A NOTE ON THE ANALYTIC FAMILIES OF COMPACT SUBMANIFOLDS OF COMPLEX MANIFOLDS

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ABSTRACT. In this paper, we prove a result of the deformation of the complex structure of a submanifold. Our result is a modification of the result of Kodaira (Ann. Math 75(1), 146–162, 1962).

1. INTRODUCTIONS

It is a classical theorem of Kodaira [1] that if the first cohomological group of the normal bundle is zero, then the deformation of a compact complex submanifold within the ambient complex manifold is unobstructed. However, it is usually difficult to check if such a cohomological group is indeed zero. Furthermore, as showed in §2, in some cases, it will never be zero.

In this short note, we are going to prove: if the deformation of the complex structure of a compact complex manifold M is unobstructed in the sense of Kodaira and Spencer, and if $M \rightarrow V$ is an embedding to the complex manifold V and $H^1(M, T_V|_M) = 0$, then any fiber in a neighborhood of the universal deformation space U at M can be embedded holomorphically to V .

Contrary to the case of normal bundle, it will be relatively easy to check the vanishing of the group $H^1(M, T_V|_M)$. For example, if the curvature of the manifold V has some kinds of positivity along M , then the group vanishes.

The typical examples of M are compact Calabi-Yau manifolds. Those manifolds admit Kähler metrics with zero Ricci curvature. In Tian [3], it is proved that the deformation of the complex structure of a Calabi-Yau manifold is unobstructed. The moduli space of a polarized Calabi-Yau manifold is then a complex orbifold.

We use the similar method as that of Kodaira [1]. That is, we construct a formal power series which gives the map we want. Then we prove the convergence of the power series. Fortunately, in our case, the original method in [1] of proving the convergence can be applied here.

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2. ANALYTIC FAMILIES OF COMPACT SUBMANIFOLDS

Suppose M is a compact complex manifold. We assume that the deformation of the complex structure of M is unobstructed. That is, the universal deformation space U of M is a complex manifold near M with complex dimension $\dim H^1(M, T_M)$, where T_M denotes the holomorphic tangent bundle of M .

If M is a Calabi-Yau manifold, then the deformation of the complex structure on M is unobstructed by the theorem of Tian [3].

We use the notations and definitions in [1].

Definition 2.1. Suppose N is a complex manifold of dimension $r + n$. By an analytic family of compact submanifolds of dimension n of N we shall mean a pair $(\mathcal{M}, U)$ of a complex manifold U and a complex analytic submanifold $\mathcal{M}$ of $N \times U$ of co-dimension r which satisfies the following two conditions:

1). For Each Point $t \in U$, the intersection $\mathcal{M} \cap N \times t$ is a connected, compact submanifold of $N \times t$ of dimension n .

2). For each point $p \in \mathcal{M}$, there exist r holomorphic functions $f_1 = f_1(w, t), \dots, f_r = f_r(w, t)$ defined on a neighborhood $\mathcal{U}_p$ of p in $N \times U$ such that

$$\text{rank} \frac{\partial(f_1, \dots, f_r)}{\partial(w^1, \dots, w^{r+d})} = r$$

and in $\mathcal{U}_p$, the submanifold $\mathcal{M}$ is defined by the simultaneous equations

$$f_1(w, t) = f_2(w, t) = \dots = f_r(w, t) = 0.$$

We call U the parameter manifold or the base space of the family (N, U) . We denote the family (N, U) simply by N when we need not indicate the base space U . For each point $t \in U$, we set

$$M_t \times t = \mathcal{M} \cap N \times t;$$

the submanifold M_t of N thus defined will be called the fiber of $\mathcal{M}$ over t . We may identify $M_t \times t$ with M_t and consider M_t as a *family consisting of compact submanifolds* M_t , $t \in U$, of N .

Definition 2.2. We say $(\mathfrak{X}, U)$ is the local total family of M , if U is a neighborhood of C^d with $d = \dim H^1(M, T_M)$ where T_M is the holomorphic tangent bundle of M , and there exists a projection

$$\pi : \mathfrak{X} \rightarrow U$$

such that it is holomorphic, surjective, of rank d and such that for all $t \in U$, $\pi^{-1}(t)$ is a deformation of complex structure of the center fiber $\pi^{-1}(0) = M$.

