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ON THE GELFAND-KIRILLOV CONJECTURE FOR QUANTUM ALGEBRAS

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ABSTRACT. Let q be a complex not a root of unity and $\mathfrak{g}$ be a semi-simple Lie $\mathbb{C}$-algebra. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of Drinfeld and Jimbo, $U_q(\mathfrak{n}^-) \otimes U^0 \otimes U_q(\mathfrak{n})$ be its triangular decomposition, and $\mathbb{C}_q[G]$ the associated quantum group. We describe explicitly $\operatorname{Fract} U_q(\mathfrak{n})$ and $\operatorname{Fract} \mathbb{C}_q[G]$ as a quantum Weyl field. We use for this a quantum analogue of the Taylor lemma.

0. Introduction

Let q be a nonzero complex number which is not a root of unity. In this article, a $\mathbb{C}$-algebra defined by generators X_i, $1 \le i \le m$, and relations $X_i X_j = q^{a_{i,j}} X_j X_i$, $1 \le i < j \le m$, $a_{i,j} \in \mathbb{Z}$, will be called "the algebra of regular functions on an affine quantum space". Its skew field of fractions will be called the quantum Weyl field. The X_i, $1 \le i \le m$, will be called a system of q-commuting generators (SQCG).

Let $\mathfrak{g}$ be a semi-simple Lie $\mathbb{C}$-algebra of rank n. Let R be the root system associated to the choice of a Cartan subalgebra $\mathfrak{h}$. We denote by $\Delta = \{\alpha_i\}$ the set of simple roots of R, P the lattice of associated weights generated by the fundamental weights ϖ_i, $1 \le i \le n$, and $P^+ := \sum_i \mathbb{N}\varpi_i$ the lattice of dominant weights. Let G be the simply connected group associated to $\mathfrak{g}$ and $U_q(\mathfrak{g})$ the Drinfeld and Jimbo's quantized enveloping algebra. We define as in the classical case its "nilpotent" subalgebra $U_q(\mathfrak{n})$ and the quantum algebra of regular functions on the group $\mathbb{C}_q[G]$. A theorem of J. Alev and F. Dumas (cf. [1]) asserts that $\operatorname{Fract} U_q(\mathfrak{n})$ is a quantum Weyl field when $\mathfrak{g}$ is of type A_n. In [15], A. Joseph proves that this property is verified for all semi-simple Lie algebras $\mathfrak{g}$ when q is generic. We prove in this article that $\operatorname{Fract} U_q(\mathfrak{n})$ and $\operatorname{Fract} \mathbb{C}_q[G]$ are quantum Weyl fields when $\mathfrak{g}$ is semi-simple and when q is not a root of one; see [9] for the case where q is a root of one. The method we used provides a system of q-commuting generators.

Inspired by [12, Theorem 3.2], we essentially used the quantum analogue of the Taylor lemma. This lemma asserts that if 1) δ is a locally nilpotent σ-derivation (cf. 1.1) on a $\mathbb{C}$-algebra A and 2) there exists an element a such that $\delta(a) = 1$, then a is (right) transcendant on the invariant algebra A^δ and $A \simeq A^\delta[a]$.

Our results are proved as follows:

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As a first step, we give (cf. Proposition 2.1) a multi-parametered version of the Taylor lemma for the locally nilpotent action (as a bialgebra) of the Borel subalgebra $U_q(\mathfrak{b})$ on an algebra A. The difficulty encountered in the quantum case is the following : the generators E_β (β being a positive root) of the Poincaré-Birkhoff-Witt base of $U_q(\mathfrak{n})$ do not act as σ-derivations on A. To get round this problem, we can, from a reduced decomposition of the longest element w_0 in the Weyl group, define a total order on the set of these generators and obtain a decreasing sequence of subalgebras $U_q(\mathfrak{n}_\beta)$ of $U_q(\mathfrak{n})$; cf. [10, Lemma 1.7]. With the help of a result of S.Z. Levendorskii and Y.S Soibelman (cf. [17, 2.4.1]) we obtain that E_β acts as a σ-derivation on the subalgebra of $U_q(\mathfrak{n}_{\beta^<})$-invariants of A, $\beta^<$ being the root preceding β. So, we can inductively apply the Taylor lemma and prove Proposition 2.1.

