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A MODIFICATION OF LOUVEAU AND VELIČKOVIČ'S CONSTRUCTION FOR F_σ-IDEALS

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ABSTRACT. We show that the construction of Louveau and Veličkovič can be modified to obtain an embedding of $([\omega]^\omega, \subset^*)$ into the preorder $(F_\sigma\text{-}ideals, \leq)$ where $\leq$ is the relation of Borel reducibility.

The notion of reducibility appeared in [1]. Generally, reducibility is a preorder on all Borel equivalence relations in Polish spaces. We will be interested here only in one Polish space $\mathcal{P}(\omega)$ (or $\mathcal{P}(A)$, where A is a countable set). We equip this space with the Tychonoff topology transfered from the Cantor cube 2^ω. For $s \in 2^{<\omega} =^{df} \bigcup_{n\in\omega}\{0,1\}^n$ let $\hat{s} = \{X \subset \omega\colon x \cap \mathrm{dom}(s) = s^{-1}\{1\}\}$. Thus $\{\hat{s}\colon s \in 2^{<\omega}\}$ is a basis of the topology defined above. By $[A]^\omega$ we will define a set of all infinite, and by $[A]^{<\omega}$ a set of all finite subsets of a set A. Let I be the Borel ideal in the space $\mathcal{P}(\omega)$. Additionally we will restrict our attention only to equivalences which are of the form $=_I$ (i.e. congruences modulo ideal I).

Thus we will define reducibility (in symbols $\leq$) and continuous reducibility (in symbols $\leq_c$) only for ideals. The definitions look as follows:

(1)
$$(I \leq J) \equiv \exists F\colon \mathcal{P}(\omega) \xrightarrow{\text{Borel}} \mathcal{P}(\omega)\ \forall x, y \in \mathcal{P}(\omega)[(x \triangle y \in I) \Leftrightarrow (F(x)\triangle F(y) \in J)]$$

and the definition of $\leq_c$ is almost the same with "Borel" replaced by "continuous".

By submeasure on a set X we mean a function $\mu\colon \mathcal{P}(X) \to [0,\infty]$ with the following properties:

$$\begin{gathered}\forall A, B \subset X[\mu(A) \leq \mu(A\cup B) \leq \mu(A) + \mu(B)]\\ \mu(\varnothing) = 0, \mu(X) > 0, \forall x \in X[\mu(\{x\}) < \infty]\\ \mu(A) = \sup_{a\in[A]^{<\omega}} \mu(a).\end{gathered} \tag{2}$$

Let us define preorder $\subset^*$ on the set $\mathcal{P}(\omega)$ by the formula:

$$S \subset^* T \Leftrightarrow S\backslash T \in [\omega]^{<\omega}.$$

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Louveau and Veličkovič found a family of $F_{\sigma\delta}$-ideals $(I'_S)_{S\in[\omega]^\omega}$ satisfying:

$$\forall S, T \in [\omega]^\omega [S \subset^* T \Leftrightarrow I'_S \leq I'_T]. \tag{3}$$

We will show that there exists a family of F_σ-ideals I_S for which the same is true.

Let us recall some basic facts about the original construction of Louveau and Veličkovič [2]. They start by partitioning ω into finite pieces $(P_n)_n$ and constructing a sequence of submeasures $(\| \, \|_n)_n$ such that for every n, $\| \, \|_n$ is originally defined on P_n and $\forall n(\|P_n\|_n \geq 1)$. These submeasures extend naturally to $\mathcal{P}(\omega)$ by the formula:

$$\|Y\|_n =^{df} \|Y \cap P_n\|_n.$$

Then they define their $F_{\sigma\delta}$-ideals $(I'_S)_{S\in[\omega]^\omega}$ by the formulas:

$$Y \in I'_S \Leftrightarrow \lim_{n\in S} \|Y\|_n = 0.$$

Our F_σ-ideals $(I_S)_{S\in[\omega]^\omega}$ will be defined by the formulas:

$$I_S = \{Y \subset \omega \colon \sup_{n\in S} \|Y\|_n < \infty\}.$$

Of course we must require something like: $\forall n(\|P_n\|_n \geq n)$ in order to obtain proper ideals.

