ON THE COMPLEXITY OF DESCRIPTION OF REPRESENTATIONS OF *-ALGEBRAS GENERATED BY IDEMPOTENTS

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ABSTRACT. In this paper, we introduce a quasiorder $\succ$ (majorization) on *-algebras with respect to the complexity of description of their representations. We show that $C^*(\mathcal{F}_2) \succ \mathfrak{A}$ for any finitely generated *-algebra $\mathfrak{A}$ (algebras $\mathfrak{B}$ such that $\mathfrak{B} \succ C^*(\mathcal{F}_2)$ are called *-wild). We show that the *-algebra generated by orthogonal projections $p, p_1, p_2, \dots, p_n$ ($p_i p_j = 0$ for $i \neq j$) is *-wild if $n \geq 2$. We also prove that *-algebras generated by a pair of idempotents and an orthogonal projection, or by a pair of idempotents q_1, q_2 ($q_1 q_2 = q_2 q_1 = 0$), etc., are *-wild.

INTRODUCTION

In Section 2 of this article, following [1]–[4], we introduce a quasiorder (majorization) $\succ$ of *-algebras with respect to the complexity of the structure of their *-representations by bounded operators. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^* -algebras, then $\mathfrak{A} \succ \mathfrak{B}$ means (see Section 2) that there is a C^* -ideal $\mathfrak{J}$ in $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{J} \approx M_n(\mathbb{C}) \otimes \mathfrak{B}$ for some $n \in \mathbb{N} \cup \{\infty\}$ ($M_\infty(\mathbb{C}) = \mathcal{K}$ is the C^* -algebra of compact operators).

In Section 2, following [1], we show that for any finitely generated *-algebra $\mathfrak{A}$, we have that $C^*(\mathcal{F}_2) \succ \mathfrak{A}$ (here $\mathcal{F}_2$ is the free group with two generators), and so that if $\mathfrak{B} \succ C^*(\mathcal{F}_2)$, then the problem of describing, up to a unitary equivalence, all *-representations of the algebra is extremely difficult (in the article such problems are called *-wild).

Then in Sections 3 and 4, we study, from the point of view developed in Section 2, the complexity of representations of some *-algebras generated by idempotents or, which is the same thing, the complexity of a unitary description of some families of idempotents $\{Q_i\}_{i=1}^n$, $Q_i \in \mathcal{L}(H)$ ($i = 1, \dots, n$), on a separable Hilbert space H (for the utility of solution of such a problem see, for example, [5] and the bibliography therein). In particular, in Section 3 we give a simpler proof of the fact that the *-algebra generated by orthogonal projections (self-adjoint idempotents) $p, q_1, \dots, q_n$ such that $q_i q_j = 0$, $i \neq j$ (“all but one” mutually orthogonal projections), is *-wild if $n \geq 2$ (see also [6]).

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In Section 4, we prove that the $*$ -algebra is $*$ -wild if it is generated by one of the following families of operators:

- (i) an idempotent and an orthogonal projection (the problem of unitary description of pairs consisting of an idempotent and an orthogonal projection);
- (ii) a pair of idempotents q_1, q_2 such that $q_1q_2 = q_2q_1 = 0$;
- (iii) a resolution of the identity into a sum of idempotents $\{q_i\}_{i=1}^n, q_1 + \dots + q_n = e$ ($n \geq 3$), or a resolution of the identity into a weighted sum of orthogonal projections $\{p_k\}_{k=1}^n, \frac{1}{2}(p_1 + \dots + p_n) = e$ ($n \geq 5$).

For corresponding enveloping C^* -algebras, if they exist, this means that their factor algebras are isomorphic to $M_n(C^*(\mathcal{F}_2))$ for some $n \in \mathbb{N}$.

Complexity of description of $*$ -representations of group C^* -algebras, their generalizations, $*$ -algebras that correspond to classes of not self-adjoint operators, Wick $*$ -algebras, etc., will be studied in a forthcoming paper.

It is noted that the problems close to the subject of this article are considered in [7].

1. DEFINITIONS AND NOTATIONS

In this article, we consider the problem of giving a description, up to a unitary equivalence, of families of idempotents $Q_1, \dots, Q_n$. As usual, two families of operators $\{X_\alpha\}_{\alpha \in \Lambda}$ in H and $\{\tilde{X}_\alpha\}_{\alpha \in \Lambda}$ in $\tilde{H}$ are unitarily equivalent if there exists a unitary operator $U: H \rightarrow \tilde{H}$ such that

$$UX_\alpha = \tilde{X}_\alpha U \quad (\alpha \in \Lambda).$$

It is natural to make such a description within the framework of the theory of representations of $*$ -algebras so as to associate to idempotents $\{Q_k\}_{k=1}^n$ a representation of the $*$ -algebra $\mathfrak{Q}_n$. This is the factor algebra of the free $*$ -algebra, with generators $q_1, \dots, q_n, q_1^*, \dots, q_n^*$, with respect to the two-sided $*$ -ideal, generated by the relations $q_k^2 = q_k, (q_k^*)^2 = q_k^* (k = 1, \dots, n)$. In the sequel, the $*$ -algebra $\mathfrak{A}$ defined by generators $x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots, x_n^*$ and relations $P_l(x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots, x_n^*) = 0, l = 1, 2, \dots, m$, will be denoted by $\mathbb{C}\langle x_1, \dots, x_n \mid P_l(x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots, x_n^*) = 0, l = 1, \dots, m \rangle$. It will also be assumed that, besides these relations, all relations obtained from them by taking $*$ are valid as well.