As a second step, we apply Proposition 2.1, see also Assertion 2.2, to the (right) regular action of $U_q(\mathfrak{b})$ on $\mathbb{C}_q[G]$. Recall (cf. 1.4) that $\mathbb{C}_q[G]$ is generated as a space by the coefficients $c^\lambda_{\mu,\nu}$ of the simple finite dimensional $U_q(\mathfrak{g})$-modules $L_q(\lambda)$, $\lambda \in P^+$. Let $w_0 = s_{i_1} \dots s_{i_N}$ be a reduced decomposition of w_0 into a product of elementary reflections. Let $\beta = \beta_l := s_{i_1} \dots s_{i_{l-1}}(\alpha_{i_l})$ and $y_l = s_{i_1} \dots s_{i_l}$. By using the Lusztig automorphisms and the Weyl character formula, we prove that $c_\beta := c^{\varpi_{i_l}}_{y_{l-1}\varpi_{i_l},\varpi_{i_l}}$, is $U_q(\mathfrak{n}_{\beta^<})$-invariant. Moreover, with the help of the $\mathcal{R}$-matrix, we prove (cf. Proposition 2.3) that the c_β q-commute, i.e. commute up to a power of q. By the quantized Taylor lemma and the Drinfeld duality, we obtain the claimed theorem for $\operatorname{Fract} U_q(\mathfrak{n})$. We may specify the description of $\operatorname{Fract} U_q(\mathfrak{n})$ as in [1, Théorème 2.15]; cf. Theorem 3.2. We give similar results for the quantum algebras S^+_w of regular functions on a Schubert variety; cf. [14, 10.3.1 (3)]. On this subject, we remark that the elements c_β belong to the Lakshmibai-Reshetikhin base of standard monomials [16]. After localization, they generate a polynomial base.

As a third step, we show that our method works for $\mathbb{C}_q[G]$. If ρ is the sum of fundamental weights, then the elements $d_\beta = c^\rho_{y_{l-1}\rho,-y_l\rho}$, $d'_\beta = c^\rho_{y_{l-1}\rho,-y_{l-1}\rho}$ and $c^{\varpi_i}_{w_0\varpi_i,\varpi_i}$ generate the quantum Weyl field $\operatorname{Fract}\mathbb{C}_q[G]$. This theorem is a consequence of the Taylor lemma for the regular action of $U_q(\mathfrak{b}) \otimes U_q(\mathfrak{b})^{opp}$ on $\mathbb{C}_q[G]$. Note that this result was proved by A.N. Panov for $G = SL_n$ and generic q [21].

In the classical case, the Gelfand-Kirillov conjecture asks if the enveloping algebra of $\mathfrak{g}$ is a Weyl field. In [12], A. Joseph gives a generalization of the Gelfand-Kirillov conjecture, replacing the enveloping algebra of $\mathfrak{g}$ by an algebra on which $\mathfrak{n}$ acts by derivations. The title of our article must be understood in the sense of this generalization. At the present time, we do not know if $\operatorname{Fract} U_q(\mathfrak{g})$ is a quantum Weyl field. As for the classical case, this assertion may be shown when $\mathfrak{g}$ has type A_n (see [19]).

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1. Preliminaries and notations

1.1. Let $\mathfrak{g}$ be a semi-simple Lie $\mathbb{C}$-algebra of rank n. We fix a Cartan sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}$ be the triangular decomposition and $\{\alpha_i\}_i$ be a base of the root system Δ resulting from this decomposition. We note $\mathfrak{b} = \mathfrak{n} + \mathfrak{h}$ and $\mathfrak{b}^- = \mathfrak{n}^- + \mathfrak{h}$, the two opposed Borel sub-algebras. Let P be the weight lattice generated by the fundamental weights ϖ_i, $1 \leq i \leq n$, and $P^+ := \sum_i \mathbb{N}\varpi_i$ the semigroup of integral dominant weights. We denote by ρ the sum of fundamental weights. Let W be the Weyl group, generated by the reflections corresponding to the simple roots s_{α_i}. Let w_0 be the longest element of W. We denote by $(\ ,\ )$ the W-invariant form on P. We have $(\alpha_j, \varpi_i) = \delta_{ij}\frac{(\alpha_i,\alpha_i)}{2}$.