Now we will give our (very close to the original) definition of P_n's and $\| \, \|_n$'s. Let us create two increasing sequences of natural numbers $(a_n)_n$ and $(b_n)_n$. Put $a_0 = b_0 = 2$, $a_{n+1} = 2^{n+1}(a_n + b_n + 2)$, $b_{n+1} = 2^{(n+1)(a_{n+1}+b_n+1)}$. Let additionally $m_n = \sum_{k<n} b_k$, $P_n = [m_n, m_{n+1})$. Then we have of course $|P_n| = b_n$ and we will define a submeasure $\| \, \|_n$ supported by P_n by the formulas:

$$\|Y\|_n = \frac{\log_2(|Y \cap P_n| + 1)}{a_n}.$$

Notice that the above definitions imply that $\forall n(\|P_n\|_n \geq n + 1)$.

Let us begin the proof of the equivalence (3) for ideals $(I_S)_{S\in[\omega]^\omega}$.

The proof of the implication "$\Rightarrow$" is the same as in [2]: If we define $\omega_S = \bigcup_{n\in S} P_n$, then the appropriate reducing function is $F(Y) = Y \cap \omega_S$.

The proof of "$\Leftarrow$": Let us take a pair S, T of infinite subsets of ω such that $I_S \leq I_T$. Assume towards a contradiction that $S \not\subseteq^* T$. As in [2] again (Lemma 2) we can observe that if there is a Borel reduction, then there exists a continuous one (possibly for smaller S) and that we can assume $S, T \in [\omega]^\omega$ are disjoint. Let us define submeasures φ_S, φ_T on ω connected with the ideals I_S, I_T, respectively:

$$\varphi_S(Y) = \sup_{n\in S} \|Y\|_n$$

and φ_T in the similar way. We will prove:

Lemma 1. *Assume that $F \colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is continuous and reduces I_S to I_T. Then we can find $K \in \omega$, $S' \in [S]^\omega$, $F' \colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ continuously reducing $I_{S'}$ to I_T such that:*

$$\forall X, Y \subset \omega[(\varphi_{S'}(X \triangle Y) \leq 1) \Rightarrow (\varphi_T(F'(X) \triangle F'(Y)) \leq K)]. \tag{$*$}$$

Proof. The proof will be split into two facts:

Fact 2. *Assume that $F \colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ continuously reduces I_S to I_T. Then there exist an $S' \in [S]^\omega$ and F' reducing $I_{S'}$ to I_T satisfying:*

$$\forall n \in \omega \; \exists m_n \in \omega \; \forall X, Y \subset \omega[((X \triangle Y) \subset n) \Rightarrow (\varphi_T(F'(X) \triangle F'(Y)) \leq m_n)]. \tag{$**$}$$

Fact 3. *Assume that F is a continuous function reducing $I_{S'}$ to I_T and satisfying $(**)$. Then F also satisfies $(*)$.*

Proof of Fact 2. We will define first a suitable dense G_δ-set and then we will proceed as in [2], Lemma 2. For $m, n \in \omega$ let:

$$C^n_m = \{x \colon \forall y[(x\backslash n = y\backslash n) \Rightarrow (\varphi_T(F(x)\triangle F(y)) \leq m)]\}.$$

