LIPSCHITZ CONTINUITY OF OBLIQUE PROJECTIONS

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(Communicated by David R. Larson)

ABSTRACT. Let W and L be complementary spaces of a finite dimensional unitary space V and let $P(W, L)$ denote the projection of V on W parallel to L . Estimates for the norm of $P(W, L) - P(W, M)$ are derived which involve the norm of the restriction of $P(W, L)$ to M or the gap between L and M .

1. INTRODUCTION AND PRELIMINARIES

Let $V = W \oplus L$ be a nontrivial direct sum decomposition of an n -dimensional unitary space V and let $P(W, L)$ denote the oblique projection on W along L . If the distance between subspaces is measured in the gap metric, then all subspaces M contained in a sufficiently small neighbourhood $U(L)$ of L are also complementary to W (see e.g. [1, p. 390] or [5]). For $M \in U(L)$ set $\pi(M) = P(W, M)$. In this note we study the map $\pi(M)$. An estimate for $\|\pi(M) - \pi(L)\|$ will be obtained which involves the restriction of $P(W, L)$ to M . A Lipschitz constant for π in [1] will be improved.

Notation. For a linear map $A: Y \rightarrow V$ the norm $\|A\|$ denotes the operator norm, i.e. $\|A\| = \sup\{\|Ay\|, y \in Y, \|y\| = 1\}$. Let P_W denote the orthogonal projection of V on W and set

$$P(W, L; M) = P(W, L)|_M.$$

We write $d(x, M)$ for the distance of $x \in V$ from M . The gap between two subspaces L and M is defined by

$$\theta(L, M) = \|P_L - P_M\|.$$

We shall need the following facts on the gap, for which we refer to [2] and [1]. First of all θ is a metric on the set of subspaces of V , and $\theta(L, M) \leq 1$. If $\theta(L, M) < 1$, then

$$(1.1) \quad \dim M = \dim L.$$

In the case of (1.1) we have

$$(1.2) \quad \theta(L, M) = \|P_L(I - P_M)\| = \|P_M(I - P_L)\|.$$

Received by the editors August 10, 1997 and, in revised form, April 28, 1998.

1991 *Mathematics Subject Classification.* Primary 51M05, 51M16, 15A45.

Key words and phrases. Oblique projections, direct complements.

Lemma 1.1. *Assume $V = W \oplus L$.*

(i) *For a subspace M of V we have*

$$(1.3) \quad \|P(W, L; M)\| = \|P(W, L)P_M\|.$$

(ii) *If $\dim M = \dim L$, then*

$$(1.4) \quad \|P(W, L; M)\| \leq \|P(W, L)\|\theta(L, M).$$

Proof. (i) For $y \in M$, $M \neq 0$, we have $P(W, L; M)y = P(W, L)P_M y$. Therefore

$$\begin{aligned} \|P(W, L; M)\| &= \max\left\{\frac{\|P(W, L)P_M y\|}{\|y\|}, y \neq 0, y \in M\right\} \leq \|P(W, L)P_M\| \\ &= \max\left\{\frac{\|P(W, L)P_M y\|}{\|y\|}, y \neq 0, y \in V\right\} \\ &\leq \max\left\{\frac{\|P(W, L)P_M y\|}{\|P_M y\|}, y \in V, P_M y \neq 0\right\} = \|P(W, L; M)\|. \end{aligned}$$

(ii) From $P(W, L) = P(W, L)(I - P_L)$ and (1.3) follows

$$\|P(W, L; M)\| = \|P(W, L)(I - P_L)P_M\|.$$

Hence (1.2) yields (1.4). □

The following observations do not seem to be widely known.

Lemma 1.2. *Assume $V = W \oplus L$.*

(i) *If $W \neq 0$, then*

$$(1.5) \quad \max_{x \in W, \|x\|=1} \frac{1}{d(x, L)} = \|P(W, L)\|.$$

(ii) *If $W \neq 0$ and $L \neq 0$, then*

$$(1.6) \quad \|P(W, L)\| = \|P(L, W)\|.$$

Proof. (i) We shall see that (1.5) is equivalent to the identity

$$(1.7) \quad \frac{1}{1 - \|P_L P_W\|^2} = \|P(W, L)\|^2,$$

which is due to Ljance [3] (see [4] or [6]). Set

$$\tau = \min_{x \in W, \|x\|=1} d(x, L) = \min_{x \in W, \|x\|=1} \|(I - P_L)P_W x\|.$$

Then the left-hand side of (1.5) is equal to $1/\tau$. If $x \in W$ and $\|x\| = 1$, then

$$\|(I - P_L)P_W x\|^2 + \|P_L P_W x\|^2 = 1.$$

Hence

$$\tau^2 = 1 - \max_{x \in W, \|x\|=1} \|P_L P_W x\|^2 = 1 - \|P_L P_W\|^2.$$

Therefore

$$\frac{1}{\tau^2} = \frac{1}{1 - \|P_L P_W\|^2},$$

and (1.5) follows from (1.7).

(ii) Since $P_L P_W = 0$ implies $P_L P_W P_L = 0$ and thus $P_W P_L = 0$, we note that either $P_L P_W = P_W P_L = 0$ or both $P_L P_W \neq 0$ and $P_W P_L \neq 0$. In each case we have $\|P_L P_W\| = \|P_W P_L\|$. Hence (1.7) implies (1.6). □

2. ESTIMATES FOR OBLIQUE PROJECTIONS

Theorem 2.1. Assume $V = W \oplus L$, $W \neq 0$, $L \neq 0$.

