ON COMPACT SUBSETS IN COECHELON SPACES OF INFINITE ORDER

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ABSTRACT. For coechelon spaces $k_\infty(v)$ of infinite order it is proved that every compact subset of $k_\infty(v)$ is contained in a closed absolutely convex hull of some null sequence if and only if the matrix v is regularly decreasing.

In connection with the study of some interesting problems on Montel maps which are closely connected to the classical Grothendieck question on completeness of regular LB-spaces [PB, Problem 13.8.6] and the problem of bornologicity of $C(K, E)$ with E an LB-space [S, Chapter IV], Dierolf and Domański [DD2, Example 3.1] gave an example of a coechelon LB-Montel space of infinite order which has compact sets not contained in closed absolutely convex hulls of any null sequence. Consequently, the well-known characterization of compact sets in Fréchet spaces [J, Theorem 9.4.2] turns out to be not generally true in the LB setting.

The purpose of this note is to show that for coechelon spaces $k_\infty(v)$ of order ∞ the condition v *regularly decreasing* [BMS, Definition 3.1] is necessary and sufficient for every compact set to be contained in a closed absolutely convex hull of some null sequence.

For more information on Montel maps and related questions the reader is referred to [DD1], [DD2], [DD3] and [D].

In what follows we recall some notation.

Let E be a Fréchet space with a fundamental system of seminorms $(\|\cdot\|_n)_n$; then the inductive dual E'_i is defined to be $\text{ind}_n E'_n$, where the E_n are the completions of the normed spaces $(E/\ker \|\cdot\|_n, \|\cdot\|_n)$. It is known that algebraically $E'_i = E'$, the inclusion map $E'_i \hookrightarrow E'_\beta$ is continuous and E'_i is the bornological space associated with E'_β , i.e., $E'_i = (E', \beta(E', E''))$ [J, Theorem 13.4.2] (E' and E'_β denote the topological dual and the strong dual of E , resp.).

We also recall that an LB-space $E = \text{ind}_n E_n$ is called *boundedly retractive* (respectively, *compactly regular*) if, and only if, for each bounded (respectively, compact) subset B of E there is $n \in \mathbf{N}$ such that $B \subset E_n$ and E_n and E induce the same topology on B (see [PB, Definitions 8.5.32–(ii), –(iii)]). It is clear that a boundedly retractive LB-space is compactly regular. On the other hand, in [N] it is proved that these conditions are also equivalent.

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Let $a = (a_n(i))_n$ and $v = (v_n(i))_n$ be matrices on an index set I , with $v_n(i) \geq v_{n+1}(i) > 0$ and $a_n(i) = \frac{1}{v_n(i)}$ for all $i \in I$ and $n \in \mathbf{N}$. The Köthe echelon space of order 1 (associated to a) and the Köthe coechelon space of order ∞ (associated to v) are defined by

$$\lambda_1(a) := \left\{ (x_i)_{i \in I} : \|(x_i)_i\|_n^1 := \sum_{i \in I} a_n(i) |x_i| < +\infty \text{ for all } n \in \mathbf{N} \right\}$$

and

$$k_\infty(v) := \left\{ (x_i)_{i \in I} : \|(x_i)_i\|_n^\infty := \sup_{i \in I} v_n(i) |x_i| < +\infty \text{ for some } n \in \mathbf{N} \right\},$$

respectively. Clearly, $k_\infty(v) = \text{ind}_n l_\infty(v_n)$ ($l_\infty(w) := \{(x_i)_{i \in I} : \sup_{i \in I} w(i) |x_i| < +\infty\}$ for any positive function w on I); also, $k_\infty(v) = (\lambda_1(a))'_i$ (see [BMS, Corollary 2.8]) and hence it is a regular and complete LB-space.