For any $n \in \omega$, $(C^n_m)_m$ is an increasing family of closed sets and $\bigcup_{m\in\omega} C^n_m = \mathcal{P}(\omega)$. Hence, by the Baire category theorem, the set $\bigcup_{m\in\omega} \operatorname{int}(C^n_m)$ is open dense for any $n \in \omega$. Our G will be of the form $\bigcap_{n\in\omega} G_n$ where $G_n = \bigcup_{n\in\omega} \operatorname{int}(C^n_m)$. Proceeding as in [2], Lemma 2, take $S' \in [S]^\omega$ and $Z \subset \omega\backslash\omega_{S'}$ such that $\forall A \subset \omega_{S'}(A \cup Z \in G)$. The set $\{A \cup Z \colon A \subset \omega_{S'}\}$ is compact and contained in every G_n. Hence:

$$\forall n\ \exists m_n[\{A \cup Z \colon A \subset \omega_{S'}\} \subset \operatorname{int}(C^n_{m_n})],$$

i.e.,

$$\forall n\ \forall x, y \in \{A \cup Z \colon A \subset \omega_{S'}\}[(x\triangle y \subset n) \rightarrow (\varphi_T(F(x)\triangle F(y)) \leq m_n)].$$

Define $F' \colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by the formula: $F'(X) = F((X \cap \omega_{S'}) \cup Z)$. Then F' is as required. □

Proof of Fact 3. We know that: $\{F(X)\triangle F(Y) \colon \varphi_S(X\triangle Y) \leq 1\}$ is a compact set covered by the countable union of closed sets: $\bigcup_{n\in\omega}\{A \colon \varphi_T(A) \leq n\}$. Hence, by the Baire category theorem applied to this space there exist a $u \in 2^{<\omega}$ and $m_1 \in \omega$ such that $\varnothing \neq \{F(X)\triangle F(Y) \colon \varphi_S(X\triangle Y) \leq 1\} \cap \hat{u} \subset \{A \colon \varphi_T(A) \leq m_1\}$. By the continuity of F we can also find $s_1, t_1 \in 2^{<\omega}$ such that $lh(s_1) = lh(t_1)$, $\varphi_S(s_1^{-1}\{1\}\triangle t_1^{-1}\{1\}) \leq 1$ and

$$\{F(X)\triangle F(Y) \colon \varphi_S(X\triangle Y) \leq 1, X \in \hat{s}_1, Y \in \hat{t}_1\} \subset \{A \colon \varphi_T(A) \leq m_1\}.$$

Take any $X, Y \subset \omega$ such that $\varphi_S(X\triangle Y) \leq 1$. Let $C = \bigcup_{k\in\omega \colon P_k\cap \operatorname{dom}(s_1)\neq\varnothing} P_k$, $X_1 = s_1^{-1}\{1\} \cup (X\backslash C)$, $Y_1 = t_1^{-1}\{1\} \cup (Y\backslash C)$. Let $n = \sup(C)$. Using Fact 2 we can find m_2 such that: $\forall Z_1, Z_2[(Z_1\triangle Z_2 \subset n) \Rightarrow (\varphi_T(F(Z_1)\triangle F(Z_2)) \leq m_2)]$. Now we have:

$$\varphi_T(F(X)\triangle F(X_1)) \leq m_2,$$
$$\varphi_T(F(X_1)\triangle F(Y_1)) \leq m_1,$$
$$\varphi_T(F(Y_1)\triangle F(Y)) \leq m_2.$$

From the above we infer that: $\varphi_T(F(X)\Delta F(Y)) \leq m_1 + 2m_2$, which concludes the proof of Fact 3 and the lemma. □

Next we will prove two interesting properties of the sequence $(P_n, \| \, \|_n)_n$.

Lemma 4. *Let $n < m$ and let $(A_k)_{k<l\leq b_n}$ be a family of subsets of P_m. Then*

$$\left\|\bigcup_{k<l} A_k\right\|_m \leq \sup_{k<l} \|A_k\|_m + \frac{1}{2^{n+1}}.$$

Proof.

$$\log_2\left(\left|\bigcup_{k<l} A_k\right| + 1\right) \leq \log_2(l) + \sup_{k<l} \log_2\left(|A_k| + \frac{1}{l}\right)$$
$$\leq \log_2(b_n) + \sup_{k<l}[\log_2(|A_k| + 1)].$$