1.2. Let q be a nonzero complex number not a root of unity and $U_q(\mathfrak{g})$ be the simply connected quantized enveloping algebra, defined as in [14, 3.2.9]. Let $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{n}^-)$, be the subalgebra generated by the canonical generators E_{α_i}, resp. F_{α_i}, of positive, resp. negative, weights. For all λ in P, let $\tau(\lambda)$ be the corresponding element in the algebra U^0 of the torus of $U_q(\mathfrak{g})$. We have the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U^0 \otimes U_q(\mathfrak{n})$. We set

$$U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \otimes U^0, \qquad U_q(\mathfrak{b}^-) = U_q(\mathfrak{n}^-) \otimes U^0. \tag{1.2.1}$$

$U_q(\mathfrak{g})$ is endowed with a structure of Hopf algebra with comultiplication Δ, and antipode S.

We fix the following notations, where t is a complex not root of one, n a nonnegative integer and α a positive root : $[n]_t = \frac{1-q^t}{1-q}$, $[n]_t! = [n]_t[n-1]_t \dots [1]_t$, $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$.

1.3. For w in W, let T_w be the Lusztig automorphism [18] associated to w. We fix a decomposition of the longest element of the Weyl group $w_0 = s_{i_1} \dots s_{i_N}$, where $N = dim\,\mathfrak{n}$. This decomposition settles an order, denoted $<$, into the set Δ^+ of positive roots : $\beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}), \dots, \beta_2 = s_{i_1}(\alpha_{i_2}), \beta_1 = \alpha_{i_1}$. Then, we introduce the following elements in $U_q(\mathfrak{n})$: $E_{\beta_s} = T_{i_1} \dots T_{i_{s-1}}(E_{i_s})$. We define in the same way $F_{\beta_s} = T_{i_1} \dots T_{i_{s-1}}(F_{i_s})$.

We know (cf. [18]) that these elements generate a Poincaré-Birkoff-Witt base of $U_q(\mathfrak{n})$. We have, by [22], see also [10, Lemma 1.7]:

Proposition. *Let $U_q(\mathfrak{n}_\beta)$ be the space generated by the ordered products $\prod_{\rightarrow} E_\alpha^{k_\alpha}$, $\alpha \in \Delta^+$, $\alpha \leq \beta$, $k_\alpha \in \mathbb{N}$. Then $U_q(\mathfrak{n}_\beta)$ is a subalgebra of $U_q(\mathfrak{n})$. Moreover, if $\mu < \beta$, we have $E_\mu E_\beta - q^{-(\mu,\beta)} E_\beta E_\mu \in \sum_{\alpha<\beta} U_q(\mathfrak{n}_\beta)E_\alpha$.* □

1.4. The dual $U_q(\mathfrak{g})^*$ is endowed with a structure of a left, resp. right, $U_q(\mathfrak{g})$-module by $u.c(a) = c(au)$, resp. $c.u(a) = c(ua)$, u, $a \in U_q(\mathfrak{g})$, $c \in U_q(\mathfrak{g})^*$. In the same way, if M is a left $U_q(\mathfrak{g})$-module, we endow the dual M^* with the structure of a right $U_q(\mathfrak{g})$-module by $\xi u(v) = \xi(uv)$, $u \in U_q(\mathfrak{g})$, $\xi \in M^*$, $v \in M$.

For all λ in P^+, let $L_q(\lambda)$ be the simple $U_q(\mathfrak{g})$-module with highest weight λ. We know that $L_q(\lambda)$ verifies the Weyl character formula, for all w in W we denote by $v_{w\lambda}$ the extremal vector of weight $w\lambda$. For all integral dominant weight λ, we fix a weight base (v_μ), $\mu \in \Omega(L_q(\lambda))$, of $L_q(\lambda)$. We denote by (v_μ^*) its dual base. From [14, 10.2], we have the assertion

Assertion. *Let λ be an integral dominant weight and w an element of the Weyl group. Fix a space M and an isomorphism $\phi : M \to L_q(\lambda)^*$. We can endow M with the structure of a right $U_q(\mathfrak{g})$-module by : $v^*.u = \phi^{-1}(\phi(v^*)T_w(u))$, $v^* \in M$. Then the $U_q(\mathfrak{g})$-module M is isomorphic to $L_q(\lambda)^*$ and $\phi^{-1}(v_{w\lambda}^*)$, resp. $\phi^{-1}(v_{ww_0\lambda}^*)$, is its highest weight, resp. lowest weight, vector.* □

For all ξ in $L_q(\lambda)^*$ and v in $L_q(\lambda)$, let $c_{\xi,v}^\lambda$ in $U_q(\mathfrak{g})^*$ given by $c_{\xi,v}^\lambda(u) = \xi(uv)$, $u \in U_q(\mathfrak{g})$. Then we have $u.c_{\xi,v}^\lambda = c_{\xi,uv}^\lambda$ and $c_{\xi,v}^\lambda.u = c_{\xi u,v}^\lambda$. If ξ, resp. v, has weight