We state the main result of this paper.

Theorem 2.1. Suppose M is a compact complex manifold whose deformation of its complex structure is unobstructed. Let N be another complex manifold. Suppose that

$$i : M \rightarrow N$$

is a holomorphic embedding. If $H^1(M, T_N|_M) = 0$ where $T_N|_M$ is the restriction of the holomorphic tangent bundle of N to M , then there is a holomorphic map

$$f : \mathfrak{X} \rightarrow N$$

such that $f|_M = i$. Here $(\mathfrak{X}, U)$ is the local total family of M . Furthermore, $f|_{\pi^{-1}(t)}$ is an embedding of $\pi^{-1}(t)$ to N for $t \in U$.

Proof. Suppose $\bigcup_{j \in I} U_j \supset \mathfrak{X}$ is an open covering. On each U_j , $j \in I$, suppose $(z_j^1, \dots, z_j^n, t)$ is a local coordinate such that

$$\pi(z_j^1, \dots, z_j^n, t) = t, \quad t \in U \subset C^d.$$

It is obvious that, for fixed t , $(z_j^1, \dots, z_j^n)$ will be local coordinate for $\pi^{-1}(t) = M_t$. We further assume that U_j is defined by

$$U_j = \{|z_j| = \text{Max}_\alpha |z_j^\alpha| < 1\}.$$

We have, however, holomorphic functions g_{jk} such that

$$z_j^\alpha = g_{jk}^\alpha(z_k, t), \quad j, k \in I,$$

for $\alpha = 1, \dots, n$ and for $U_j \cap U_k \neq \emptyset$.

Now we suppose $\bigcup V_A \supset N$ is an open covering of N . Suppose

$$i(U_j) \subset V_{A(j)}, \quad j \in I,$$

for some $A(j)$ of j . Suppose $(w_A^1, \dots, w_A^r)$ is the local holomorphic coordinate chart of N on V_A . And we have transition functions h_{AB} on $V_A \cap V_B \neq \emptyset$,

$$w_A^s = h_{AB}^s(w_B)$$

for $s = 1, \dots, r + n = \dim N$. And again, we assume

$$V_j = \{|w_j| = \text{Max}|w_j^s| < 1\}.$$

For the sake of simplicity, we denote j for $A(j)$. In order to construct f , we need only have to construct holomorphic mappings (f_j) such that

$$f_j : U_j \rightarrow V_j, \quad j \in I,$$

satisfying

$$(2.1) \quad h_{jk}^s(f_k(z_k, t)) = f_j^s(g_{jk}(z_k, t), t) \quad \text{on } U_j \cap U_k \neq \emptyset, \quad j, k \in I,$$

for $s = 1, \dots, r + n$.

We set up some notations. Let

$$f_k(z, t) = f_{k|0} + f_{k|1} + \dots + f_{k|m} + \dots$$

be the decomposition of $f_k(z_k, t)$ into homogeneous polynomials of t of degree m . Of course, each f_k 's and $f_{k|m}$'s are vector valued functions $f_k = (f_k^s), f_{k|m} = (f_{k|m}^s), s = 1, \dots, r + n$. Suppose

$$f_k^m = f_{k|0} + \dots + f_{k|m}, \quad k \in I,$$

and let $a \equiv_m b$ mean $a - b$ is a polynomial of t of degree bigger than or equal to $m + 1$.

We construct $f_{k|m}$ inductively. First, set

$$f_{j|0} = i_j(z_j), \quad j \in I.$$

It is easy to check that

$$h_{jk}(f_k^0(z_k, t)) \equiv_0 f_j^0(g_{jk}(z_k, t), t), \quad U_j \cap U_k \neq \emptyset,$$

where $h_{jk} = (h_{jk}^s)_{s=1, \dots, r+n}$.

