ON p -HYPONORMAL OPERATORS

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ABSTRACT. In this paper we show that p -hyponormal operators with $0 \notin \sigma(|T|_r^{\frac{1}{2}})$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

1. INTRODUCTION

Let H and K be separable complex Hilbert spaces and let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from H to K . If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H, K)$.

An operator $T \in \mathcal{L}(H)$ is said to be p -hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where T^* is the adjoint of T . If $p = 1$, T is called hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [Xi]), and p -hyponormal operators for a general p , $0 < p < 1$, have been studied by Aluthge. Any p -hyponormal operators are q -hyponormal if $q \leq p$. But there are examples to show that the converse of the above statement is not true (see [Al]).

A bounded linear operator S on H is called scalar of order m if it has a spectral distribution of order m , i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m , $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace. We now define the weaker form of a subscalar operator. An operator $T \in \mathcal{L}(H)$ is quasi-subscalar if there exists a one-to-one $V \in \mathcal{L}(H, K)$ such that $VT = SV$ where $S (= \Phi(z))$ in the above definition) is a scalar operator.

This paper has been divided into three sections. Section 2 deals with some preliminary facts. In section 3, we show that p -hyponormal operators with the property $0 \notin \sigma(|T|_r^{\frac{1}{2}})$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

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2. PRELIMINARIES

Let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let H be a complex separable Hilbert space, and let D be a bounded open disc in $\mathbf{C}$. We shall denote by $L^2(D, H)$ the Hilbert space of measurable functions $f : D \rightarrow H$, such that

$$\|f\|_{2,D} = \left(\int_D \|f(z)\|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(D, H)$ which are analytic functions in D (i.e., $\bar{\partial}f = 0$) is defined by

$$A^2(D, H) = L^2(D, H) \cap \mathcal{O}(D, H)$$

where $\mathcal{O}(D, H)$ denotes the Fréchet space of H -valued analytic functions on D with respect to uniform topology. $A^2(D, H)$ is called the Bergman space for D . Note that $A^2(D, H)$ is a Hilbert space. The operator $T - z$ on the space $\mathcal{O}(D, H)$ has property (β) , which means by definition that $T - z$ is one-to-one and has closed range for every disc D .

Let us define now a Sobolev type space, called $W^2(D, H)$ where D is a bounded disc in $\mathbf{C}$. $W^2(D, H)$ will be the space of those functions $f \in L^2(D, H)$ whose derivatives $\bar{\partial}f, \bar{\partial}^2f$ in the sense of distributions still belong to $L^2(D, H)$. Endowed with the norm $\|f\|_{W^2}^2 = \sum_{i=0}^2 \|\bar{\partial}^i f\|_{2,D}^2$, $W^2(D, H)$ becomes a Hilbert space contained continuously in $L^2(D, H)$.

Now for $f \in C_0^2(\mathbf{C})$, let M_f denote the operator on $W^2(D, H)$ given by multiplication by f . This has a spectral distribution of order 2, defined by the functional calculus

$$\Phi_M : C_0^2(\mathbf{C}) \longrightarrow \mathcal{L}(W^2(D, H)), \quad \Phi_M(f) = M_f.$$

Therefore M_z is a scalar operator of order 2. In fact, it can be shown [Pu] that M_z is subnormal.

3. SUBSCALARITY

This section deals with the characterization for some p -hyponormal operators. Recall that an operator $T \in \mathcal{L}(H)$ is said to be p -hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where T^* is the adjoint of T .

We need the following lemmas to give a proof of the main theorem.

Lemma 1 ([Xi], Lemma 2.1). *Let $T = U|T|_r$ be the polar decomposition of T , $Q = |T|_r - |T|_l$, $z = \rho e^{i\theta}$, $0 < \rho$, and $|e^{i\theta}| = 1$ where $|T|_r = (T^*T)^{\frac{1}{2}}$ and $|T|_l = (TT^*)^{\frac{1}{2}}$. Then*

$$\|(T - z)f\|_{2,D}^2 = \|(|T|_r - \rho)f\|_{2,D}^2 + \rho \| |T|^{\frac{1}{2}} (U - e^{i\theta})^* f \|_{2,D}^2 + \rho \langle Qf, f \rangle$$

for all $f \in L^2(D, H)$.