Dividing $\log_2(b_n) + \sup_{k<l}[\log_2(|A_k| + 1)]$ by a_m for $m > n$, and noting that $\frac{\log_2(b_n)}{a_{n+1}} \leq \frac{1}{2^{n+1}}$, we infer the lemma. □

Lemma 5. *Let $n < m$ and assume that $f\colon \mathcal{P}(P_n) \to \mathcal{P}(P_m)$ satisfies:*

$$\forall A, B \subset P_n[(\|A \triangle B\|_n \leq 1) \Rightarrow (\|f(A) \triangle f(B)\|_m \leq K)].$$

Then

$$\forall A, B \subset P_n \left(\|f(A) \triangle f(B)\|_m \leq K + \frac{1}{2^{n+1}} \right).$$

Proof. Enumerate $A \triangle B = \{t_k\colon k < l\}$ where $l \leq b_n$. For $p \leq l$ let $U_p = A \triangle \{t_k\colon k < p\}$. We have $U_0 = A$, $U_l = B$. For every $p < l$, $|U_p \triangle U_{p+1}| = 1$, hence $\|U_p \triangle U_{p+1}\|_n \leq 1$. Thus from our assumptions on f we have $\|f(U_p) \triangle f(U_{p+1})\|_m \leq K$. Using Lemma 4 we can calculate:

$$\begin{aligned}\|f(A) \triangle f(B)\|_m &= \left\| \mathop{\triangle}_{p=0}^{l-1} (f(U_p) \triangle f(U_{p+1})) \right\|_m \\ &\leq \left\| \bigcup_{p=0}^{l-1} (f(U_p) \triangle f(U_{p+1})) \right\|_m \underset{\text{Lemma 4}}{\leq} K + \frac{1}{2^{n+1}}. \quad \square\end{aligned}$$

Now we want to construct a sequence of natural numbers $(i_n)_n \subset S$ and two sequences $(A_n)_n$ and $(B_n)_n$ of subsets of ω such that

i) $A_n, B_n \subset m_{i_n}; A_{n+1} \cap m_{i_n} = A_n; B_{n+1} \cap m_{i_n} = B_n$,

ii) $\|A_{n+1} \triangle B_{n+1}\|_{i_n} \geq n$,

iii) $\forall X \subset \omega \backslash m_{i_n} [\varphi_T[F(A_n \cup X) \triangle F(B_n \cup X)] \leq K + 1 - \frac{1}{2^n}]$.

We start the construction by taking $i_1 =$ the first element of S and $A_1, B_1 \subset m_{i_1}$ such that $\varphi_S(A_1 \triangle B_1) \leq 1$. Now we describe how to do the inductive step. Let us find a family $\mathcal{F} \subset \mathcal{P}(P_{i_n})$ such that $|\mathcal{F}| \geq 2^{m_{i_n}} + 1$, consisting of disjoint sets, each of cardinality $2^{n a_{i_n}}$. This is possible because:

$$b_{i_n} = |P_{i_n}| = 2^{i_n(b_{i_n-1} + a_{i_n} + 1)} = 2^{i_n + i_n b_{i_n-1}} 2^{i_n a_{i_n}} \geq (2^{m_{i_n}} + 1) 2^{n a_{i_n}}.$$

From the pigeon-hole principle it follows that we can find $A, B \in \mathcal{F}$ such that:

$$F(A_n \cup A) \cap m_{i_n} = F(A_n \cup B) \cap m_{i_n}.$$

Let us choose $i_{n+1} > i_n$, $i_{n+1} \in S$, such that for any $X \subset \omega \backslash i_{n+1}$

$$F(A_n \cup A \cup X) \cap m_{i_n} = F(A_n \cup A) \cap m_{i_n},$$

$$F(A_n \cup B \cup X) \cap m_{i_n} = F(A_n \cup B) \cap m_{i_n}.$$

Finally we put:

$$A_{n+1} = A_n \cup A,$$

$$B_{n+1} = B_n \cup B.$$

By the properties of the family $\mathcal{F}, A$ and B are disjoint and the submeasure $\| \ \|_{i_n}$ is $\geq n$ on both of them. Therefore $\|A_{n+1} \triangle B_{n+1}\|_{i_n} \geq \|A \triangle B\|_{i_n} \geq n$. Now we want to check if iii) holds for $n + 1$. Take any $X \subset \omega \backslash i_{n+1}$ and $m \in T$. We want to show that:

$$\|F(A_{n+1} \cup X) \triangle F(B_{n+1} \cup X)\|_m \leq K + 1 - \frac{1}{2^{n+1}}.$$

We can partition this symmetric difference introducing the intermediate factor $F(A_n \cup B \cup X)$. We obtain:

$$F(A_{n+1} \cup X) \triangle F(B_{n+1} \cup X)$$
$$= [F(A_{n+1} \cup X) \underset{(I)}{\triangle} F(A_n \cup B \cup X)] \triangle [F(A_n \cup B \cup X) \underset{(II)}{\triangle} F(B_{n+1} \cup X)].$$

For our $m \in T$ there are two possibilities: $m < i_n$ and $m > i_n$. Recall that because $i_n \in S$ and S and T are disjoint, the case $m = i_n$ is impossible. When $m < i_n$, then $\|(I)\|_m = 0$ and $\|(II)\|_m$ is small by our inductive assumption iii). When $m > i_n$, then, if we take $f\colon \mathcal{P}(P_{i_n}) \to \mathcal{P}(P_m)$ defined by $f(C) = F(A_n \cup C \cup X) \cap P_m$, then from Lemma 5 we have $\|(I)\|_m \leq K + \frac{1}{2^{i_n}} \leq K + \frac{1}{2^n}$ and from our inductive assumption $\|(II)\|_m \leq K + 1 - \frac{1}{2^n}$. From Lemma 4 we obtain:

$$\|(I) \triangle (II)\|_m \leq \|(I) \cup (II)\|_m \leq \sup(\|(I)\|_m, \|(II)\|_m) + \frac{1}{2^{n+1}} \leq K + 1 - \frac{1}{2^{n+1}}.$$

Finally, if we put $\widetilde{A} = \bigcup_{n\in\omega} A_n$, $\widetilde{B} = \bigcup_{n\in\omega} B_n$, then $\widetilde{A} \triangle \widetilde{B} \notin I_S$ but $\varphi_T[F(\widetilde{A}) \triangle F(\widetilde{B})] \leq K + 1$. Thus the pair $\widetilde{A}, \widetilde{B}$ is an example showing that F does not reduce I_S to I_T. □

Let us recall the definitions of two important Borel ideals:

$$Fin \times \varnothing =^{df} \{x \subset \omega^2 \colon \exists n [x \subset n \times \omega]\},$$
$$\varnothing \times Fin =^{df} \{x \subset \omega^2 \colon \forall m \exists n \forall k \geq n \langle m, k\rangle \notin x\}.$$

It is not difficult to prove that the original Louveau and Veličkovič family of ideals $(I'_S)_{S\in[\omega]^\omega}$ satisfies:

$$\forall S \in [\omega]^\omega [I'_S \geq \varnothing \times Fin].$$

Similarly the family $(I_S)_{S\in[\omega]^\omega}$ constructed above satisfies:

$$\forall S \in [\omega]^\omega [I_S \geq Fin \times \varnothing].$$

Thus, in connection with the results of Solecki (see [3], Theorems 2.1 and 3.3), stating that every ideal not greater in the sense of reducibility from either $\varnothing \times Fin$ or $Fin \times \varnothing$ is a p-ideal of the class F_σ, it is interesting to ask the following

Question 6. *Does there exist a family of p-ideals of the class F_σ, $(I^*_S)_{S\in[\omega]^\omega}$, satisfying the formula analogous to* (3)?

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