ν, resp μ, we set (if no confusion occurs) $c^\lambda_{\nu,\mu} = c^\lambda_{\xi,v}$. For all integral dominant weight λ, let $C(\lambda)$, resp. $C^+(\lambda)$, be the space generated by the $c^\lambda_{\xi,v}$, resp. $c^\lambda_{\xi,\lambda}$, $\xi \in L_q(\lambda)^*$, $v \in L_q(\lambda)$. We note $R = \mathbb{C}_q[G] = \bigoplus_{\lambda\in P^+} C(\lambda)$, $R^+ = \bigoplus_{\lambda\in P^+} C^+(\lambda)$. R^+ and R are subalgebras of the Hopf dual of $U_q(\mathfrak{g})$.

For w in W, we define the quantized algebra S^+_w of regular functions on the Schubert variety (see [14], [15] for details) : S^+_w is the inductive limit of $(c^\lambda_{w\lambda,\lambda})^{-1}V^+_w(\lambda)^*$, for λ in P^+, where $V^+_w(\lambda)^*$ is the dual of the Demazure module $V_w(\lambda)$, naturally identified as a quotient of $C^+(\lambda)$.

1.5. We know that $U_q(\mathfrak{g})$ is an almost cocommutative Hopf algebra; cf. [11]. Let $\mathcal{R} = \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}$ be the $\mathcal{R}$-matrix of $U_q(\mathfrak{g})$. This satisfies $\mathcal{R}\Delta = \Delta^t\mathcal{R}$, where t is the twist. From this property it follows easily that:

$$(c.\mathcal{R}_{(1)})(c'.\mathcal{R}_{(2)}) = (\mathcal{R}_{(2)}.c')(\mathcal{R}_{(1)}.c), \quad c,\, c' \in \mathbb{C}_q[G]. \tag{1.5.1}$$

We recall the expression of the $\mathcal{R}$-matrix as an ordered product [17, 3.3]:

$$\mathcal{R} = (\prod_{\alpha\in\Delta^+} exp_{q_\alpha^{-2}}((1-q_\alpha^{-2})E_\alpha \otimes F_\alpha))\tau(\gamma)\otimes\tau(\gamma), \tag{1.5.2}$$

where $\gamma \in P$, $exp_t(x) = \sum_{n\geq 0} \frac{x^n}{[n]_t!}$.

2. A quantum Taylor lemma

2.1. We have the following lemma, whose proof is an analogue to [20, 1.1], [8, Proposition 1.1]:

Lemma. *Let A be an $\mathbb{C}$-algebra, σ a $\mathbb{C}$-automorphism of A, δ a σ-derivation of A, i.e. $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$, $a, b \in A$. Let A^δ be the algebra of δ-invariants in A. Suppose that 1) δ is locally nilpotent, 2) $\sigma\delta\sigma^{-1} = Q\delta$, $Q \in \mathbb{C}^*$, Q not root of one, 3) there exists a in A such that $\delta(a) \in \mathbb{C}^*$. Then $A = A^\delta[a]$ and a is (right) transcendantal on A^δ, i.e. $A = \bigoplus_{p\geq 0} A^\delta a^p$.*

Proof. By 2), A^δ is σ-stable. Moreover, we have : $\delta^p(a^p) = [p]_{Q^{-1}}!\delta(a)^p$. This implies the direct sum in the claimed equality. Let u be in A, with degree p, i.e. p is the greatest integer such that $u_0 := \delta^p(u) \neq 0$. Clearly, the element u_0 is in A^δ. We prove the result by induction on p by considering $u - \frac{1}{[p]_{Q^{-1}}!\delta(a)}\sigma^{-p}(u_0)a^p$, of degree $\leq p-1$. $\square$

Let A be an $\mathbb{C}$-algebra such that $U_q(\mathfrak{b})$ acts (as a bialgebra) on A, i.e. A is a $U_q(\mathfrak{b})$-module and $a(uv) = a_{(1)}ua_{(2)}v$, $u, v \in A$, $a \in U_q(\mathfrak{b})$, $\Delta(a) = a_{(1)} \otimes a_{(2)}$. Suppose that this action is locally finite. We set $A^0 = A$ and we note A^l, $1 \leq l \leq N$, the algebra of $U_q(\mathfrak{n}_{\beta_l})$-invariants in A. This proposition follows from the lemma.

