HILBERT COEFFICIENTS AND THE ASSOCIATED GRADED RINGS

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ABSTRACT. Let $(R, \mathfrak{m})$ be a d -dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R . In this paper, we prove that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1$ for some minimal reduction J of I , then $\text{depth } G(I) \geq d - 2$.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a d -dimensional Cohen-Macaulay local ring with infinite residue field and I be an $\mathfrak{m}$ -primary ideal of R . Let $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of R . During the past years, many commutative algebraists tried to estimate the depth of $G(I)$ for ideals I having good properties. In 1978, Valabrega and Valla obtained in [6] that $G(I)$ is Cohen-Macaulay if and only if there exists a minimal reduction J of I such that $I^n \cap J = I^{n-1}J$ for all n . Later on, Guerrieri studied the so called Valabrega-Valla module and made the following conjecture in her paper [1].

Conjecture 1. *If $\sum_{n=1}^{\infty} \lambda(I^n \cap J/I^{n-1}J) = t$ for some minimal reduction J of I , then $\text{depth } G(I) \geq d - t$.*

On the other hand, Sally in [5] studied the depth of $G(\mathfrak{m})$ by considering the classical bound of Abhyankar on the multiplicity e of R ; namely, $e \geq \mu(\mathfrak{m}) - d + 1$, where $\mu(I)$ stands for the minimal number of a generating set of I . She first studied the case of rings with minimal multiplicity, i.e., $e = \mu(\mathfrak{m}) - d + 1$, then the cases $e - (\mu(\mathfrak{m}) - d + 1) = 1, 2$. Recently, Huckaba and Marley showed in [4] that if one considers the first coefficient $e_1(I)$, then $e_1(I)$ is bounded above by $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$ for any minimal reduction J of I , and later Huckaba and Vaz Pinto independently showed that $\text{depth } G(I) \geq d - 1$ if $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$ for some minimal reduction J of I .

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In a similar fashion to what Sally did with the Abhyankar's bound, we can raise the following conjecture on the depth of $G(I)$ by considering the difference of $e_1(I)$ and $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$.

Conjecture 2. *If $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = t$, then $\text{depth } G(I) \geq d - 1 - t$.*

One can see in section 2 that Conjecture 1 holds if we can give an affirmative answer to Conjecture 2. In this paper, we are able to show, by using a method developed in [8] concerning the *Sally module* defined in [7], that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1$, then $\text{depth } G(I) \geq d - 2$. Hence the Conjecture 1 holds if $t \leq 2$.

2. PRELIMINARIES

Throughout, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R and J a minimal reduction of I . Let $G(I)$ be the associated graded ring of R . An element $x \in I \setminus I^2$ is called *superficial* for I if $(0 :_{G(I)} x^*)_n = 0$ for all n sufficiently large. Here, x^* denotes the image of x in $I/I^2 \subseteq G(I)$. A sequence $x_1, \dots, x_k$ is called *superficial sequence* for I if x_1 is superficial for I and x_i is superficial for $I/(x_1, \dots, x_{i-1})$. In [3], Huckaba proved that if $\dim R = 1$, then $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$ for any minimal reduction J of I ; therefore it is easy to see the following:

Lemma 2.1. *If $\dim R = d$ and $x_1, \dots, x_{d-1} \in J$ is a superficial sequence for I , then*

$$e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_1, \dots, x_{d-1}))).$$

In [4], Huckaba and Marley gave in Lemma 2.2 a sufficient conditions for $G(I)$ having positive depth. We restate it here in the following special form.

Lemma 2.2. *Let $x \in J$ be a superficial element for I . If $\text{depth } G(I/(x)) > 0$, then $\text{depth } G(I) > 0$.*

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R and J be a minimal reduction of I . Suppose that*

$$\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1.$$

Let $x \in J$ be a superficial element for I . If $\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x))) = e_1(I)$, then $\text{depth } G(I) > 0$.

Proof. The conclusion follows from Lemma 2.2 and the fact ([3, Theorem 3.1]) that if $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) = e_1(I)$, then $\text{depth } G(I) \geq \dim R - 1$. $\square$

The following two lemmas are easy to derive; we leave the proofs to the reader.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R and $J = (x_1, x_2, x_3)$ be a minimal reduction of I . Let $N \geq 2$. Suppose that $I^n \cap (x_i, x_j) \subseteq I^{n-1}J \ \forall n < N$ and $\forall i, j \in \{1, 2, 3\}$. Then $\forall n < N$ and $\forall m \geq 1$,*

- (1) $I^n : x_i = I^{n-1} \ \forall i$.
- (2) $I^n J^m : x_i = I^n J^{m-1} \ \forall i$.
- (3) $I^n : x_2 = I^n : x_3 = I^{n-1} \pmod{x_1}$.
- (4) $I^n J^m : x_2 = I^n J^m : x_3 = I^n J^{m-1} \pmod{x_1}$.
- (5) Let $\lambda_0, \dots, \lambda_t$ be either unit or 0 but not all 0. Let $s \in R$ be such that $s(\sum_{i=0}^t \lambda_i x_1^{n-i} x_2^i) \in I^n J^m$. Then $s \in I^n J^{m-t}$ if $m \geq t$ or $s \in I^{n-t+m}$ if $m < t$.