If $\bar{V}$ is the maximal Nachbin family of positive functions on I associated to v which is defined by

$$\bar{V} := \left\{ \bar{v} : I \rightarrow [0, +\infty[: \sup_{i \in I} \frac{\bar{v}(i)}{v_n(i)} < +\infty \text{ for all } n \in \mathbf{N} \right\},$$

then, by [BMS, Theorem 2.7], $(\lambda_1(a))'_\beta$ coincides algebraically and topologically with

$$K_\infty(\bar{V}) := \text{proj}_{\bar{v} \in \bar{V}} l_\infty(\bar{v}).$$

So, algebraically $k_\infty(v) = K_\infty(\bar{V})$, the inclusion map $k_\infty(v) \hookrightarrow K_\infty(\bar{V})$ is continuous and they have the same bounded sets. In particular, since every compact set in $K_\infty(\bar{V}) = (\lambda_1(a))'_\beta$ is separable and metrizable by [CO, Corollary 1.2], the topology of $K_\infty(\bar{V})$ coincides with the one of $k_\infty(v)$ on these sets (cf. [PB, Proposition 8.3.12]); hence, every compact set in $K_\infty(\bar{V})$ is also a compact set in $k_\infty(v)$.

Finally, we recall that the matrix v is called *regularly decreasing* [BMS, Definition 3.1] if, and only if, given $n \in \mathbf{N}$ there is $m \geq n$ such that

$$\forall I_0 \subset I \text{ with } \inf_{i \in I_0} \frac{v_m(i)}{v_n(i)} > 0, \text{ we have also } \inf_{i \in I_0} \frac{v_k(i)}{v_n(i)} > 0 \text{ for all } k \geq m.$$

By [BMS, Theorem 3.4] v is regularly decreasing if, and only if, $\lambda_1(a)$ is quasi-normable if, and only if, $k_\infty(v)$ is a boundedly retractive LB-space and hence if, and only if, it is a compactly regular LB-space.

For all undefined notation we refer to [PB] and [BMS].

We are now able to state and prove our result.

Theorem. *Let $k_\infty(v)$ be a coechelon space of order ∞ . Then the following conditions are equivalent:*

- (i) *v is regularly decreasing;*
- (ii) *every compact set in $k_\infty(v)$ is contained in a closed absolutely convex hull of some null sequence.*

Proof. We establish that (i) implies (ii). Let K be a compact set in $k_\infty(v)$. Since v is regularly decreasing, $k_\infty(v)$ is boundedly retractive. Thus K is contained in $l_\infty(v_n)$ for some $n \in \mathbf{N}$ and $l_\infty(v_n)$ and $k_\infty(v)$ induce the same topology on K ; hence, it is a compact subset of $l_\infty(v_n)$. Since $l_\infty(v_n)$ is a Banach space, by [J, Theorem 9.4.2]

K is contained in a closed absolutely convex hull of some null sequence in $l_\infty(v_n)$ and the result follows.

Next, we show that (ii) implies (i). Suppose that v is not regularly decreasing. Then:

$\exists n_o \in \mathbf{N} \forall m > n_o \exists J \subset I$ and $\exists k > m$ such that

$$\inf_{i \in J} \frac{v_m(i)}{v_{n_o}(i)} > 0 \text{ and } \inf_{i \in J} \frac{v_k(i)}{v_{n_o}(i)} = 0.$$

Proceeding by induction as in [V, p. 232], we can find a sectional subspace of $k_\infty(v)$ which is isomorphic to $k_\infty(\tilde{v})$, with $\tilde{v} = (\tilde{v}_n(i, j))_n$ a matrix on $\mathbf{N} \times \mathbf{N}$ satisfying:

$$(1) \quad \tilde{v}_1(i, j) = 1, \quad \forall i, j \in \mathbf{N};$$

$$(2) \quad \lim_{j \rightarrow \infty} \tilde{v}_{n+1}(n, j) = 0, \quad \forall n \in \mathbf{N};$$

$$(3) \quad \forall n \in \mathbf{N} \forall i \geq n \exists \alpha_{n,i} > 0 \inf_{j \in \mathbf{N}} \tilde{v}_n(i, j) > \alpha_{n,i}.$$

