THE FIRST OCCURRENCE FOR THE IRREDUCIBLE MODULES OF GENERAL LINEAR GROUPS IN THE POLYNOMIAL ALGEBRA

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ABSTRACT. Let p be a prime number and let GL_n be the group of all invertible matrices over the prime field $\mathbb{F}_p$. It is known that every irreducible GL_n -module can occur as a submodule of P , the polynomial algebra with n variables over $\mathbb{F}_p$. Given an irreducible GL_n -module ρ , the purpose of this paper is to find out the first value of the degree d of which ρ occurs as a submodule of P_d , the subset of P consisting of homogeneous polynomials of degree d . This generalizes Schwartz-Tri's result to the case of any prime p .

1. INTRODUCTION

Let p be a prime number and let $GL_n = GL(n, \mathbb{F}_p)$ be the group of all invertible $(n \times n)$ -matrices over the prime field $\mathbb{F}_p$. Denote by $P = \mathbb{F}_p[x_1, \dots, x_n]$ the polynomial algebra with variables $x_1, \dots, x_n$ over $\mathbb{F}_p$. There is then an action of GL_n on P given by

$$\begin{aligned} \sigma x_i &= \sum_{j=1}^n a_{ji} x_j, \quad 1 \leq i \leq n, \\ \sigma f(x_1, \dots, x_n) &= f(\sigma x_1, \dots, \sigma x_n) \end{aligned}$$

with $\sigma = (a_{ij})_{1 \leq i, j \leq n} \in GL_n, f(x_1, \dots, x_n) \in P$.

For an irreducible GL_n -module ρ , it is known (see *e.g.* [1]) that ρ is isomorphic to a submodule of P . One may ask about the first value of the degree d for which ρ occurs as a submodule of P_d , the subset of P consisting of homogeneous polynomials of degree d . This first occurrence problem also arises from topology, as it is related to the description of the cohomology of certain spaces as modules over the Steenrod algebra (see *e.g.* [7]). For the prime 2, the problem has been solved independently by Schwartz [4] and Tri [6]. The purpose of this paper is to solve this problem by generalizing the argument used in [6] to the case of any prime p .

Let us recall (see *e.g.* [2, Chapter 8]) that the irreducible GL_n -modules are indexed, up to isomorphism, by column regular partitions, *i.e.*, by sequences of non-negative integers

$$\alpha_1, \dots, \alpha_n$$

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satisfying $0 \leq \alpha_i - \alpha_{i+1} \leq p-1$ for $1 \leq i \leq n-1$ and $0 \leq \alpha_n \leq p-1$. Namely, the irreducible GL_n -modules are indexed by column regular partitions $\alpha_1, \dots, \alpha_n$ satisfying $\alpha_n \leq p-2$. Recently, in [5], they were also obtained by means of modular invariants, as follows. For $1 \leq i \leq n$, let

$$V_i = \prod_{\lambda_j \in \mathbb{F}_p} (\lambda_1 x_1 + \dots + \lambda_{i-1} x_{i-1} + x_i)$$

be the Mui invariant and let

$$L_i = V_1 \cdots V_i$$

be the Dickson invariant. It is clear that

$$L_n = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1^p & x_2^p & \dots & x_n^p \\ \vdots & \vdots & \dots & \vdots \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \dots & x_n^{p^{n-1}} \end{vmatrix}$$

and ${}^\sigma L_n = \det \sigma \cdot L_n$ for $\sigma \in GL_n$. Let $\beta = (\beta_1, \dots, \beta_n)$ with $0 \leq \beta_i \leq p-1$ and set $L^\beta = \prod_{i=1}^n L_i^{\beta_i} \in P$. Denote by $H_\beta = H_\beta(GL_n)$ the GL_n -module generated by L^β (so H_β is nothing but the $\mathbb{F}_p$ -vector space generated by the set $\{{}^\sigma L^\beta | \sigma \in GL_n\}$). We have

Theorem A ([5, Corollary 1.2]). $\{H_\beta | \beta = (\beta_1, \dots, \beta_n), 0 \leq \beta_i \leq p-1, \beta_n \neq 0\}$ is a complete set of $(p-1)p^{n-1}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_n]$.