If, moreover, $I^N \cap (x_1) \subseteq I^{N-1}J$, then $I^N : x_1 = I^{N-1}$.

Lemma 2.5. *Let $N \geq 2$. If $a_1, \dots, a_n \in I^N$ not all in $I^{N-1}J$, then there are only finite number of units λ such that $\sum_{i=1}^n a_i \lambda^{i-1} \in I^{N-1}J$.*

The following proposition presents a relation between the two conjectures stated in the previous section.

Proposition 2.6. *If Conjecture 2 has a positive answer, then so does Conjecture 1.*

Suppose that Conjecture 2 holds. Let I be an $\mathfrak{m}$ -primary ideal of R and let J be a minimal reduction of I . Let $t = \sum_{n=1}^{\infty} \lambda(\frac{I^n \cap J}{I^{n-1}J})$. Then, by Lemma 2.1, for any superficial sequence $x_1, \dots, x_{d-1} \in J$ for I ,

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) &= \sum_{n=1}^{\infty} \lambda(\frac{I^n \cap (x_1, \dots, x_{d-1}) + I^{n-1}J}{I^{n-1}J}) \\ &\leq \sum_{n=1}^{\infty} \lambda(\frac{I^n \cap J}{I^{n-1}J}) = t. \end{aligned}$$

Let k be the least integer such that $\lambda(\frac{I^k \cap J}{I^{k-1}J}) \neq 0$. Then, by [2, Lemma 3.1], $\lambda(\frac{I^k \cap (x_1, \dots, x_{d-1}) + I^{k-1}J}{I^{k-1}J}) < \lambda(\frac{I^k \cap J}{I^{k-1}J})$, so that $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) \leq t - 1$. Therefore, $\text{depth } G(I) \geq d - t$ by assumption. This shows that Conjecture 1 holds.

3. MAIN THEORY

The goal of this section is to prove the following:

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R . Suppose that there is a minimal reduction J of I such that $\sum_{n=0}^{\infty} \lambda(I^{n+1}/I^n J) - e_1(I) = 1$. Then $\text{depth } G(I) \geq d - 2$.*

By [5], it suffices to consider the case $d = 3$, so we assume in the following that $d = 3$. We also assume now that Theorem 3.1 doesn't hold. We shall reach a contradiction later.

Let $x_1, x_2 \in J$ be a superficial sequence of I ; then, by Corollary 2.3, we have for $i = 1, 2$,

$$(1) \quad \sum_{n=1}^{\infty} \lambda(I^n / (I^{n-1}J + I^n \cap (x_i))) - e_1(I) = 1.$$

Moreover, by Lemma 2.1, we have

$$(2) \quad \sum_{n=1}^{\infty} \lambda(I^n / (I^{n-1}J + I^n \cap (x_1, x_2))) - e_1(I) = 0.$$

Let $\{x, y, z\}$ be a minimal generating set of J . Consider the exact sequence:

$$0 \longrightarrow T_{k,n} \longrightarrow \bigoplus^{\binom{n+2}{2}} I^k / I^{k-1}J \xrightarrow{\phi_n} S_{k,n} = I^k J^n / I^{k-1}J^{n+1} \longrightarrow 0,$$

where $\phi_n = (x^n, x^{n-1}y, x^{n-1}z, \dots, z^n)$ and $T_{k,n} = \ker(\phi_n)$. From the proof of [8, Theorem 2.4], we see that there is a unique integer $N \geq 2$ such that $T_{N,n} \neq 0$ for some positive integer n . Notice that N is independent of the choice of a generating set of J since $S_{k,n}$ and $I^k / I^{k-1}J$ are. As $R/\mathfrak{m}$ is infinite, we may, after elementary transformation of x, y and z , require that $\{x, y, z\}$ satisfies the following conditions.