Since $k_\infty(\tilde{v})$ is complemented in $k_\infty(v)$ as it is easy to see, we can restrict our attention to $k_\infty(\tilde{v})$ ($k_\infty(v)$ satisfies (ii) $\Rightarrow$ every complemented subspace of $k_\infty(v)$ satisfies (ii)). For technical reasons we modify the weights by defining $v_n^*(i, j) := i^{-2} \tilde{v}_n(i, j)$. Clearly, $k_\infty(\tilde{v})$ is isomorphic to $k_\infty(v^*)$, with $v^* = (v_n^*(i, j))_n$. Moreover, v^* satisfies the following conditions:

$$(a) \quad v_1^*(i, j) = i^{-2}, \quad \forall i, j \in \mathbf{N};$$

$$(b) \quad \lim_{j \rightarrow \infty} v_{n+1}^*(n, j) = 0, \quad \forall n \in \mathbf{N};$$

$$(c) \quad \forall n \in \mathbf{N} \forall i \geq n \exists \alpha_{n,i} > 0 \inf_j v_n^*(i, j) > i^{-2} \alpha_{n,i}.$$

We will show that the set $C := \{(x_{ij})_{i,j \in \mathbf{N}} : \sup_{i,j \in \mathbf{N}} |x_{ij}| i^{-1} \leq 1\}$ is a compact set in $k_\infty(v^*)$ which is not contained in the closed absolutely convex hull of any null sequence (clearly, $C \subset B_1$, where B_1 denotes the closed unit ball of $l_\infty(v_1^*)$).

Let $\bar{V}$ be the maximal Nachbin family of positive functions on $\mathbf{N} \times \mathbf{N}$ associated to v^* . Since every compact set in $K_\infty(\bar{V})$ is also a compact set in $k_\infty(v^*)$, we have only to show that C is compact in $K_\infty(\bar{V})$, i.e., C is compact in $l_\infty(\bar{v})$ for every $\bar{v} \in \bar{V}$.

Now, let $\bar{v} \in \bar{V}$. By (a) and (b), it follows that:

$$(4) \quad \lim_{i \rightarrow \infty} i \sup_{j \in \mathbf{N}} \bar{v}(i, j) = 0 \text{ and } i \bar{v}(i, j) \leq c < +\infty, \quad \forall i, j \in \mathbf{N};$$

$$(5) \quad \lim_{j \rightarrow \infty} n \bar{v}(n, j) = 0, \quad \forall n \in \mathbf{N}.$$

Next, let $(x^r)_r$ be a sequence of C . Then, for each i and $j \in \mathbf{N}$, $\sup_r i^{-1} |x_{ij}^r| \leq 1$. This implies that (first by passing to subsequences and then by diagonalization) there are a sequence $x = (x_{ij})_{i,j \in \mathbf{N}}$ in $\mathbf{R}$ and a subsequence $(x^{r'})_{r'}$ of $(x^r)_r$ so that, for each i and $j \in \mathbf{N}$,

$$(6) \quad \lim_{r' \rightarrow \infty} i^{-1} |x_{ij}^{r'} - x_{ij}| = 0;$$

hence, for each i and $j \in \mathbf{N}$,

$$(7) \quad i^{-1} |x_{ij}| \leq 1 \quad \text{and} \quad i^{-1} |x_{ij}^{r'} - x_{ij}| \leq 2.$$

Clearly, $x \in C$. We check that $(x^{r'})_{r'}$ converges to x in $l_\infty(\bar{v})$. Given any $\varepsilon > 0$, by (4), there is $i_0 \in \mathbf{N}$ such that

$$\sup_{j \in \mathbf{N}} i\bar{v}(i, j) < \varepsilon/6 \quad \text{for all } i \geq i_0.$$