By noting that $H_{(\beta_1, \dots, \beta_{n-1}, p-1)} \cong H_{(\beta_1, \dots, \beta_{n-1}, 0)}$, we can restate the theorem as follows.

Theorem B (compare [1, Proposition 1.3]). $\{H_\beta | \beta = (\beta_1, \dots, \beta_n), 0 \leq \beta_i \leq p-1, \beta_n \neq p-1\}$ is a complete set of $(p-1)p^{n-1}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_n]$.

We should note that, in terms of column regular partitions, for $0 \leq \beta_i \leq p-1, \beta_n \neq p-1$, H_β corresponds to the sequence $\lambda_1, \dots, \lambda_n$ with $\lambda_i = \beta_i + \dots + \beta_n, 1 \leq i \leq n$.

Generalizing Schwartz-Tri's result ([4], [6]) to the case of any prime, we prove

Theorem C. *With β given in Theorem B, the first occurrence of H_β as a submodule of P_d is for $d = \deg L^\beta$.*

2. PRELIMINARIES

Let T_n be the group consisting of upper triangular matrices with 1 on the diagonal. It follows from [3] that

$$P^{T_n} = \mathbb{F}_p[V_1, \dots, V_n].$$

We shall use the following notation. Given $g = g(x_1, \dots, x_n) \in P, \lambda \in \mathbb{F}_p$ and $1 \leq i, k \leq n$, then ${}^{\sigma_{i,k,\lambda}} g$ (resp. ${}^{\eta_{i,\lambda}} g, {}^{\tau_{i,k}} g$) denotes the polynomial obtained from g by replacing x_i by $x_i + \lambda x_k$ (resp. by replacing x_i by λx_i , by interchanging x_i and x_k). Consider a homogeneous element $f = f(x_1, \dots, x_n)$ of P^{T_n} . It is clear that V_i is a factor of f if and only if x_i is also. We have

Lemma 1. *If ${}^{\eta_{i,\lambda}} f = \lambda^\ell f$ with $1 \leq \ell < p-1$ and $\lambda \neq 0, 1$, then f contains x_i as a factor.*

Proof. Write $f = \sum_{r=0}^m x_i^r f_r(x_1, \dots, x_n)$ with f_r free of x_i . It follows that $\sum_{r=0}^m x_i^r (\lambda^\ell f_r - \lambda^r f_r) = 0$. So $\lambda^\ell f_0 - f_0 = 0$, which implies that $f_0 = 0$. Therefore f contains x_i as a factor. The lemma follows. $\square$

The case $p = 2$ in the following lemma was treated in [6]. However, our proof given here, for any prime p , is much simpler.

Lemma 2. *If $i < k$, then f contains x_i as a factor provided that one of the following conditions is satisfied:*

- (i) $\sum_{r=0}^{p-1} \sigma_{i,k,r} f = -\tau_{i,k} f$ and $\eta_{i,\mu} f = f$ for every $\mu \neq 0$;
- (ii) f contains x_k as a factor and $\sum_{r=0}^{p-1} \sigma_{i,k,r} f = \alpha \cdot \tau_{i,k} f$ with $\alpha \in \{0, 1, -1\}$.

