_1+1} x_1^{i_1} \cdots (1+x_s)^{i_s+1} x_s^{i_s} (1+x_{s+1})^{j_1} x_{s+1}^{j_1} \cdots (1+x_{s+t})^{j_t} x_{s+t}^{j_t}; \end{aligned}$$

by Proposition 1 and the proof of Corollary 1, we have

$$\begin{aligned} (3.4) \quad & S_\omega\{M^n\} = f_\omega(x_1, \dots, x_n)[M^n] \\ &= \sum_{k \leq \frac{3}{5}n} \frac{f_\omega(1+y_1, \dots, 1+y_k, \overbrace{0, \dots, 0}^{n-k})}{(1+y_1) \cdots (1+y_k)} [F^{n-k}] \\ & \quad + \sum_{k > \frac{3}{5}n} \frac{f_\omega(1+y_1, \dots, 1+y_k, z_1, \dots, z_{n-k})}{(1+y_1) \cdots (1+y_k)} [F^{n-k}]. \end{aligned}$$

As in the proof of Theorem A, first we have

$$(3.5) \quad \sum_{k \leq \frac{3}{5}n} \frac{f_\omega(1+y_1, \dots, 1+y_k, \overbrace{0, \dots, 0}^{n-k})}{(1+y_1) \cdots (1+y_k)} [F^{n-k}] = 0.$$

Notice that $k > n/2$ for all k since $n > 2\dim F$. Next, since ω is a non-dyadic partition of n , we have $i_\alpha \geq 2$ ($\alpha = 1, \dots, s$), and thus

$$n = \sum_{\alpha=1}^s (2i_\alpha + 1) + \sum_{\beta=1}^t 2j_\beta \geq \sum_{\alpha=1}^s (2i_\alpha + 1) \geq 5s,$$

i.e., $s \leq \frac{1}{5}n$. Furthermore, for $k > \frac{3}{5}n$,

$$\begin{aligned} \min\{\deg f_\omega(1+y_1, \dots, 1+y_k, z_1, \dots, z_{n-k})\} &\geq i_1 + \dots + i_s + j_1 + \dots + j_t \\ &= \frac{n-s}{2} \\ &\geq \frac{2}{5}n \\ &> n-k. \end{aligned}$$

Hence

$$(3.6) \quad \sum_{k > \frac{3}{5}n} \frac{f_\omega(1+y_1, \dots, 1+y_k, z_1, \dots, z_{n-k})}{(1+y_1) \cdots (1+y_k)} [F^{n-k}] = 0.$$

From (3.4), (3.5) and (3.6), it follows at once that $S_\omega\{M^n\} = 0$, i.e., M^n bounds. This completes the proof of the lemma. $\square$

Being given an involution (T, M^n) , let $\Gamma^1(M^n)$ denote an $(n+1)$ -dimensional manifold formed from the product $S^1 \times M^n$ of the circle with M^n by identifying (z, x) with $(-z, Tx)$, with the involution T_1 on $\Gamma^1(M^n)$ induced by $(z, x) \rightarrow (\bar{z}, x)$, as defined in [3]. Let $(T_0, \Gamma^0(M^n)) = (T, M^n)$, and let $(T_r, \Gamma^r(M^n))$ be the r -th iterate of (T, M^n) . Thus a sequence of involutions $(T_r, \Gamma^r(M^n))$ is constructed. In particular, we also know from [3] that the normal bundle to the fixed point set of $(T_r, \Gamma^r(M^n))$ is

$$\bigsqcup_k (\nu^k \oplus r\mathbb{R} \rightarrow F^{n-k}) \sqcup \left(\bigsqcup_{j=0}^{r-1} (r-j)\mathbb{R} \rightarrow \Gamma^j(M^n) \right)$$

where $\nu \rightarrow F = \bigsqcup_k (\nu^k \rightarrow F^{n-k})$ is the original normal bundle to F in M^n .

Let $MO_m(BO(p))$ denote the unoriented bordism group of real p -dimensional plane bundles over closed smooth m -dimensional manifolds.

Proof of Theorem B. Consider the involution $(T_1, \Gamma^1(M^n))$ with the fixed point data $\bigsqcup_k (\nu^k \oplus \mathbb{R} \rightarrow F^{n-k}) \sqcup (\mathbb{R} \rightarrow M^n)$. Since M^n bounds (by Lemma 3.1), we have that the fixed data of $(T_1, \Gamma^1(M^n))$ is bordant to $\bigsqcup_k (\nu^k \oplus \mathbb{R} \rightarrow F^{n-k})$. Again by Lemma 3.1, we obtain that $\Gamma^1(M^n)$ bounds. Continuing this process, it is easy to see that, for each $r \geq 1$, the fixed data of $(T_r, \Gamma^r(M^n))$ is bordant to $\bigsqcup_k (\nu^k \oplus r\mathbb{R} \rightarrow F^{n-k})$, and $\Gamma^r(M^n)$ bounds by Lemma 3.1. The stronger Boardman 5/2-Theorem makes $(T_r, \Gamma^r(M^n))$ bound for large r (e.g., choose $r \geq [\frac{1}{2} \dim F] +$

1) and thus for each k , the bordism class of $\nu^k \oplus r\mathbb{R} \rightarrow F^{n-k}$ is equal to 0 in $MO_{n-k}(BO(k+r))$. Since

$$\oplus r\mathbb{R}: MO_{n-k}(BO(k)) \rightarrow MO_{n-k}(BO(k+r))$$

is a monomorphism, this implies that for each k , $\nu^k \rightarrow F^{n-k}$ bounds, i.e., $\nu \rightarrow F = \bigsqcup_k (\nu^k \rightarrow F^{n-k})$ bounds. Thus (T, M^n) bounds equivariantly. This completes the proof of Theorem B. $\square$

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DEPARTMENT OF APPLIED MATHEMATICS, TSINGHUA UNIVERSITY, BEIJING, 100084, PEOPLE'S REPUBLIC OF CHINA

Current address: Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mail address: zlu@ms326kaz.ms.u-tokyo.ac.jp