Proposition. *Let A be an algebra defined as above. Suppose that, for all β in Δ^+, there exists a_β in A such that $E_\alpha.a_\beta = \delta_{\alpha\beta}$, $\alpha \leq \beta$, where $\delta_{\alpha\beta}$ is the Kroenecker symbol.*

Then, for all l, $1 \leq l \leq N$, we have

$$A = \bigoplus_{(k_1,\dots,k_l)\in\mathbb{N}^l} A^l a^{k_l}_{\beta_l} \dots a^{k_1}_{\beta_1}.$$

Proof. We note $\phi : U_q(\mathfrak{b}) \to End(A)$, the natural morphism for this action. By Proposition 1.3 and [17, 2.4.1], $\delta := \phi(E_{\beta_l})$ is a $\phi(\tau(\beta_l))$-derivation on A^{l-1}. The conditions of the previous lemma are satisfied because 1) δ is locally nilpotent on A^{l-1}, 2) $\tau(\beta_l)$ and E_{β_l} q-commute, 3) $a := a_{\beta_l}$ is in A^{l-1} and satisfies $\delta(a) = 1$ by the definition. The proposition is obtained by induction on l using the previous lemma. □

2.2. We shall see that if A is one of the algebras considered in the introduction, then the elements a_β of Proposition 2.1 exist in some localization of A, and not in the algebra A. For the classical case, cf. [12, Theorem 2.6], it is enough to localize by a set S generated by $\mathfrak{n}$-invariant elements in A. We can then apply the Taylor lemma to A_S. In the quantum case, the Taylor lemma needs some refinements. We slightly modify Lemma 2.1 to get

Assertion. *Let C be a noetherian domain on $\mathbb{C}$, σ a $\mathbb{C}$-automorphism of C and δ be a σ-derivation on C which satisfies 1) and 2) of Lemma 2.1. Suppose s (nonzero) and s' are q-commuting in C and such that $\delta(s') = s \in C^\delta$. Let $a = s's^{-1}$ in* Fract C *and $M = \bigcup_{p\geq 0} Cs^{-p}$. Then δ acts on M and (as spaces) $C \subset M = \bigoplus_{p\geq 0} M^\delta a^p$.*

Proof. The element s in δ-invariant, so δ extends as a locally nilpotent derivation on M. Clearly, C is a subset of M and $\delta(a) = 1$. The direct sum is proved as in Lemma 2.1. As s and s' q-commute, we have $\bigoplus_{p\geq 0} M^\delta a^p \subset M$. The reverse inclusion is an easy induction as in the proof of Lemma 2.1. □

We now give a condition on A which implies that Fract A is isomorphic to a quantum Weyl field.

Definition. Let A be a noetherian domain. We say that A verifies the property $(\mathcal{P})$ if the following hypotheses are verified:

(i) $U_q(\mathfrak{b})$ acts (as a Hopf algebra) on the $\mathbb{C}$-algebra A and this action is locally finite. Let $B = A^N$ be the subalgebra of $U_q(\mathfrak{n})$-invariant elements in A.

(ii) B is generated by elements c^i, $1 \leq i \leq m$, and $B = \mathbb{C}[c^m]\dots[c^2][c^1]$ is an algebra of functions on a quantum affine space with SQCG $\{c^1, \dots, c^m\}$.

(iii) There exist nonzero elements c_β, $\beta \in \Delta^+$, in A which q-commute, q-commute with c^i, $1 \leq i \leq m$, and satisfy : $E_\alpha.c_\beta = \delta_{\alpha\beta} c_\beta^>$, $\alpha \leq \beta$, where $c_\beta^>$ is either c_γ, $\gamma > \beta$, E_β-invariant, or c^j, $1 \leq j \leq m$.

Proposition. *Let A be a noetherian domain. If A satisfies the property $(\mathcal{P})$, then* Fract A *is isomorphic to a quantum Weyl field. To be precise, if B is the algebra of $U_q(\mathfrak{n})$-invariant elements in A, then* Fract A *is isomorphic to the skew field of fractions of $B[c_{\beta_N}]\dots[c_{\beta_2}][c_{\beta_1}]$.*

Proof. Suppose that A satisfies the property $(\mathcal{P})$. For all $\beta = \beta_i$ in Δ^+, note S_i the multiplicative set generated by $s_i := c_\beta^>$; cf. (iii). We can define the following elements : $a_\beta = (c_\beta^>)^{-1} c_\beta \in$ Fract A.