For reference, we quote Lemma 2 from [Pu].

Lemma 2 ([Pu], Proposition 2.1). *For every bounded disk D in $\mathbf{C}$ there is a constant C_D , such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$ we have*

$$\|(I - P)f\|_{2,D} \leq C_D (\|(T - z)^* \bar{\partial}f\|_{2,D} + \|(T - z)^* \bar{\partial}^2f\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

For p -hyponormal operator $T = U|T|$, Aluthge ([Al]) introduced the operator $\tilde{T} = |T|^{\frac{1}{2}}_r U |T|^{\frac{1}{2}}_r$ and showed very interesting results on $\tilde{T}$.

Lemma 3 ([Al]). *Let $T = U|T|_r$ be a p -hyponormal operator, $0 < p < 1$, and U unitary. Then the operator $\tilde{T} = |T|^{\frac{1}{2}}_r U |T|^{\frac{1}{2}}_r$ is hyponormal if $\frac{1}{2} \leq p < 1$, and $(p + \frac{1}{2})$ -hyponormal if $0 < p < \frac{1}{2}$.*

Lemma 4. *Let $T = U|T|_r$ be semi-hyponormal and let U be unitary. Let D be a bounded disk which contains $\sigma(T)$. Then the map $V : H \rightarrow H(D)$ defined by $Vh = \widetilde{1 \otimes h} (\equiv 1 \otimes h + \overline{(T - z)W^2(D, H)})$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h .*

Proof. Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$(1) \quad \lim_{n \rightarrow \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0.$$

Then by the definition of the norm of Sobolev space (1) implies

$$(2) \quad \lim_{n \rightarrow \infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2$. Since T is a semi-hyponormal operator, Lemma 1 and equation (2) imply

$$(3) \quad \begin{cases} \lim_{n \rightarrow \infty} \|(|T|_r - \rho)\bar{\partial}^i f_n\|_{2,D} = 0, \\ \lim_{n \rightarrow \infty} \rho \| |T|^{\frac{1}{2}}_r (U - e^{i\theta})^* \bar{\partial}^i f_n \|_{2,D} = 0, \quad \text{and} \\ \lim_{n \rightarrow \infty} \rho \langle Q \bar{\partial}^i f_n, \bar{\partial}^i f_n \rangle = 0. \end{cases}$$

We note that for $i = 1, 2$

$$(4) \quad \begin{aligned} (T - z)^* \bar{\partial}^i f_n &= |T|^{\frac{1}{2}}_r [|T|^{\frac{1}{2}}_r (U - e^{i\theta})^* \bar{\partial}^i f_n] \\ &\quad + e^{-i\theta} [(|T|_r - \rho) \bar{\partial}^i f_n]. \end{aligned}$$

By equations (3) and (4), we get

$$(5) \quad \lim_{n \rightarrow \infty} \|(T - z)^* \bar{\partial}^i f_n\|_{2,D} = 0.$$

Lemma 2 and equation (5) imply

$$(6) \quad \lim_{n \rightarrow \infty} \|(I - P)f_n\|_{2,D} = 0,$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. Then by (1)

$$\lim_{n \rightarrow \infty} \|(T - z)Pf_n + 1 \otimes h_n\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n \rightarrow \infty} \|Pf_n(z) + (T - z)^{-1}(1 \otimes h_n)\| = 0$$

uniformly. Hence

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) dz + h_n \right\| = 0.$$

But by Cauchy's theorem,

$$\int_{\Gamma} Pf_n(z) dz = 0.$$

Hence $\lim_{n \rightarrow \infty} h_n = 0$. Thus V is one-to-one and has closed range. $\square$