(i) Let M be a subspace of V with $\dim M = \dim L$ and

$$(2.1) \quad \mu = \|P(W, L; M)\| < 1.$$

Then

$$(2.2) \quad V = W \oplus M$$

and

$$(2.3) \quad \|P(W, M) - P(W, L)\| \leq \frac{\mu}{1 - \mu} \|P(W, L)\|.$$

(ii) If a subspace M satisfies

$$(2.4) \quad \theta(L, M) \leq (1 - c) \|P(W, L)\|^{-1}, \quad 0 < c < 1,$$

then we have (2.2) and

$$(2.5) \quad \|P(W, M) - P(W, L)\| \leq \frac{1}{c} \|P(W, L)\|^2 \theta(L, M).$$

Proof. (i) Suppose $x \neq 0$ for some $x \in W \cap M$. Then $P(W, L; M)x = x$. Hence $\|P(W, L; M)\| \geq 1$, which contradicts (2.1). Therefore we have $W \cap M = 0$ and (2.2). Now put $S = P(M, W)P_L$. Then

$$P(M, W)[I - P_L P(L, W)] = P(M, W)P(W, L) = 0$$

implies $P(M, W) = SP(L, W)$. Using $P(W, L) = I - P(L, W)$ and (1.6) we obtain

$$\begin{aligned} \|P(W, M) - P(W, L)\| &= \|P(M, W) - P(L, W)\| \\ &= \|SP(L, W) - P_L P(L, W)\| \leq \|S - P_L\| \|P(W, L)\|. \end{aligned}$$

Thus our target inequality is

$$(2.6) \quad \|S - P_L\| \leq \frac{\mu}{1 - \mu}.$$

Since $P(L, W)P(W, M)x = 0$ for all $x \in V$, we have $P(L, W)[I - P(M, W)] = 0$ or $P(L, W)P(M, W) = P(L, W)$. Similarly $P(L, W)P_L = P_L$. Hence

$$P(L, W)P(M, W)P_L = P(L, W)S = P_L,$$

and we obtain $S - P_L = [I - P(L, W)]S = P(W, L)P_M S$. Then

$$\begin{aligned} (S - P_L)^*(S - P_L) &= S^*S - P_L S - S^*P_L + P_L \\ (2.7) \quad &= S^*P_M P(W, L)^* P(W, L)P_M S, \end{aligned}$$

which implies

$$(2.8) \quad P_L S + S^*P_L = S^*[I - P_M P(W, L)^* P(W, L)P_M]S + P_L.$$

For the left-hand side of (2.8) we obtain

$$\|P_L S + S^*P_L\| \leq 2\|P_L S\| \leq 2\|P_L\| \|S\| \leq 2\|S\|.$$

Put $T = I - P_M P(W, L)^* P(W, L)P_M$ such that the right-hand side of (2.8) equals $R = S^*TS + P_L$. Since P_L is the identity map on L and $SP_L = S$, it is not difficult to show that $\|R\| = \|S^*TS\| + 1$. Now (2.1) implies that T is positive definite and that $1 - \|P(W, L)P_M\| = 1 - \mu^2 > 0$ is the smallest eigenvalue of T . Hence $\|S^*TS\| \geq \|S\|^2(1 - \mu^2)$. Thus $\|S\|$ satisfies

$$0 \leq \|S\|^2(1 - \mu^2) + 1 \leq 2\|S\|,$$

which is equivalent to

$$(2.9) \quad 0 < \frac{1}{1+\mu} \leq \|S\| \leq \frac{1}{1-\mu}.$$

Then (2.7) yields

$$\|S - P_L\| \leq \|S\| \|P(W, L)P_M\| \leq \frac{\mu}{1-\mu},$$

and we have (2.6), which completes the proof of (2.3).

(ii) From (2.4) and (1.4) we obtain

$$(2.10) \quad \mu \leq \|P(W, L)\|\theta(L, M) \leq 1 - c < 1.$$

Then $\|P(W, L)\| \geq 1$ implies $\theta(L, M) < 1$ and $\dim M = \dim L$. Because of $\mu < 1$ we can use (i) and conclude that $P(W, M)$ exists. Since $0 \leq \mu \leq 1 - c$ is equivalent to

$$0 \leq \frac{1}{1-\mu} \leq \frac{1}{c},$$

the estimate (2.5) follows immediately from (2.3). $\square$

In the neighbourhood of L given by (2.4) the estimate (2.5) yields a Lipschitz constant for $P(W, M)$ of the form

$$(2.11) \quad \frac{1}{c} \|P(W, L)\|^2.$$

In [1, p. 390] we find for sufficiently small $\theta(L, M)$ an estimate

$$\|P(W, M) - P(W, L)\| \leq K\theta(L, M)$$

with

$$(2.12) \quad K = 2\|P(W, L)\| \max_{x \in W, \|x\|=1} \frac{1}{d(x, L)}.$$

According to Lemma 1.2 the Lipschitz constant K in (2.12) is equal to (2.11) with $c = 1/2$.

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