In the context of the previous assertion, we set (improperly) $C_S = \bigcup_{p\geq 0} Cs^{-p}$, where S is the multiplicative set generated by s. Recall that A is a domain. The conditions of the assertion are easily verified from the property $(\mathcal{P})$. We have:

$$(*) \qquad \begin{aligned} A = & A^0 \subset A^1_{S_1}[a_{\beta_1}] \subset A^2_{S_2}[a_{\beta_2}][s_1^{-1}, a_{\beta_1}] \\ & \subset \dots \subset A^N_{S_N}[a_{\beta_N}][s_{N-1}^{-1}, a_{\beta_{N-1}}, \dots, s_1^{-1}, a_{\beta_1}]. \end{aligned}$$

Moreover, from (iii), $c^>_{\beta_N} \in B$, so $c_{\beta_N} \in B[a_{\beta_N}]$. Inductively, we can prove that $B[a_{\beta_N}] \dots [a_{\beta_1}]$ contains all the s_i. This and (*) imply that $\operatorname{Fract} A$ is isomorphic to the skew field of fractions of $B[a_{\beta_N}] \dots [a_{\beta_1}]$. The property $(\mathcal{P})$ (iii) asserts that the a_α are in the skew field of fractions of $B[c_{\beta_N}] \dots [c_{\beta_2}][c_{\beta_1}]$. By the Taylor lemma, these extensions are (right) transcendantal. Our proposition follows. □

2.3. Fix $1 \le l \le N$ and $\beta = \beta_l$. The reduced decomposition of w_0 being fixed as in 1.3, we define the elements y_l of the Weyl group : $y_0 = Id$, $y_l = s_{i_1} s_{i_2} \dots s_{i_l}$, $l > 0$. Then, we introduce in R^+ : $c_\beta = c^{\varpi_{i_l}}_{y_{l-1}\varpi_{i_l}, \varpi_{i_l}}$, $c^>_\beta = c_\beta . E_\beta$.

Lemma. *Let $I_l = \{p, l < p \le N \mid i_p = i_l\}$. If I_l is empty, $c^>_{\beta_l} = c^{\varpi_{i_l}}_{w_0\varpi_{i_l}, \varpi_{i_l}}$. If not, let l' be the minimal element in I_l; then $c^>_{\beta_l} = c_{\beta_{l'}}$ (up to a multiplicative scalar).*

Proof. Fix l. Set $j = i_l$. We show the second assertion of the lemma; the first is similar. We have $y_{l'-1}(\varpi_j) = y_{l-1} s_{i_l} \dots s_{i_{l'-1}}(\varpi_j) = y_{l-1} s_{i_l}(\varpi_j) = y_{l-1}(\varpi_j - \alpha_j) = y_{l-1}(\varpi_j) - \beta$. From Assertion 1.4, with $\lambda = \varpi_j$ and $w = y_{l-1}$, it is enough to prove that $v^*_{\varpi_{i_l}} . E_{\varpi_{i_l}} = v_{s_{i_l}\varpi_{i_l}}$. This is clear by the Weyl character formula and we can conclude the lemma. □

Proposition. *Let S be the multiplicative set generated by the $c^>_\alpha$, $\alpha \in \Delta^+$. S is a Ore set in R^+. Let $a_\beta = (c^>_\beta)^{-1} c_\beta \in S^{-1}R^+$. We have $a_{\alpha'} a_\alpha = q^{(\alpha, \alpha')} a_\alpha a_{\alpha'}$, α, $\alpha' \in \Delta^+$, $\alpha' < \alpha$.*

Proof. As in the previous proof, we fix $\beta = \beta_l$ in Δ^+, $1 \le l \le N$, and $i_l = j$. From Assertion 1.4, the extremal vector $v^*_{y_{l-1}\varpi_j}$ in $L_q(\varpi_j)^*$ is annihilated by the right action of E_α, $\alpha > \beta$, and F_α, $\alpha \le \beta$. So, this holds for c_β. From (1.5.1) and (1.5.2), we can deduce that the c_β, $\beta \in \Delta^+$, q-commute. By Lemma 2.3 and by [14, Corollary 9.1.4], this is also true for the $c^>_\beta$, $\beta \in \Delta^+$. The first assertion of the proposition follows from loc. cit., [Lemma A.2.9], and loc. cit., [Lemma 9.1.10]. We used loc. cit., [Proposition 9.1.5] to calculate the exponent of q in the formula. □

Remark. The fact that S is a Ore set is not essential for the next section. Indeed, the elements a_β defined above exist at least in $\operatorname{Fract} R^+$, because R^+ is a noetherian domain, cf. [14, 9.1.11].