Proposition 5. *Let $T = U|T|_r$ be a p -hyponormal operator with the property $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, $0 < p < 1$, and U unitary. Let D be a bounded disk which contains $\sigma(T)$. Then the map $V : H \rightarrow H(D)$ defined by $Vh = \widetilde{1 \otimes h} (\equiv 1 \otimes h + \overline{(T - z)W^2(D, H)})$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h .*

Proof. Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$(7) \quad \lim_{n \rightarrow \infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0.$$

Then equation (7) implies

$$(8) \quad \lim_{n \rightarrow \infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2$.

(a) If $\frac{1}{2} \leq p < 1$, then T is semi-hyponormal. Therefore, Proposition 5 follows from Lemma 4.

(b) Let $0 < p < \frac{1}{2}$. Since $T = U|T|_r$,

$$\lim_{n \rightarrow \infty} \||T|_r^{\frac{1}{2}}(U|T|_r - z)\bar{\partial}^i f_n\|_{2,D} = 0.$$

Since $\tilde{T} = |T|_r^{\frac{1}{2}}U|T|_r^{\frac{1}{2}}$, we have

$$(9) \quad \lim_{n \rightarrow \infty} \|(\tilde{T} - z)\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n)\|_{2,D} = 0.$$

Since $\tilde{T}$ is $(p + \frac{1}{2})$ -hyponormal by Lemma 3, $\tilde{T}$ is semi-hyponormal. Let $\tilde{T} = W|\tilde{T}|_r$ be the polar decomposition. Lemma 1 and equation (9) imply

$$(10) \quad \begin{cases} \lim_{n \rightarrow \infty} \|(|\tilde{T}|_r - \rho)\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n)\|_{2,D} = 0, \\ \lim_{n \rightarrow \infty} \rho\| |\tilde{T}|_r^{\frac{1}{2}}(W - e^{i\theta})^*\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n)\|_{2,D} = 0, \quad \text{and} \\ \lim_{n \rightarrow \infty} \rho\langle Q\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n), \bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n) \rangle = 0. \end{cases}$$

Now we note that for $i = 1, 2$

$$(11) \quad \begin{aligned} (\tilde{T} - z)^*\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n) &= |\tilde{T}|_r^{\frac{1}{2}}[|\tilde{T}|_r^{\frac{1}{2}}(W - e^{i\theta})^*\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n)] \\ &\quad + e^{-i\theta}[(|\tilde{T}|_r - \rho)\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n)]. \end{aligned}$$

By (10) and (11), we get

$$(12) \quad \lim_{n \rightarrow \infty} \|(\tilde{T} - z)^*\bar{\partial}^i(|T|_r^{\frac{1}{2}}f_n)\|_{2,D} = 0.$$

Lemma 2 and equation (12) imply

$$(13) \quad \lim_{n \rightarrow \infty} \|(I - P)|T|_r^{\frac{1}{2}}f_n\|_{2,D} = 0.$$

Since $|T|_r^{\frac{1}{2}}(T - z) = (\tilde{T} - z)|T|_r^{\frac{1}{2}}$ and $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, it follows from (7) that $\sigma(T) = \sigma(\tilde{T})$ and

$$(14) \quad \lim_{n \rightarrow \infty} \|(\tilde{T} - z)|T|_r^{\frac{1}{2}}f_n + |T|_r^{\frac{1}{2}}(1 \otimes h_n)\|_{2,D} = 0.$$

By (13) and (14), we have

$$\lim_{n \rightarrow \infty} \|(\tilde{T} - z)P(|T|_r^{\frac{1}{2}}f_n) + |T|_r^{\frac{1}{2}}(1 \otimes h_n)\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$ ($= \sigma(\tilde{T})$). Then for $z \in \Gamma$

$$\lim_{n \rightarrow \infty} \|P(|T|_r^{\frac{1}{2}} f_n(z)) + (\tilde{T} - z)^{-1}(|T|_r^{\frac{1}{2}}(1 \otimes h_n))\| = 0$$

uniformly. Hence

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} P(|T|_r^{\frac{1}{2}} f_n(z)) dz + |T|_r^{\frac{1}{2}} h_n \right\| = 0.$$