Proposition 3.2. *There is a generating set $\{x, y, z\}$ of J satisfying the following conditions:*

- (i) $\{x, y\}$, $\{x, z\}$, $\{y, z\}$ and $\{z\}$ are all superficial sequences for I .
- (ii) $I^n \cap (x)$, $I^n \cap (y)$ and $I^n \cap (z)$ are all contained in $I^{n-1}J \forall n$.
- (iii) $I^n \cap (x, y)$, $I^n \cap (x, z)$ and $I^n \cap (y, z)$ are all contained in $I^{n-1}J \forall n \neq N$.

Moreover,

$$\begin{aligned} \lambda\left(\frac{I^N \cap (x, y) + I^{N-1}J}{I^{N-1}J}\right) &= \lambda\left(\frac{I^N \cap (x, z) + I^{N-1}J}{I^{N-1}J}\right) \\ &= \lambda\left(\frac{I^N \cap (y, z) + I^{N-1}J}{I^{N-1}J}\right) = 1. \end{aligned}$$

Proof. (Sketch.) Notice that (ii) follows from (i) and (1); therefore we need only to show (i) and (iii).

Let $\{x, y, z\}$ be a generating set of J . Let n be an integer such that $T_{N,n} \neq 0$. Then there are $a_{ijk} \in I^N$ not all in $I^{N-1}J$ such that $\sum_{i+j+k=n} a_{ijk}x^i y^j z^k \in$

$I^{N-1}J^{n+1}$. By Lemma 2.5, we may, after elementary transformation of x, y and z , assume that a_{n00} , a_{0n0} and a_{00n} are not in $I^{N-1}J$.

Next, we can use prime avoidance and Corollary 2.3 to replace $\{x, y, z\}$ by elements of the set $\{x + \alpha y + \beta z\}$ so that the condition (i) holds without changing the condition that the coefficients of x^n , y^n and z^n are not in $I^{N-1}J$.

Since $\sum_{i+j+k=n} a_{ijk}x^i y^j z^k \in (I^{N-1}J)J^n$, there are $a_1, a_2, a_3 \in I^{N-1}J$ such that $a_{n00} - a_1 \in I^N \cap (y, z)$, $a_{0n0} - a_2 \in I^N \cap (x, z)$ and $a_{00n} - a_3 \in I^N \cap (x, y)$. Thus the condition (iii) holds by condition (i) and (2). $\square$

Let $\{x, y, z\}$ be a generating set of J satisfying the conditions of Proposition 3.2. Let t be chosen least such that $T_{N,n} \neq 0$ for $n \geq t$. Then there are elements $\{a_{ijk} \mid i+j+k=t\}$ not all in $I^{N-1}J$ such that $\sum a_{ijk}x^i y^j z^k \in I^{N-1}J^{t+1}$. If $\{a_{0jk} \mid j+k=t\} \subseteq I^{N-1}J$, then $x(\sum a_{ijk}x^{i-1}y^j z^k) \in I^{N-1}J^{t+1}$, so that by Lemma 2.4, $\sum a_{ijk}x^{i-1}y^j z^k \in I^{N-1}J^t$; it follows that $T_{N,t-1} \neq 0$, which contradicts the choice of t . Therefore, $\{a_{0jk} \mid j+k=t\}$ are not all in $I^{N-1}J$.

Let the overbars denote mod (x) in the following. Consider the exact sequence

$$0 \longrightarrow \bar{T}_{k,n} \longrightarrow \bigoplus_{i=0}^{n+1} \bar{I}^k / \bar{I}^{k-1} \bar{J} \xrightarrow{\bar{\phi}_n} \bar{I}^k \bar{J}^n / \bar{I}^{k-1} \bar{J}^{n+1} \longrightarrow 0,$$

where $\bar{\phi}_n = (\bar{y}^n, \bar{y}^{n-1}\bar{z}, \dots, \bar{z}^n)$ and $\bar{T}_{k,n} = \ker(\bar{\phi}_n)$. Since $\sum_{n=1}^{\infty} \lambda(\bar{I}^n / \bar{I}^{n-1} \bar{J}) - e_1(\bar{I}) = 1$, there is a unique integer N' such that $\bar{T}_{N',n} \neq 0$ for some n . However, by the following remark, $\{\bar{a}_{0jk} \mid j+k=t\}$ are not all in $\bar{I}^{N-1} \bar{J}$. Since $\sum \bar{a}_{0jk} \bar{y}^j \bar{z}^k \in \bar{I}^{N-1} \bar{J}^{t+1}$, we see that $N' = N$.

Remark 3.3. Let $b \in I^N \setminus I^{N-1}J$; then $\bar{b} \notin \bar{I}^{N-1} \bar{J}$ by the fact that $I^N : x = I^{N-1}$.