It follows, by (7), that

$$(8) \quad \begin{aligned} \sup_{i, j \in \mathbf{N}} |x_{ij}^{r'} - x_{ij}| \bar{v}(i, j) &\leq \sup_{i < i_0} \sup_{j \in \mathbf{N}} |x_{ij}^{r'} - x_{ij}| \bar{v}(i, j) + \sup_{i \geq i_0} \sup_{j \in \mathbf{N}} i^{-1} |x_{ij}^{r'} - x_{ij}| i\bar{v}(i, j) \\ &\leq \sup_{i < i_0} \sup_{j \in \mathbf{N}} |x_{ij}^{r'} - x_{ij}| \bar{v}(i, j) + 2 \sup_{i \geq i_0} \sup_{j \in \mathbf{N}} i\bar{v}(i, j) \\ &\leq \sup_{i < i_0} \sup_{j \in \mathbf{N}} |x_{ij}^{r'} - x_{ij}| \bar{v}(i, j) + \varepsilon/3. \end{aligned}$$

for every $r' \in \mathbf{N}$. On the other hand, by (5), $\lim_{j \rightarrow \infty} i\bar{v}(i, j) = 0$ for $i = 1, \dots, i_0 - 1$ and hence there is $j_0 \in \mathbf{N}$ such that $i\bar{v}(i, j) < \varepsilon/6$ for each $i = 1, \dots, i_0 - 1$ and $j \geq j_0$. This implies by (4), (7) and (8) that, for each $r' \in \mathbf{N}$,

$$(9) \quad \begin{aligned} \sup_{i, j \in \mathbf{N}} |x_{ij}^{r'} - x_{ij}| \bar{v}(i, j) &\leq \sup_{i < i_0} \sup_{j < j_0} i^{-1} |x_{ij}^{r'} - x_{ij}| i\bar{v}(i, j) \\ &\quad + \sup_{i < i_0} \sup_{j \geq j_0} i^{-1} |x_{ij}^{r'} - x_{ij}| i\bar{v}(i, j) + \varepsilon/3 \\ &\leq c \sup_{i < i_0} \sup_{j < j_0} i^{-1} |x_{ij}^{r'} - x_{ij}| + 2 \sup_{i < i_0} \sup_{j \geq j_0} i\bar{v}(i, j) + \varepsilon/3 \\ &\leq c \sup_{i < i_0} \sup_{j < j_0} i^{-1} |x_{ij}^{r'} - x_{ij}| + 2\varepsilon/3, \end{aligned}$$

where by (6) $\lim_{r' \rightarrow \infty} \sup_{i < i_0} \sup_{j < j_0} i^{-1} |x_{ij}^{r'} - x_{ij}| = 0$ and hence there is $r'_0 \in \mathbf{N}$ such that, for each $r' \geq r'_0$,

$$\sup_{i < i_0} \sup_{j < j_0} i^{-1} |x_{ij}^{r'} - x_{ij}| < \varepsilon/3c.$$

Thus, by (9), we get that

$$\sup_{i, j \in \mathbf{N}} |x_{ij}^{r'} - x_{ij}| \bar{v}(i, j) < \varepsilon$$

for all $r' \geq r'_0$.

Since ε is arbitrary, this means that $(x^{r'})_{r'}$ converges to x in $l_\infty(\bar{v})$, where $x \in C$. We can therefore conclude that C is a compact subset of $l_\infty(\bar{v})$. Since $\bar{v}$ is also arbitrary, C is a compact subset of $K_\infty(\bar{V})$ too.

It remains to show that C is not contained in the closed absolutely convex hull of any null sequence in $k_\infty(v^*)$. For this we proceed in a similar way as in Example 3.1 [DD2], which is based on an idea of L. Frerick and J. Wengenroth.

Assume that

$$C \subset \overline{\text{absconv}\{y_k : k \in \mathbf{N}\}} := C_1,$$

where $(y_k)_k \in c_0(k_\infty(v^*))$. Since $k_\infty(v^*)$ is a regular LB-space, $C_1 \subseteq \rho B_n$ for some $n > 1$ and $\rho > 0$ (B_n denotes the closed unit ball of $l_\infty(v_n^*)$).