Now suppose for integer m , f_k^m is constructed, and

$$(2.2) \quad h_{jk}(f_k^m(z_k, t)) \equiv_m f_j^m(g_{jk}(z_k, t), t), \quad U_j \cap U_k \neq \emptyset.$$

Define

$$(2.3) \quad \Psi_{jk}(z_k, t) \equiv_{m+1} h_{jk}(f_k^m(z_k, t)) - f_j^m(g_{jk}(z_k, t), t), \quad U_j \cap U_k \neq \emptyset.$$

Then

Claim.

$$\Psi_{ik} = \Psi_{ij} + \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s \quad \text{on } U_i \cap U_j \cap U_k \neq \emptyset.$$

Proof of the Claim. By Equation (2.3), we have

$$f_i^m(g_{ik}(z_k, t), t) \equiv_{m+1} h_{ik}(f_k^m(z_k, t)) - \Psi_{ik}(z_k, t), \quad U_i \cap U_k \neq \emptyset.$$

Thus on $U_i \cap U_j \cap U_k \neq \emptyset$, we have

$$\begin{aligned} h_{ij}(f_j^m(g_{jk}(z_k, t), t)) &\equiv_{m+1} h_{ij}(h_{jk}(f_k^m(z_k, t)) - \Psi_{jk}(z_k, t)) \\ &\equiv_{m+1} h_{ij}(h_{jk}(f_k^m(z_k, t))) - \frac{\partial w_i}{\partial w_j^s} \Psi_{jk}^s(z_k, t) \\ &\equiv_{m+1} h_{ik}(f_k^m(z_k, t)) - \frac{\partial w_i}{\partial w_j^s} \Psi_{jk}^s(z_k, t). \end{aligned}$$

Note that $\Psi_{jk}^s(z_k, t) \equiv_m 0$. So

$$\frac{\partial w_i}{\partial w_j^s} \Psi_{jk}^s(z_k, t) \equiv_{m+1} \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s(z_k, t).$$

We have

$$\Psi_{ij}(g_{jk}(z_k, t), t) \equiv_{m+1} \Psi_{ik}(z_k, t) - \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s(z_k, t), \quad U_j \cap U_k \neq \emptyset.$$

On the other hand

$$g_{jk}(z_k, t) \equiv_0 z_j, \quad U_j \cap U_k \neq \emptyset.$$

So

$$\Psi_{ij}(z_j, t) \equiv_{m+1} \Psi_{ij}(g_{jk}(z_k, t), t).$$

Thus we have

$$(2.4) \quad \Psi_{ij}(z_j, t) = \Psi_{ik}(z_k, t) - \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s(z_k, t), \quad U_j \cap U_k \neq \emptyset.$$

The claim is proved. $\square$

Suppose $\mathcal{U} = (U_j)$ is the covering of M ; we see from Equation (2.4) $\{\Psi_{ij}\}_{i,j \in I}$ defined a cocycle of $Z^1(\mathcal{U}, T_N|_M)$. Thus $\{\Psi_{ij}\}_{i,j \in I} \in H^1(\mathcal{U}, T_N|_M)$. With a good covering $\mathcal{U}$, we have $H^1(\mathcal{U}, T_N|_M) = H^1(M, T_N|_M)$ and by the assumption, the latter is zero. So we can find $\{\Psi_j\}$ such that

$$\Psi_{jk} = \Psi_j - \frac{\partial w_j}{\partial w_k^s} \Big|_{t=0} \Psi_k^s, \quad U_j \cap U_k \neq \emptyset.$$

We then define f_k^{m+1} inductively as

$$(2.5) \quad f_k^{m+1}(z_k, t) = f_k^m(z_k, t) + \Psi_k(z_k, t), \quad k \in I.$$