In the sequel, let R^0 denote the set $\{\text{units of } R\} \cup \{0\}$ and R_n ($n \geq 0$) denote the set $\{f \mid f = \sum_{i=0}^n \lambda_i y^{n-i} z^i \text{ for some } \lambda_i \in R^0\}$.

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R and $J = (y, z)$ be a minimal reduction of I . Let $N \geq 2$. Suppose that*

- (i) $\lambda(\frac{I^N \cap (y) + I^{N-1}J}{I^{N-1}J}) = \lambda(\frac{I^N \cap (z) + I^{N-1}J}{I^{N-1}J}) = 1$, and
- (ii) $\forall n < N$ and $\forall m \geq 1$, $I^n : y = I^n : z = I^{n-1}$ and $I^n J^m : y = I^n J^m : z = I^n J^{m-1}$.

Consider the exact sequence

$$0 \longrightarrow T_{N,n} \longrightarrow \bigoplus_{i=0}^{n+1} I^N / I^{N-1} J \xrightarrow{\phi_n} I^N J^n / I^{N-1} J^{n+1} \longrightarrow 0,$$

where $\phi_n = (y^n, y^{n-1}z, \dots, z^n)$ and $T_{N,n} = \ker(\phi_n)$.

Suppose that $T_{N,n} \neq 0$ for some n . Let l be the least integer such that $T_{N,n} \neq 0$ for all $n \geq l$. Let $a_0, \dots, a_l \in I^N$ not all in $I^{N-1}J$ such that $\sum_{i=0}^l a_i y^{l-i} z^i \in I^{N-1}J^{l+1}$.

Then the following hold:

- (1) $a_0 \notin I^{N-1}J$ and $\mathfrak{m}a_i \subseteq I^{N-1}J \forall i$.
- (2) If $\sum_{i=0}^n b_i y^{n-i} z^i \in I^{N-1}J^{n+1}$ for some $b_i \in I^N$ and for some n , then $\mathfrak{m}b_i \subseteq$

$$I^{N-1}J \forall i \text{ and there are } \lambda_0, \dots, \lambda_{n-l} \in R^0 \text{ such that } b_j - \sum_{i=0}^j a_i \lambda_{j-i} \in I^{N-1}J.$$

(Conventions: $a_i = 0$ if $i > l$ and $\lambda_i = 0$ if $i > n-l$.)

Proof. By the choice of l and the fact that $I^{N-1}J^m : z = I^{N-1}J^{m-1}$, we obtain that $a_0 \notin I^{N-1}J$. Since $a_0 \in I^N \cap (z) + I^{N-1}J$, we have, by assumption, $\mathfrak{m}a_0 \subseteq I^{N-1}J$.

Furthermore, let $w \in \mathfrak{m}$; then $\sum_{i=1}^l (wa_i)y^{l-i}z^{i-1} \in I^{N-1}J^l$ by the assumption that $I^{N-1}J^m : z = I^{N-1}J^{m-1}$. Again, by the choice of l , wa_i must belongs to $I^{N-1}J$ for all i . This proves (1).

To see (2), we may assume that $n \geq l$ and $b_0 \in I^N \setminus I^{N-1}J$. Then $b_0 \in I^N \cap (z) + I^{N-1}J$. Since $a_0 \in I^N \cap (z) + I^{N-1}J$; there is a unit λ_0 such that $b_0 - \lambda_0 a_0 \in I^{N-1}J$, therefore, $\sum_{i=0}^{n-1} (b_{i+1} - \lambda_0 a_{i+1})y^{n-1-i}z^i \in I^{N-1}J^n$.

If $n = l$, then, by the choice of l , $b_i - \lambda_0 a_i \in I^{N-1}J \forall i$; hence $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$. If $n > l$, then by induction $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$ and there are $\lambda_1, \dots, \lambda_{n-l} \in R^0$ such that $\forall j \geq 1$ $b_j - \lambda_0 a_j = \sum_{i=0}^{j-1} a_i \lambda_{j-i}$. This proves (2). $\square$

Let $\{x, y, z\}$ be a generating set of J satisfying the conditions of Proposition 3.2. Let the overbars denote $\text{mod}(x)$ in the following. By Lemma 2.4, it is easy to check that $\bar{R}$, $\bar{y}$ and $\bar{z}$ satisfy all the assumptions of Lemma 3.4. Let $l = \min\{n \mid \bar{T}_{N,n} \neq 0\}$, where $\bar{T}_{N,n}$ is defined as the above; then there are $a_0, \dots, a_l \in I^N$ not all in $I^{N-1}J$ such that $\sum \bar{a}_i \bar{y}^{l-i} \bar{z}^i \in \bar{I}^{N-1} \bar{J}^{l+1}$. Let $u = \sum_{i=0}^l a_i y^{l-i} z^i$; then u has the following property.