Proof. For $1 \leq r \leq p-1$, since f is T_n -invariant, we have

$$\begin{aligned} & f(x_1, \dots, x_{i-1}, rx_k, x_{i+1}, \dots, x_k, \dots, x_n) \\ &= f(x_1, \dots, x_{i-1}, rx_k, x_{i+1}, \dots, x_k - x_k, \dots, x_n) \\ &= f(x_1, \dots, x_{i-1}, rx_k, x_{i+1}, \dots, 0, \dots, x_n) \\ &= \begin{cases} f(x_1, \dots, x_{i-1}, x_k, x_{i+1}, \dots, 0, \dots, x_n), & \text{if } \eta_{i,r} f = f, \\ 0, & \text{if } f \text{ contains } x_k \text{ as a factor.} \end{cases} \end{aligned}$$

Consider $\sum_{r=0}^{p-1} \sigma_{i,k,r} f$ and set $x_i = 0$. (i) and the above equality imply that

$$\begin{aligned} \sum_{r=0}^{p-1} \sigma_{i,k,r} f &= f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_k, \dots, x_n) \\ &\quad - f(x_1, \dots, x_{i-1}, x_k, x_{i+1}, \dots, 0, \dots, x_n) \\ &= -f(x_1, \dots, x_{i-1}, x_k, x_{i+1}, \dots, 0, \dots, x_n), \end{aligned}$$

so $f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_k, \dots, x_n) = 0$; on the other hand, (ii) implies that

$$\begin{aligned} \sum_{r=0}^{p-1} \sigma_{i,k,r} f &= f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_k, \dots, x_n) \\ &= \alpha f(x_1, \dots, x_{i-1}, x_k, x_{i+1}, \dots, 0, \dots, x_n) \\ &= 0, \quad \text{since } f \text{ contains } x_k \text{ as a factor.} \end{aligned}$$

In any case, f contains x_i as a factor. The lemma follows. $\square$

3. PROOF OF THEOREM C

We proceed by induction on n . The proof is trivial for $n = 1$. Assume that the theorem holds for $n - 1$.

Let W be an irreducible GL_n -module of homogeneous polynomials. There exist then $\beta = (\beta_1, \dots, \beta_n)$ with $0 \leq \beta_i \leq p-1$, $\beta_n \neq p-1$ and a GL_n -isomorphism $\zeta : W \rightarrow H_\beta(GL_n)$. Set $f = f(x_1, \dots, x_n) = \zeta^{-1}(L^\beta)$. We need to prove that $\deg f \geq \deg L^\beta$. It is clear that f is invariant under the action of T_n and W is the GL_n -module generated by f . For convenience, write $W = W_f(GL_n)$.

Denote by $(\beta_{n_1}, \dots, \beta_{n_k})$ the subsequence of β consisting of all non-zero elements, so $L^\beta = L_{n_1}^{\beta_1} \cdots L_{n_k}^{\beta_{n_k}}$. First, we prove that f contains L_{n_k} as a factor (note that $\beta_{n_k} < p-1$ if $n_k = n$). Let m be an integer satisfying $n_{k-1} < m \leq n_k$. For every $\mu \neq 0$, as $\eta_{m,\mu} L^\beta = \mu^{\beta_{n_k}} L^\beta$, it follows that $\eta_{m,\mu} f = \mu^{\beta_{n_k}} f$. If $\beta_{n_k} < p-1$, Lemma 1

implies that f contains x_m , therefore V_m , as a factor. If $\beta_{n_k} = p-1$ (which implies $n_k < n$), then $\eta_{m,\mu} f = f$ and

$$\begin{aligned} \sigma_{m,n,r} L_{n_k} &= L_{n_k}(x_1, \dots, x_{m-1}, x_m + rx_n, x_{m+1}, \dots, x_{n_k}) \\ &= L_{n_k}(x_1, \dots, x_{m-1}, x_m, x_{m+1}, \dots, x_{n_k}) \\ &\quad + rL_{n_k}(x_1, \dots, x_{m-1}, x_n, x_{m+1}, \dots, x_{n_k}) \end{aligned}$$