A representation of a $*$ -algebra $\mathfrak{A}$ is a $*$ -homomorphism $\pi: \mathfrak{A} \rightarrow L(H)$ into the $*$ -algebra $L(H)$ of bounded operators in a complex separable Hilbert space H . By $\text{Rep } \mathfrak{A}$ we denote the category, objects of which are representations of the algebra $\mathfrak{A}$ and morphisms (intertwining operators). Each representation π of the $*$ -algebra $\mathfrak{A} = \mathbb{C}\langle x_1, \dots, x_n \mid P_l(x_1, \dots, x_n, x_1^*, \dots, x_n^*) = 0, l = 1, \dots, m \rangle$ determines the family of bounded operators $\{X_k\} = \{\pi(x_k)\}, k = 1, \dots, n$, such that

$$(1) \quad P_l(X_1, \dots, X_n, X_1^*, \dots, X_n^*) = 0, \quad l = 1, \dots, m.$$

Conversely, a given family of operators $\{X_k\}, k = 1, \dots, n$, such that

$$P_l(X_1, \dots, X_n, X_1^*, \dots, X_n^*) = 0, \quad l = 1, \dots, m,$$

uniquely defines a representation of the whole $*$ -algebra $\mathfrak{A}$. Thus the problem of a unitary description of families of operators $\{X_k\}, k = 1, \dots, n$, satisfying relations (1) is a problem of description, up to a unitary equivalence, of representations of the $*$ -algebra $\mathfrak{A}$.

In the sequel, we will be considering the unitary classification problems for representations of the following *-algebras (and, correspondingly, the unitary classification problems for the following families of operators):

- 1) $\mathfrak{S}_n = \mathbb{C}\langle a_1, \dots, a_n \mid a_i = a_i^*, i = 1, \dots, n \rangle$ (classification problem for n self-adjoint operators);
- 2) $\mathfrak{U}_n = \mathbb{C}\langle u_1, \dots, u_n \mid u_i u_i^* = u_i^* u_i = e, i = 1, \dots, n \rangle$ (classification problem for n unitary operators);
- 3) $\mathfrak{P}_n = \mathbb{C}\langle p_1, \dots, p_n \mid p_i^2 = p_i = p_i^*, i = 1, \dots, n \rangle$ (classification problem for n orthogonal projections);
- 4) $\mathfrak{C} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^2 = p_i = p_i^*, i = 1, 2, 3; p_1 p_2 = p_2 p_1 = 0 \rangle$ (classification problem for a triple of orthogonal projections, two of which are orthogonal);
- 5) $\mathfrak{Q}_n = \mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i, i = 1, \dots, n \rangle$ (classification problem for n idempotents);
- 6) $\mathfrak{D} = \mathbb{C}\langle q, p \mid q^2 = q, p^2 = p = p^* \rangle$ (classification problem for a pair of operators, one of which is idempotent and the other is an orthogonal projection);
- 7) $\mathfrak{Q}_{n,\perp} = \mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i, i = 1, \dots, n; q_i q_j = 0 \text{ for } i \neq j \rangle$ (classification problem for n mutually orthogonal idempotents).

2. MAJORIZATION OF C^* -ALGEBRAS AND *-ALGEBRAS WITH RESPECT TO COMPLEXITY OF THEIR REPRESENTATIONS

Before passing to the problem of unitary description of representations of *-algebras generated by idempotents, we give definitions and some results concerning the ideology and methodology of *-wildness. In the theory of representations of algebras, it was suggested ([8]) that the representation problem be considered wild if it contains a standard difficult problem of the representation theory, e. g. the problem to describe, up to similarity, a pair of matrices without relations. To define an analogue of wildness for *-algebras (*-wildness), it was suggested in [1] to choose, for a standard difficult problem in the theory of *-representations, the problem of describing pairs of self-adjoint (or unitary) operators up to a unitary equivalence (free *-algebra $\mathfrak{S}_2$ (or $\mathfrak{U}_2$) generated by a pair of self-adjoint (or unitary) generators); and there were indications that suggested that the problems, which contain the standard *-wild problems, be regarded as *-wild. One can prove that these problems contain as a subproblem the problem of describing *-representations of any affine *-algebra.

We give exact definitions, examples, and statements necessary for what follows.