Lemma 3.5. *If $\sum_{i=0}^n b_i y^{n-i} z^i \in (x) + I^{N-1}J^{n+1}$ for some $b_i \in I^N$, then*

- (1) $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$.
- (2) *There is an $f \in R_{n-l}$ such that $\sum_{i=0}^n b_i y^{n-i} z^i - fu \in I^{N-1}J^{n+1}$.*

Proof. By Lemma 3.4, $\mathfrak{m}\bar{a}_i \in \bar{I}^{N-1} \bar{J}$; hence $\mathfrak{m}a_i \subseteq I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J$.

Moreover, there are $\lambda_0, \dots, \lambda_{n-l} \in R^0$ such that $\bar{b}_j - \sum_{i=0}^j \bar{a}_i \lambda_{j-i} \in \bar{I}^{N-1} \bar{J}$; then

$b_j - \sum_{i=0}^j a_j \lambda_{j-i} \in I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J$. Therefore $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$. Let $f = \sum_{i=0}^{n-l} \lambda_i y^{n-l-i} z^i \in R_{n-l}$. Then

$$\sum_{i=0}^n b_i y^{n-i} z^i - fu = \sum (b_j - \sum_{i=0}^j a_j \lambda_{j-i}) y^{n-j} z^j \in I^{N-1}J^{n+1}.$$

$\square$

Remark 3.6. If, in Lemma 3.5, at least one of the b_i is not in $I^{N-1}J$, then $n \geq l$ and we can choose f to be a nonzero element of R_{n-l} .

Since $\text{depth } G(I) = 0$, $I^n : J \neq I^{n-1}$ for some n . Let $N' = \min\{n \mid I^n : J \neq I^{n-1}\}$. Since $I^n : x = I^{n-1} \forall n \leq N$, $N' > N$. Let $s \notin I^{N'-1}$ such that $sJ \subseteq I^{N'}$.

Lemma 3.7. *There exists an element s' with $s - s' \in I^{N'-1}$ such that $s'y \in I^N J^{N'-N}$, $s'z \in I^N J^{N'-N}$ and $s'x \in I^N(y, z)^{N'-N}$.*

Proof. Suppose we have shown for some $k > N$ that there is an element s' with $s - s' \in I^{N'-1}$ such that $s'y \in I^k J^{N'-k}$, $s'z \in I^k J^{N'-k}$ and $s'x \in I^k(y, z)^{N'-k}$. Then there are $a_i, b_{ijk} \in I^k$ such that $s'x = \sum a_i y^{t-i} z^i$ and $s'y = \sum b_{ijk} x^i y^j z^k$, where $t = N' - k$. Therefore,

$$\sum a_i y^{t-i+1} z^i - \sum b_{ijk} x^{i+1} y^j z^k = 0 \in I^{k-1} J^{t+2}.$$

Since $k \neq N$, $T_{k,t+1} = 0$; therefore, a_i, b_{ijk} are in $I^{k-1} J$. It follows that $s'x \in I^{k-1} J^{t+1}$ and $s'y \in I^{k-1} J^{t+1}$. Similarly, we can get $s'z \in I^{k-1} J^{t+1}$.

Finally, from the expression $s'x \in I^{k-1} J^{t+1}$, we see that there is a $w \in I^{k-1} J^t \subseteq I^{N'-1}$ such that $(s' - w)x \in I^{k-1}(y, z)^{t+1}$ and $(s' - w)J \subseteq I^{k-1} J^{t+1}$. This completes the proof. $\square$

By Lemma 3.7 we may assume that s satisfies the conditions $sJ \in I^N J^t$ and $sx \in I^N(y, z)^t$, where $t = N' - N$. Since $s \notin I^{N'-1}$, $sx \notin I^{N-1} J^{t+1}$ by Lemma 2.4. Therefore, by Remark 3.6 and the condition $sx \in I^N(y, z)^t$, we obtain that $t \geq l$. Hence by Lemma 3.5, there is a nonzero element $f_0 \in R_{t-l}$ such that

$$(3) \quad sx - f_0 u \in I^{N-1} J^{t+1}.$$

In what follows, let $\mathbf{k}$ be the residue field of R . If $f = \sum_{i=0}^n \lambda_i y^{n-i} z^i \in R_n$, then we associate to f a homogeneous polynomial $T(f) = F = \sum_{i=0}^n \bar{\lambda}_i Y^{n-i} Z^i$ in $\mathbf{k}[Y, Z]$.