for $r \in \mathbb{F}_p$. Since

$$\begin{aligned} &\sum_{r=0}^{p-1} (L_{n_k}(x_1, \dots, x_{m-1}, x_m, x_{m+1}, \dots, x_{n_k}) \\ &\quad + rL_{n_k}(x_1, \dots, x_{m-1}, x_n, x_{m+1}, \dots, x_{n_k}))^\lambda \\ (*) &= \begin{cases} 0, & \text{if } \lambda < p-1, \\ -L_{n_k}^{p-1}(x_1, \dots, x_{m-1}, x_n, x_{m+1}, \dots, x_{n_k}), & \text{if } \lambda = p-1, \end{cases} \end{aligned}$$

we have $\sum_{r=0}^{p-1} \sigma_{m,n,r} L_{n_k}^{p-1} = -L_{n_k}^{p-1}(x_1, \dots, x_{m-1}, x_n, x_{m+1}, \dots, x_{n_k})$ which implies that $\sum_{r=0}^{p-1} \sigma_{m,n,r} L^\beta = -\tau_{m,n} L^\beta$. Hence $\sum_{r=0}^{p-1} \sigma_{m,n,r} f = -\tau_{m,n} f$. It follows from Lemma 2 that f contains x_m , hence V_m , as a factor.

Set $n_0 = 0$. Suppose that f contains $x_{t+1}, \dots, x_{n_k}$ as factors with $n_{\ell-1} < t \leq n_\ell < n_k$. It follows from (*) that

$$\sum_{r=0}^{p-1} \sigma_{t,n_\ell+1,r} L_{n_\ell}^{\beta_{n_\ell}} = \begin{cases} 0, & \text{if } \beta_{n_\ell} < p-1, \\ -\tau_{t,n_\ell+1} L_{n_\ell}^{p-1}, & \text{if } \beta_{n_\ell} = p-1. \end{cases}$$

Therefore

$$\sum_{r=0}^{p-1} \sigma_{t,n_\ell+1,r} L^\beta = \begin{cases} 0, & \text{if } \beta_{n_\ell} < p-1, \\ \alpha^{\tau_{t,n_\ell+1}} L^\beta, & \text{if } \beta_{n_\ell} = p-1, \end{cases}$$

with $\alpha = \pm 1$. So

$$\sum_{r=0}^{p-1} \sigma_{t,n_\ell+1,r} f = \begin{cases} 0, & \text{if } \beta_{n_\ell} < p-1, \\ \alpha^{\tau_{t,n_\ell+1}} f, & \text{if } \beta_{n_\ell} = p-1. \end{cases}$$

By Lemma 2, f contains x_t , hence V_t , as a factor. Therefore f contains L_{n_k} as a factor.

Write $f = L_{n_k} g$ with $g = g(x_1, \dots, x_n) \in P$. So $g \in P^{T_n}$. Consider GL_{n_k} as a subgroup of GL_n in the usual way. $H_\beta(GL_{n_k})$ and $W = W_f(GL_{n_k})$ are then GL_{n_k} -modules and $H_\beta(GL_{n_k}) \stackrel{\zeta}{\cong} W_f(GL_{n_k}) \cong \det \otimes W_g(GL_{n_k})$. Set $\beta' = (\beta_1, \dots, \beta_{n_k}-1)$ and $\beta'' = (\beta_1, \dots, \beta_{n_k-1})$. As GL_{n_k} -modules, $H_\beta(GL_{n_k}) \cong \det \otimes H_{\beta'}(GL_{n_k})$, so $W_g(GL_{n_k}) \cong H_{\beta'}(GL_{n_k})$. If $n_k < n$, it follows from the inductive hypothesis that $\deg g \geq \deg L^{\beta'}$, so $\deg f \geq \deg L^\beta$. Suppose that $n_k = n$. The argument used above shows that g contains L_n as a factor; hence f contains $L_n^{\beta_n}$ as a factor.

By writing $f = L_n^{\beta_n} h$, we have $W_h(GL_n) \cong H_{\beta''}(GL_n)$. By replacing f by h , arguing as in the case $n_k < n$ yields $\deg h \geq \deg L^{\beta''}$; hence $\deg f \geq \deg L^\beta$. The theorem is proved.

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