3. Applications

3.1. In this section, we give a list of quantum algebras which satisfy the desired property.

Theorem. *The skew field of fractions of the algebra R^+, resp. S^+_w, resp. $U_q(\mathfrak{n})$, resp. $U_q(\mathfrak{b})$, is isomorphic to a quantum Weyl field of dimension $N + n$, resp. $l(w) + n$, resp. N, resp. $N + n$.*

Proof. By [14, Chapter 7, Chapter 9], all the algebras in the claim are noetherian domains. Let's verify the assertions of the property $(\mathcal{P})$.

For R^+, the action of (i) is the right regular action, which is locally finite. The algebra of $U_q(\mathfrak{n})$-invariant elements in R^+ is generated by the $c^{\varpi_i}_{w_0\varpi_i, \varpi_i}$, $1 \le i \le n$, which q-commute by [14, 9.1.4]. Then, (iii) is given by Lemma 2.3 and Proposition 2.3.

The assertion for S^+_w is similar. We may consider, without loss of generality, the case where $w = s_{i_{N-l(w)+1}} \dots s_{i_N}$. Set $\beta := \beta_{N-l(w)+1}$. We prove the theorem with the help of (i) the right action of $U_q(\mathfrak{n})$, (ii) the $U_q(\mathfrak{n})$-invariant elements $c^{\varpi_i}_{w\varpi_i, \varpi_i}$,

$1 \leq i \leq n$, (iii) the q-commuting elements c_α, $\alpha \leq \beta$. Indeed, (i) and (iii) are clear, and (ii) follows from the fact that $(V_w^+(\lambda)^*)^{U_q(\mathfrak{n})}$ is generated by $c^\lambda_{w\lambda,\lambda}$.

Let φ be the restriction homomorphism from $U_q(\mathfrak{g})^*$ to $U_q(\mathfrak{b}^-)^*$ and $J^- = Ker\varphi \cap R$. We know (cf. [14, 9.1.10, 9.2.11]) that φ restricts to an embedding from R^+ to R/J^-, and this map is surjective up to localization. It is also well known (cf. [3]) that there exists an algebra antihomomorphism $R/J^- \simeq U_q(\mathfrak{b})$. The theorem being true for R^+, it is true for $U_q(\mathfrak{b})$. Let's consider now the previous isomorphism extent to the skew field of fractions. We remark (cf. [4, Lemme 3.4], [5, I Prop. 4.2]) that the image of a_β is in $\operatorname{Fract} U_q(\mathfrak{n})$. Thus, we can conclude by (1.2.1). □

As in [15, Corollaire 6], we have the following corollary.

Corollary. *Let P minimal primitive ideal of $U_q(\mathfrak{g})$. Then $\operatorname{Fract} U_q(\mathfrak{g})/P$ is isomorphic to a quantum Weyl field of dimension $2N$.* □

3.2. In this section section, we make more explicit the system of q-commuting generators (SQCG) of $\operatorname{Fract} U_q(\mathfrak{n})$ for the classical simple Lie algebras $\mathfrak{g}$. Recall that these generators are, from the proof of Theorem 3.1, the images of the elements c_β, $\beta \in \mathcal{R}^+$, by the Drinfeld (anti)homomorphism, followed by the natural projection on $U_q(\mathfrak{n})$.

We know [6] that there is a natural embedding from $L_q(\lambda)^*$ to $U_q(\mathfrak{n})$ which maps the lowest weight vector of $L_q(\lambda)^*$ to 1. It maps the highest weight vector to an element e_λ of the q-center of $U_q(\mathfrak{n})$; cf. [7]. An element of $U_q(\mathfrak{n})$ will be called almost maximal, and be noted e^i_λ, if it is the image of a vector $v_{-s_i w_0 \lambda} \in L_q(\lambda)^*$, $1 \leq i \leq n$, $\lambda \in P^+$. Remark that those elements can be explicitly computed with the help of [6, Lemme 3.3].