With this definition, we have

$$\begin{aligned}
h_{jk}(f_k^{m+1}(z_k, t)) &\equiv_{m+1} h_{jk}(f_k^m(z_k, t) + \Psi_k(z_k, t)) \\
&\equiv_{m+1} h_{jk}(f_k^m(z_k, t)) + \frac{\partial w_j}{\partial w_k^s} \Psi_k^s(z_k, t) \\
&\equiv_{m+1} \Psi_{jk}(z_k, t) + \frac{\partial w_j}{\partial w_k^s} \Psi_k^s(z_k, t) + f_j^m(g_{jk}(z_k, t), t) \\
&\equiv_{m+1} \Psi_j(z_j, t) + f_j^m(g_{jk}(z_k, t), t) \\
&\equiv_{m+1} f_j^{m+1}(g_{jk}(z_k, t), t).
\end{aligned}$$

Now we have got a formal series

$$f_{k|0} + f_{k|1} + \cdots + f_{k|m} + \cdots, \quad k \in I,$$

which satisfies Equation (2.2) for any m . If it converges, then f_k 's and f will be holomorphic and f will satisfy Equation (2.1). We put off the proof of the convergence to the next section. At this moment, we assume the convergence is true.

By continuity, for fixed t , t being sufficiently small, f will be an immersion on $\pi^{-1}(t)$. We claim that for sufficiently small t , f is an embedding. Suppose not, then we can find $t_n \rightarrow 0$ and $x_n, y_n \in \pi^{-1}(t_n)$ such that $x_n \neq y_n$ but $f(x_n, t) = f(y_n, t)$. Suppose $x_n \rightarrow x$ and $y_n \rightarrow y$. We see $x = y$; otherwise it will contradict the fact that i is an embedding. But if $x = y$, it will contradict the fact that f is an immersion on each fiber.

Now we give an important example of manifolds such that $H^1(M, T_N|_M) = 0$.

Proposition 2.1. *Let M be a simply connected Calabi-Yau threefold. $M \rightarrow \mathbb{C}P^D$ is an embedding. Then we have*

$$H^1(M, T_{\mathbb{C}P^D}|_M) = 0.$$

Proof. From the Euler exact sequence

$$(2.6) \quad 0 \rightarrow \mathcal{O} \rightarrow \oplus_{D+1} \mathcal{O}(1) \rightarrow T_{\mathbb{C}P^D} \rightarrow 0.$$

We have the long exact sequence

$$\cdots \rightarrow H^1(M, \oplus_{d+1} \mathcal{O}(1)) \rightarrow H^1(M, T_{\mathbb{C}P^D}|_M) \rightarrow H^2(M, \mathcal{O}) \rightarrow \cdots.$$

By the Kodaira Vanishing theorem

$$H^1(M, \oplus_{D+1} \mathcal{O}(1)) = 0.$$

By the Dolbeault theorem and Serre Duality $H^2(M, \mathcal{O}) = 0$, we have

$$H^1(M, T_{\mathbb{C}P^D}|_M) = 0.$$

□

Corollary 2.1. *Suppose M is a simply connected Calabi-Yau threefold. If M is embedded to some $\mathbb{C}P^D$, then $M_t = \pi^{-1}(t)$ can also be embedded to the same $\mathbb{C}P^D$ for small t .*

The following example showed that, in general, $H^1(M, T_{\mathbb{C}P^D}|_M/T_M) \neq 0$.

Example. Suppose $\mathcal{N} = T_{\mathbb{C}P^D}|_M/T_M$ is the normal bundle of M in $\mathbb{C}P^D$ in the previous proposition. Then, in general, $H^1(M, \mathcal{N}) \neq 0$.

Proof. We have the exact sequence:

$$0 \rightarrow T_M \rightarrow T_{CP^D}|_M \rightarrow \mathcal{N}|_M \rightarrow 0.$$

So the long exact sequence gives

$$\begin{aligned} \cdots \rightarrow H^1(M, T_{CP^D}|_M) &\rightarrow H^1(M, \mathcal{N}) \rightarrow H^2(M, T_M) \\ &\rightarrow H^2(M, T_{CP^D}|_M) \rightarrow \cdots \end{aligned}$$

However, by Serre Duality, $H^2(M, T_M) = H^{1,1}(M)$. And it is easy to see from the Euler Sequence (2.6) that $\dim H^2(M, T_{CP^D}|_M) = 1$. Thus in general $\dim H^1(M, \mathcal{N}) \geq \dim H^{1,1}(M) - 1$ and is not zero.