But by Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{\Gamma} P(|T|_r^{\frac{1}{2}} f_n(z)) dz = 0.$$

Therefore $\lim_{n \rightarrow \infty} |T|_r^{\frac{1}{2}} h_n = 0$. Since $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, $|T|_r^{\frac{1}{2}}$ is bounded below. Hence $\lim_{n \rightarrow \infty} h_n = 0$. $\square$

Theorem 6. *Let $T = U|T|_r$ be p -hyponormal, $0 < p < 1$, and U unitary. If $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, then T is subscalar of order 2.*

Proof. Consider an arbitrary bounded open disk D in the complex plane $\mathbf{C}$ and the quotient space

$$H(D) = W^2(D, H) / \overline{(T - z)W^2(D, H)}$$

endowed with the Hilbert space norm. The class of a vector f or an operator on $H(D)$ will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Let M be the operator of multiplication by z on $W^2(D, H)$. As noted at the end of section 2, M is a scalar of order 2 and has a spectral distribution Φ . Let $S \equiv \tilde{M}$. Since $\overline{(T - z)W^2(D, H)}$ is invariant under every operator $M_f, f \in C^2(D)$, we infer that S is a scalar operator of order 2 with spectral distribution $\tilde{\Phi}$.

Consider the natural map $V : H \rightarrow H(D)$ defined by $Vh = \widetilde{1 \otimes h}$, for $h \in H$, where $1 \otimes h$ denotes the constant function identically equal to h . Note that $VT = SV$. In particular $\text{ran } V$ is an invariant subspace for S . Since V is one-to-one and has closed range by Proposition 5, T is subscalar of order 2. $\square$

Corollary 7. *Every invertible p -hyponormal operator is subscalar of order 2.*

Proof. Assume $T = U|T|_r$ is an invertible p -hyponormal operator where U is unitary. Then $|T|_r$ is invertible. By [Ru, Theorem 12.33], $|T|_r^{\frac{1}{2}}$ is invertible. Therefore, $0 \notin \sigma(|T|_r^{\frac{1}{2}})$. By Theorem 6, T is subscalar of order 2. $\square$

Corollary 8. *Let $T = U|T|_r$ be a p -hyponormal operator with the property $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, $0 < p < 1$, and U unitary. If $\sigma(T)$ has interior in the plane, then T has a non-trivial invariant subspace.*

Proof. The corollary follows from Theorem 6 and [Es]. $\square$

Corollary 9. *Let T be as in Corollary 8. Then T has the property (β) .*

Proof. Since every subscalar operator has the property (β) , the corollary follows from Theorem 6. $\square$

Recall that an X in $\mathcal{L}(H, K)$ is called a quasi-affinity if it has trivial kernel and dense range. An operator A in $\mathcal{L}(H)$ is said to be a quasi-affine transform of an operator T in $\mathcal{L}(K)$ if there is a quasi-affinity X in $\mathcal{L}(H, K)$ such that $XA = TX$ (notation: $A \prec T$).

Corollary 10. *Let T be as in Corollary 8. If A is any operator such that $A \prec T$, then $\sigma(T) \subseteq \sigma(A)$.*

Proof. This is clear from [Ko, Theorem 3.2] and Corollary 9. $\square$

Corollary 11. *Under the same hypothesis as Corollary 10, $A \in \mathcal{L}(H)$ is quasi-subscalar.*

Proof. Let $X \in \mathcal{L}(H, K)$ be a quasi-affinity such that $XA = TX$. Since V (in the construction of V and S) and X are one-to-one, VX is one-to-one. Therefore VX implements the quasi-subscalar properties. Thus A is quasi-subscalar. $\square$

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