(Here, the overbars denote $\text{mod}(\mathfrak{m})$.)

Remark 3.8. From (3) and Lemma 3.5, we obtain $\mathfrak{m}sx \subseteq I^{N-1} J^{t+1}$, and therefore $\mathfrak{m}s \subseteq I^{N-1} J^t$. Moreover, if f and g are two elements of R_n such that $F = G$, then $f - g \in \mathfrak{m}R_n$; it follows that $sf - sg \in I^{N-1} J^{n+t}$.

From (3), Lemma 3.5 and Remark 3.6 we have the following corollary.

Corollary 3.9. *Let $B_0 = \sum b_i y^{n-i} z^i$ with $b_i \in I^N$ not all in $I^{N-1} J$ and $B_j \in (y, z)^{n-j} I^N$. Suppose that $B_0 + B_1 x + \dots \in I^{N-1} J^{n+1}$. Then there is a nonzero element $h \in R_{n-l}$ such that $B_0 + hu \in I^{N-1} J^{n+1}$ and $sh - f_0(B_1 + B_2 x + \dots) \in I^{N-1} J^{n+t-l}$.*

Proof. By Lemma 3.5 and Remark 3.6, there is a nonzero element $h \in R_{n-l}$ such that $B_0 + hu \in I^{N-1} J^{n+1}$. By (3), $shx - f_0 x(B_1 + B_2 x + \dots) \in I^{N-1} J^{n+t-l+1}$. It follows by Lemma 2.4 that $sh - f_0(B_1 + B_2 x + \dots) \in I^{N-1} J^{n+t-l}$. $\square$

Let Q_j ($j \geq 0$) be the set of all integers n such that there exists a nonzero element $f \in R_n$ with $sf \in \sum_{i=0}^j x^i(y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n}$. (For example, since $sy \in I^N J^t$, $1 \in Q_t$.)

Let $m = \min\{j \mid Q_j \neq \emptyset\}$. Let k be the smallest integer in Q_m ; then there is a nonzero element $f \in R_k$ with $sf \in \sum_{i=0}^m x^i(y, z)^{t+k-1-i} I^N + I^{N-1} J^{t+k}$, so that

there are $A_i \in (y, z)^{t+k-1-i} I^N$ such that

$$(4) \quad sf - \sum_{i=0}^m A_i x^i \in I^{N-1} J^{t+k}.$$

Lemma 3.10. *Let $C_i \in (y, z)^{n-i} I^N$ such that $\sum_{i=0}^m C_i x^i \in I^{N-1} J^{n+1}$. Then $C_i \in I^{N-1} J^{n+1-i}$ and all the coefficients of C_i are all in $I^{N-1} J$.*

Proof. If $m = 0$, then $C_0 \in I^{N-1} J^{n+1}$. If the coefficients of C_0 are not all in $I^{N-1} J$, then by Corollary 3.9 there is a nonzero element $h \in R_{n-l}$ such that $sh \in I^{N-1} J^{n+t-l}$. This gives the contradiction $s \in I^{N-1} J^t$ by Lemma 2.4.

Assume that $m \geq 1$ and the assertion is false. Let j be the least integer such that the coefficients of C_j are not all in $I^{N-1} J$. Then, by Lemma 2.4, $\sum_{i=j}^m C_i x^{i-j} \in I^{N-1} J^{n+1-j}$; hence by Corollary 3.9 there is a nonzero element $h \in R_{n-j-l}$ such that $sh - f_0(\sum_{i=j+1}^m C_i x^{i-j-1}) \in I^{N-1} J^{n+t-j-l}$, which contradicts the choice of m . The assertion now follows. $\square$

Let p be the maximal integer such that $Z^p | F$. Let $F' = F/Z^p$.

Lemma 3.11. *If g is a nonzero element of R_n such that $sg \in \sum_{i=0}^m x^i(y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n}$, then $F | G$.*

Proof. Let $B_i \in (y, z)^{t+n-1-i} I^N$ such that

$$(5) \quad sg - \sum_{i=0}^m B_i x^i \in I^{N-1} J^{t+n}.$$

Write $G = G'Z^q$ with $(G', Z) = 1$. Let $g' \in R_{n-q}$ and $f' \in R_{k-p}$ such that $T(g') = G'$ and $T(f') = F'$. By Remark 3.8, we may assume that $f = f'z^p$ and $g = g'z^q$.