3. THE CONVERGENCE

Now we shall show that the power series $\sum f_k(z_k, t)$ in Equation (2.5) for all $k \in I$, converge for $|t| < \varepsilon_0$, ε_0 being a sufficiently small positive number, provided that we choose for each m the homogeneous polynomials $f_{k|m+1}(z_k, t)$, $k \in I$, satisfying (2.3) in a proper manner.

For any vector $\xi = (\xi^1, \xi^2, \dots, \xi^\lambda, \dots)$, we define

$$|\xi| = \max_\lambda |\xi^\lambda|.$$

Consider a power series

$$\xi(z, u) = \sum \xi_{lm, \dots, n} u_1^l u_2^m \cdots u_q^n$$

in $u_1, \dots, u_q$ whose coefficients $\xi_{lm, \dots, n}$ are vector valued functions of z and a power series

$$a(u) = a_{lm, \dots, n} u_1^l u_2^m \cdots u_q^n, \quad a_{lm, \dots, n} \geq 0.$$

We indicate by writing $\xi(z, u) \ll a(u)$ that

$$|\xi_{lm, \dots, n}(z)| \leq a_{lm, \dots, n}.$$

Let

$$A(t) = \frac{a}{16b} \sum_{n=1}^{\infty} \frac{1}{n^2} b^n (t_1 + \cdots + t_l)^n,$$

where a and b are positive constants. We have

$$A(t)^\gamma \ll \left(\frac{a}{b}\right)^{\gamma-1} A(t), \quad \gamma = 2, 3, \dots$$

For our purpose it suffices to prove the inequalities

$$f_k(z_k, t) \ll A(t), \quad i \in I.$$

In what follows we denote by $c_0, c_1, c_2, \dots$ positive constants *which are greater than 1*. We may assume that

$$\left| \frac{\partial h_{ij}^\lambda}{\partial w_j^\mu} \right| < c_0, \quad c_0 > 1.$$

For the sake of simplicity, we denote $f_k^m(z_k, t) - i(z_k)$ by $f_k^m(z_k, t)$. Then for sufficiently large a , we have

$$f_k^1(z_k, t) \ll \frac{a}{16} (t_1 + \cdots + t_d) \ll A(t), \quad k \in I.$$

Now assuming the inequalities

$$f_k^m(z_k, t) \ll A(t), \quad k \in I,$$

for an integer $m \geq 1$, we shall estimate the coefficients of the homogeneous polynomials $\Psi_{ik}(z, t)$. We expand h_{ik} and g_{ik} into power series, whose coefficients are vector valued holomorphic functions:

$$h_{ik}(w_k) \ll \sum_{\alpha=0}^{\infty} c_1^\alpha (w_k^1 + \cdots + w_k^{r+n})^\alpha,$$

$$g_{ik}(z_k) \ll \sum_{\alpha=0}^{\infty} c_1^\alpha (z_k^1 + \cdots + z_k^n)^\alpha$$

for $i, k \in I$.

Recall that in Equation (2.3)

$$\Psi_{jk} = [h_{jk}(f_k^m(z_k, t))]_{m+1} - [f_j^m(g_{jk}(z_k, t), t)]_{m+1},$$

where $[a]_{m+1}$ is a polynomial of t of degree bigger than m . First we estimate $[h_{jk}(f_k^m(z_k, t))]_{m+1}$. The terms which are linear in h_{jk} contribute nothing to $[h_{jk}(f_k^m(z_k, t))]_{m+1}$. So we have

$$[h_{jk}(f_k^m(z_k, t))]_{m+1} \ll \sum_{\alpha=2}^{\infty} c_1^\alpha (r+n)^\alpha A(t)^\alpha$$

$$\ll c_1(r+n)A(t) \sum_{\alpha=1}^{\infty} \left(\frac{c_1(r+n)a}{b}\right)^\alpha.$$