With the standard notations of [2, Planches I à IV], we recall the canonical embeddings of Dynkin diagrams:

$$A_1 \subset \ldots \subset A_n, \;\; A_1 \subset B_2 \subset \ldots \subset B_n, \;\; A_1 \subset C_2 \subset \ldots \subset C_n, \;\; A_3 \subset D_4 \subset \ldots \subset D_n.$$

If X is a Dynkin label for a classical simple Lie algebra, we denote by X^- the previous label for the embedding sequence above. We can inductively define the reduced decomposition of the longest element of the Weyl group $w_0(X)$ for the Lie algebra of type X by :

$$w_0(X) = w_0(X^-). \begin{cases} s_n \ldots s_2 s_1, & \text{if } X = A_n, \\ s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1 & \text{if } X = B_n \text{ or } C_n, \\ s_1 \ldots s_{n-2} s_n s_{n-1} \ldots s_1 & \text{if } X = D_n. \end{cases}$$

This decomposition of w_0 permits us to obtain inductively our SQCG.

Theorem. *The system of q-commuting generators corresponding to the simple classical Lie algebra $\mathfrak{g}$ of type X is inductively given by*

$$SQCG(X) = SQCG(X^-) \cup \bigcup_{i=1}^{n} e_{\varpi_i} \cup \begin{cases} \varnothing & \textit{if } X = A_n, \\ \bigcup_{i=1}^{n-1} e^i_{\varpi_i} & \textit{if } X = B_n \textit{ or } C_n, \\ \bigcup_{i=1}^{n-2} e^i_{\varpi_i} & \textit{if } X = D_n. \end{cases}$$

□

Remark. This theorem is a generalization of [1, Théorème 2.15] for the classical Lie algebras. Note that, except for e_{ϖ_n} if $\mathfrak{g}$ has type B_n, C_n, D_n and $e_{\varpi_{n-1}}$, if $\mathfrak{g}$ has type D_n, all those elements can be obtained as quantum determinants of a "basic" matrix. This can be obtained as in [6] (see also [14, 7.5.5]) by considering exterior powers of $L_q(\varpi_1)$.

3.3. Now, we give the proof of a similar theorem for the algebra $R = \mathbb{C}_q[G]$. For all i, j, $1 \leq i, j \leq N$, we denote by $R^{i,j}$ the subalgebra of elements in R which are invariant for the right action of $U_q(\mathfrak{n}_{\beta_j})$ and for the left action of $U_q(\mathfrak{n}_{\beta_i})$ (recall that these actions commute). Set $B = R^{N,N}$.

Fix $\beta = \beta_l$. We define the following elements in $C(\rho)$:

$$d_\beta = c_{y_{l-1}\rho, -y_l\rho}, \qquad d'_\beta = c_{y_{l-1}\rho, -y_{l-1}\rho}.$$

By Assertion 1.4, we prove that d_β is invariant for the left action of E_α, $\alpha \leq \beta$, and of F_α, $\alpha > \beta$. Moreover, d_β is invariant for the right action of E_α, $\alpha < \beta$, and of F_α, $\alpha \geq \beta$. In the same way, d'_β is invariant for the left and right action of E_α, $\alpha < \beta$, and of F_α, $\alpha \geq \beta$. Hence, by (1.5.1) and (1.5.2), the elements d_α and d'_α, $\alpha \in \Delta^+$, q-commute. We have clearly $d_\beta \in R^{l,l-1}$ and $d_\beta \in R^{l-1,l-1}$. Moreover, up to a nonzero multiplicative scalar, $E_\beta.d'_\beta = d_\beta$ and $d_\beta.E_\beta = d'_{\beta_{l+1}}$, $l \neq N$, $d_\beta.E_\beta = c_{w_0\rho,\rho} \in B$, $l = N$. $\mathbb{C}_q[G]$ being noetherian [14, 9.2.2], we can easily modify the proof of Proposition 2.2 to obtain:

Theorem. $\operatorname{Fract}\mathbb{C}_q[G]$ *is isomorphic to a quantum Weyl field of dimension* $2N+n$. *To be more precise,* $\operatorname{Fract}\mathbb{C}_q[G]$ *is isomorphic to the skew Weyl field of fractions of* $B[d_{\beta_N}][d'_{\beta_N}]\dots[d_{\beta_2}][d'_{\beta_2}][d_{\beta_1}][d'_{\beta_1}]$. □

The generators of B, the d_β and d'_β, can be easily expressed as a product of elements of $C(\varpi_i)$, $1 \leq i \leq n$. Consider the algebra $\mathbb{C}_q[SL_2]$ generated by the elements of the quantum matrix : $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then the system of q-commuting generators provided by the theorem is $\{b, a, c\}$.

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