Assume that $q < p$. Then from (4) and (5), we obtain that

$$(6) \quad \sum_{i=0}^m (f'z^{p-q}B_i - g'A_i)x^i \in I^{N-1} J^{t+k+n-q},$$

so that by Lemma 3.10 all the coefficients of $f'z^{p-q}B_i - g'A_i$ are all in $I^{N-1} J$. Since $(G', Z) = 1$, there are $A'_i \in (y, z)^{t+k-2-i} I^N$ such that $A_i - zA'_i \in I^{N-1} J^{t+k-i}$; it follows from (4) that $s(f'z^{p-1}) - \sum_{i=0}^m A'_i x^i \in I^{N-1} J^{t+k-1}$, which contradicts the choice of k . Hence $q \geq p$.

Write $G = G'Z^p$. Then $G' = F'Q + Z^{n-k+1}G''$ for some Q and G'' . Suppose $G'' \neq 0$. Then $\deg G'' = \deg F' - 1$. Let $g'' \in R_{k-p-1}$ such that $T(g'') = G''$; then

$$(7) \quad sg''z^{n-k+1+p} - \sum_{i=0}^m C_i x^i \in I^{N-1} J^{t+n}$$

for some $C_i \in (y, z)^{t+n-1-i} I^N$.

From (4) and (7), we obtain that

$$\sum_{i=0}^m (g'' z^{n-k+1} A_i - f' C_i) x^i \in I^{N-1} J^{t+n+k-p},$$

therefore, by Lemma 3.10, all the coefficients of $g'' z^{n-k+1} A_i - f' C_i$ are in $I^{N-1} J$. Since $(F', Z) = 1$, there are $C'_i \in (y, z)^{t+k-2-i} I^N$ such that $C_i - z^{n-k+1} C'_i \in I^{N-1} J^{t+n-i}$. Hence from (7), $sg'' z^p - \sum_{i=0}^m C'_i x^i \in I^{N-1} J^{t+k-1}$, which contradicts the choice of k . Therefore, $G'' = 0$ and $F|G$. $\square$

In fact, Lemma 3.11 can be improved as follows.

Lemma 3.12. *Let $m \leq m' \leq t$. If g is a nonzero element of R_n such that $sg \in \sum_{i=0}^{m'} x^i (y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n}$, then $F|G$.*

Proof. We use induction on m' . If $m' = m$, then this is the content of Lemma 3.11.

Assume that $m' > m$. Let $H_0 = \frac{F_0}{(F_0, F)}$ and let $n_0 = \deg H_0$. Let $h_0 \in R_{n_0}$ such that $T(h_0) = H_0$. Let $B_i \in (y, z)^{t+n-1-i} I^N$ such that

$$(8) \quad sg - \sum_{i=0}^{m'} B_i x^i \in I^{N-1} J^{t+n}.$$

Let $H = (F, G)$ and let $k' = \deg H$. Then $F = F'H$ and $G = G'H$ for some F' and G' . Let $f' \in R_{k-k'}$ and $g' \in R_{n-k'}$ such that $T(f') = F'$ and $T(g') = G'$. By Remark 3.8, we may assume that $f = f'h$ and $g = g'h$. Set $A_i = 0 \quad \forall i > m$.

From (4) and (8), we obtain that $\sum_{i=0}^{m'} (f' B_i - g' A_i) x^i \in I^{N-1} J^{t+n+k-k'}$. Therefore,

by Lemma 3.5, there is an element $h_1 \in R_{t+n+k-k'-l-1}$ such that $f' B_0 - g' A_0 + h_1 u \in I^{N-1} J^{t+n+k-k'}$, and then, by (3), $sh_1 - f_0 (\sum_{i=0}^{m'-1} (f' B_{i+1} - g' A_{i+1}) x^i) \in$

$I^{N-1} J^{2t+n+k-k'-l-1}$. Hence, by induction, $F|H_1$. Moreover, since (F_0, F) is a common factor of H_1 and F_0 , there is an element $g_1 \in R_{n_0+n+k-k'-1}$ such that

$sg_1 - h_0 (\sum_{i=0}^{m'-1} (f' B_{i+1} - g' A_{i+1}) x^i) \in I^{N-1} J^{t+n_0+n+k-k'-1}$. Therefore, by induction,

$F|G_1$. Hence there is an element $g'_1 \in R_{n_0+n-k'-1}$ such that $G_1 = G'_1 F$.

Suppose that $m = 0$. Since $f' B_0 - g' A_0 + h_1 u \in I^{N-1} J^{t+n+k-k'}$, by Lemma 3.10, $g' h A_0 - f A'_0 \in I^{N-1} J^{t+n+k}$ for some $A'_0 \in (y, z)^{t+n-1} I^N$. Therefore by (4), $sf g' h - f A'_0 \in I^{N-1} J^{t+n+k}$, and then $sg' h - A'_0 \in I^{N-1} J^{t+n}$ by Lemma 2.4, so that by Lemma 3.11, $F|G'H$. But $(F, G') = 1$; we obtain $F|H$.