Assuming that

$$b > 2c_1(r+n)a,$$

we obtain therefore

$$(3.1) \quad [h_{jk}(f_k^m(z_k, t))]_{m+1} \ll 2c_1^2 r^2 a b^{-1} A(t).$$

On the other hand

$$[f_j^m(g_{jk}(z_k, t), t)]_{m+1} = [f_j^m(g_{jk}(z_k, t), t) - f_j^m(z_j, t)]_{m+1}.$$

Denote by U_i^δ the subdomain of U_i consisting of all points $z_j = (z_j, 0)$, $|z_j| < 1 - \delta$. We fix a positive number δ such that $\{U_i^\delta | i \in I\}$ forms a covering of M . Take a point $z \in U_k \cap U_i^\delta$ and let z_k and z_j be the local coordinates of z on U_k and U_j respectively. Obviously, we have

$$z_j = g_{jk}(z_k, 0), \quad |z_k| < 1, |z_j| < 1 - \delta.$$

Letting $y = (y_1, \dots, y_n)$, we expand the coefficients of polynomial $f_i^m(z_j + y, t)$ into power series. Suppose $|y| < \delta$; we have

$$[f_j^m(z_j + y, t) - f_j^m(z_j, t)]_{m+1} \ll A(t) \left(\prod_{\alpha=1}^n \left(1 - \frac{|y_\alpha|}{\delta}\right)^{-1} - 1 \right).$$

Now if t is small, and $\mu = \max_{j,k} |g_{jk}(z_k, t) - z_j| < \delta$, then

$$(3.2) \quad [f_j^m(g_{jk}(z_k, t), t)]_{m+1} \ll \left(\left(1 - \frac{\mu}{\delta}\right)^{-n} - 1 \right) A(t).$$

So from Equation (3.1) and (3.2), we have

$$\Psi_{jk} \ll (2c_1^2(r+n)^2ab^{-1} + ((1 - \frac{\mu}{\delta})^{-n} - 1))A(t).$$

We take an arbitrary point $z \in U_k \cap U_i$ and choose a domain U_j^δ , which contains z ; then

$$\Psi_{ik} = \frac{\partial w_i}{\partial w_j} \Psi_{jk} - \frac{\partial w_i}{\partial w_j} \Psi_{ji}.$$

Thus

$$\Psi_{ik} \ll 2c_0(2c_1^2(r+n)^2ab^{-1} + ((1 - \frac{\mu}{\delta})^{-n} - 1))A(t).$$

Let $c_3 = 2c_0(2c_1^2(r+n)^2ab^{-1} + ((1 - \frac{\mu}{\delta})^{-n} - 1))$. Then

$$(3.3) \quad \Psi_{jk} \ll c_3 A(t), \quad z \in U_k \cup U_i.$$

The following lemma can be proved by an elementary consideration [2].

Lemma 3.1. *We can choose the homogeneous polynomials $\Psi_i, i \in I$, satisfying*

$$\Psi_{ik} = \Psi_i - \frac{\partial w_i}{\partial w_k^s} \Psi_k^s$$

in such a way that

$$\Psi_i \ll c_3 c_4 A(t)$$

where $c_4 > 1$ is a constant independent of m . □

Since μ is independent of m , and $\mu \rightarrow 0$ as $t \rightarrow 0$, we can choose t small enough such that $2c_0((1 - \mu/\delta)^{-n} - 1) < \frac{1}{2}$. Choosing b large enough so that $4c_0c_1^2(r+n)^2ab^{-1} < \frac{1}{2}$, we have

$$f_j^{m+1} \ll A(t), \quad j \in I.$$

The convergence of the power series is proved.

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REFERENCES

- [1] Kunihiko Kodaira *A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds* Annals of Mathematics, 75(1), 146-162, 1962 MR **24**:A3665b
- [2] Kunihiko Kodaira and D. C. Spencer. *A Theorem of Completeness of Characteristic Systems of Complete Continuous Systems*, Amer. J. Math, 81, 477-500, 1959 MR **22**:3011
- [3] Gang Tian, *Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Peterson-Weil Metric*, Mathematical aspects of string theory, (1), 629-646, Shing-Tung Yau ed, 1987, World Scientific CMP 20:04

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