Let $P: k_\infty(v^*) \rightarrow k_\infty(v^*)$ be the projection defined by

$$P\left((x_{ij})_{i,j}\right) = (z_{ij})_{i,j}, \quad z_{ij} := \begin{cases} 0, & \text{if } i \neq n; \\ x_{nj}, & \text{if } i = n. \end{cases}$$

We then have $P(C) \subseteq P(B_1) \subseteq P(B_n)$. By (c), we have that $P(B_n) \subseteq n\alpha_{n,n}^{-1}P(C)$. Put $D := P(C)$ and $E_D := \text{span } D$ (clearly, (E_D, p_D) is a Banach space).

Next, we define a map

$$T: l^1 \rightarrow E_D, \quad T((a_k)_k) := \sum_{k=1}^{\infty} a_k P y_k.$$

Since $T(B_{l^1}) \subseteq \rho P(B_n) \subseteq \rho n\alpha_{n,n}^{-1}D$, T is continuous. Also, since $(P y_k)_k \in c_0(k_\infty(v^*))$, the restriction of T to the closed unit ball of l^1 , B_{l^1} , endowed with the weak topology $\sigma(l^1, c_0)$ is continuous (see [PB, Lemma 3.2.11]) and hence $T(B_{l^1})$ is a compact absolutely convex subset of $k_\infty(v^*)$. So, we have

$$T(B_{l^1}) \supseteq \overline{\text{absconv}\{P y_k : k \in \mathbf{N}\}} \supseteq D$$

and hence T is open. Now, (E_D, p_D) is isomorphic to l^∞ and l^1 is separable, thereby obtaining a contradiction.

Thus, (ii) implies (i) and this completes the proof. $\square$

As an immediate consequence, we obtain:

Remark. Every coechelon LB-Montel space $k_\infty(v)$ of order ∞ which is not a (DFS)-space (and hence $(k_\infty(v))'_\beta = \lambda_1(a)$ is not quasinormable) has compact subsets not contained in closed absolutely convex hulls of any null sequence and hence so does the coechelon LB-Montel space constructed by Grothendieck and Köthe and considered in Example 3.1 of [DD2].

REFERENCES

- [BMS] K. D. Bierstedt, R. Meise, W. Summers, *Köthe sets and Köthe sequence spaces*, pp. 27–91 in: “Functional Analysis, Holomorphy and Approximation Theory”, North-Holland Math. Studies **71**, Amsterdam 1982. MR **84f**:46011
- [CO] B. Cascales and J. Orihuela, *Metrizability of precompact subsets in (LF)-spaces*, Proc. Roy. Soc. Edinburgh **103A** (1986), 293–299. MR **88b**:46003
- [DD1] S. Dierolf and P. Domański, *Factorization of Montel operators*, Studia Math. **107** (1993), 15–32. MR **94i**:46004
- [DD2] S. Dierolf and P. Domański, *Null Sequences in Coechelon Spaces*, Math. Nachr. **184** (1997), 167–176. MR **98a**:46013
- [DD3] S. Dierolf and P. Domański, *Bornological Spaces of Null Sequences*, Arch. Math. **65** (1995), 46–52. MR **96d**:46002
- [D] P. Domański, *On Spaces of Continuous Functions with Values in Coechelon Spaces*, to appear in Rev. Real Acad. Sci. Exactas, Madrid.
- [J] H. Jarchow, *Locally convex spaces*, Teubner Verlag, Stuttgart 1981. MR **83h**:46008
- [N] H. Neus, *Über die Regularitätsbegriffe induktiver lokalkonvexer Sequenzen*, Manuscripta Math. **25** (1978), 135–145. MR **58**:2125
- [PB] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Math. Studies **131**, Amsterdam 1987. MR **88j**:46003

- [S] J. Schmets, *Spaces of Vector-Valued Continuous Functions*, Lect. Notes Math. **1003**, Springer Verlag, Berlin 1983. MR **85g**:46046
- [V] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Math. Studies **67**, Amsterdam 1982. MR **84i**:46007

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