Assume now that $m \geq 1$. We claim: There are integers $n_1, \dots, n_{m'-m}, d_1, \dots, d_{m'-m}$ and elements $h_j \in R_{n_j}$, $g_j \in R_{d_j}$ and $g'_j \in R_{d_j-k}$ satisfy the following conditions:

(i) $G_j = G'_j F$.

(ii) $h_0^{j-1} (f' B_{j-1} - g' A_{j-1}) + h_0^{j-2} g'_1 A_{j-2} - \dots - g'_{j-1} A_0 + h_j u \in I^{N-1} J^{n_j+l+1}$.

$$(iii) \quad sg_j - \sum_{i=0}^{m'-j} (h_0^j(f'B_{i+j} - g'A_{i+j}) - h_0^{j-1}g'_1A_{i+j-1} - \cdots - h_0g'_{j-1}A_{i+1})x^i \in I^{N-1}J^{d_j+t}.$$

Suppose that we have constructed, for some $j \geq 1$, h_j , g_j and g'_j . Then from (i), (iii) and (4), we see that the element

$$\begin{aligned} & h_0^j(f'B_j - g'A_j) - h_0^{j-1}g'_1A_{j-1} - \cdots - g'_jA_0 \\ & + \sum_{i=0}^{m'-j-1} (h_0^j(f'B_{i+j+1} - g'A_{i+j+1}) - h_0^{j-1}g'_1A_{i+j} - \cdots - g'_jA_{i+1})x^{i+1} \end{aligned}$$

is in $I^{N-1}J^{d_j+t}$. Therefore, by Lemma 3.5, there is an element $h_{j+1} \in R_{n_{j+1}}$ for some n_{j+1} such that (ii) holds for $j+1$ and $F|H_{j+1}$ (cf. the construction of h_1) by induction. Moreover, from the construction of g_1 , it is easy to see that there is an element $g_{j+1} \in R_{d_{j+1}}$ for some d_{j+1} such that (iii) holds for $j+1$. Since by induction $F|G_{j+1}$, there is an $g'_{j+1} \in R_{d_{j+1}-k}$ such that (i) holds. This proves the claim.

Set $j = m' - m$ in (iii) of the claim and compare with (4). We obtain that the element

$$\sum_{i=0}^m (h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g'_1A_{i+m'-m-1} - \cdots - g'_{m'-m}A_i)x^i$$

is in $I^{N-1}J^{d_{m'-m}+t}$. However by Lemma 3.10, $\forall i \leq m$,

$$(9) \quad h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g'_1A_{i+m'-m-1} - \cdots - g'_{m'-m}A_i \in I^{N-1}J^{d_{m'-m}+t-i}$$

From (ii) and (9), it is not hard to see that for $0 \leq i \leq m$ there are $A'_i \in (y, z)^{e_i}I^N$ for some e_i such that $hh_0^i g'^{i+1}A_i - fA'_i \in I^{N-1}J^{e_i+k+1}$. Therefore

by (4), $shh_0^m g'^{m+1}f - f(\sum_{i=0}^m h_0^{m-i} g'^{m-i} A'_i x^i) \in I^{N-1}J^e$ for some e ; it follows by Lemma 2.4 that

$$shh_0^m g'^{m+1} - \sum_{i=0}^m h_0^{m-i} g'^{m-i} A'_i x^i \in I^{N-1}J^{e-k}.$$

Finally, by Lemma 3.11, $F|G'^{m+1}HH_0^m$. Since $(F, G') = (F, H_0) = 1$, we have $F|H$. This completes the proof. $\square$

Now, choose a unit λ so that $(Y + \lambda Z, F) = 1$. Since $s(y + \lambda z) \in \sum_{i=0}^t x^i(y, z)^{t-i}I^N + I^{N-1}J^{t+1}$, by Lemma 3.12, $F|Y + \lambda Z$, which contradicts the choice of λ . This proves Theorem 3.1.

By Proposition 2.6 and Theorem 3.1, we have the following corollary.

Corollary 3.13. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field. Let I be an $\mathfrak{m}$ -primary ideal of R . Suppose that there is a minimal reduction J of I such that $\sum_{n=1}^{\infty} \lambda(I^n \cap J / I^{n-1}J) = 2$. Then $\text{depth } G(I) \geq d - 2$.*

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