Definition 1. Let $\mathfrak{A}$ be a *-algebra. A pair $(\tilde{\mathfrak{A}}; \phi: \mathfrak{A} \rightarrow \tilde{\mathfrak{A}})$, where $\tilde{\mathfrak{A}}$ is a *-algebra and ϕ is a *-homomorphism, is called an enveloping *-algebra of the algebra $\mathfrak{A}$ if, for any *-representation $\pi: \mathfrak{A} \rightarrow L(H)$ of the algebra $\mathfrak{A}$, there exists a unique *-representation $\tilde{\pi}: \tilde{\mathfrak{A}} \rightarrow L(H)$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathfrak{A}} & & \\ \uparrow \phi & \searrow \tilde{\pi} & \\ \mathfrak{A} & \xrightarrow{\pi} & L(H) \end{array}$$

is commutative, and any operator $X: H_1 \rightarrow H_2$ which intertwines two representations $\pi_1: \mathfrak{A} \rightarrow L(H_1)$, $\pi_2: \mathfrak{A} \rightarrow L(H_2)$ of the algebra $\mathfrak{A}$ is also an intertwining operator for the representations $\tilde{\pi}_1, \tilde{\pi}_2$ of the algebra $\tilde{\mathfrak{A}}$.

The following examples of enveloping $*$ -algebras will be used in the sequel.

1) $\tilde{\mathfrak{A}} = \mathfrak{A}$, ϕ is the identity mapping;

2) Let Σ be any set of elements of an algebra $\mathfrak{A}$, the images of which are invertible operators for any representation $\pi: \mathfrak{A} \rightarrow L(H)$. Let $\tilde{\mathfrak{A}} = \mathfrak{A}[\Sigma^{-1}]$ be the quotient algebra (see [9]) of the algebra $\mathfrak{A}$ with respect to the set Σ , and let ϕ be the natural imbedding of $\mathfrak{A}$ into $\mathfrak{A}[\Sigma^{-1}]$;

3) Let $\mathfrak{A}$ be a star-bounded $*$ -algebra, $\tilde{\mathfrak{A}}$ its enveloping C^* -algebra, and ϕ its canonical $*$ -homomorphism of $\mathfrak{A}$ into $\tilde{\mathfrak{A}}$, defined by a faithful representation (see, for example, [10]).

Let $M_n(\tilde{\mathfrak{A}})$ be the matrix algebra over $\tilde{\mathfrak{A}}$ with the naturally given $*$ -structure. Any representation $\pi: \mathfrak{A} \rightarrow L(H)$ induces the representation

$$\pi_n: M_n(\mathfrak{A}) \rightarrow L(H \oplus H \oplus \cdots \oplus H)$$

and, hence, the representation

$$\tilde{\pi}_n: M_n(\tilde{\mathfrak{A}}) \rightarrow L(H \oplus H \oplus \cdots \oplus H)$$

of the algebra $\tilde{M}_n(\mathfrak{A})$ enveloping the algebra $M_n(\mathfrak{A})$. If $\psi: \mathfrak{B} \rightarrow M_n(\tilde{\mathfrak{A}})$ is a $*$ -homomorphism of the algebras, then there is a natural way to construct the functor $F_\psi: \text{Rep}(\mathfrak{A}) \rightarrow \text{Rep}(\mathfrak{B})$. By definition, $F_\psi(\pi) = \tilde{\pi}_n \circ \psi$ and, if $\alpha: \pi_1 \rightarrow \pi_2$ is a morphism of representations, then $F_\psi(\alpha) = \text{diag}(\alpha, \alpha, \dots, \alpha)$.

Definition 2. A $*$ -algebra $\mathfrak{B}$ majorizes a $*$ -algebra $\mathfrak{A}$ ($\mathfrak{B} \succ \mathfrak{A}$) if there exist $n \in \mathbb{N}$, an enveloping algebra $\tilde{\mathfrak{A}}$, and a $*$ -homomorphism $\psi: \mathfrak{B} \rightarrow M_n(\tilde{\mathfrak{A}})$ such that the functor $F_\psi: \text{Rep}(\mathfrak{A}) \rightarrow \text{Rep}(\mathfrak{B})$ is full and faithful.

Remark 1. For a class of C^* -algebras, it is possible to consider only homomorphisms $\psi: \mathfrak{B} \rightarrow M_n(\mathfrak{A})$ in Definition 2, since for a C^* -algebra $\mathfrak{A}$, $M_n(\mathfrak{A})$ is also a C^* -algebra, and the unique enveloping algebra of a C^* -algebra is the algebra itself. Moreover, we have the following. *A C^* -algebra $\mathfrak{B}$ majorizes a C^* -algebra $\mathfrak{A}$ if and only if $\mathfrak{B}$ contains an ideal $\mathfrak{I}$ such that $\mathfrak{B}/\mathfrak{I} \approx M_n(\mathfrak{A})$. This isomorphism is defined precisely by the mapping ψ . A proof of this claim is not difficult and will be given elsewhere.*

Remark 2. It is also possible to show that the majorization is a quasiorder relation: if $\mathfrak{C} \succ \mathfrak{B}$ and $\mathfrak{B} \succ \mathfrak{A}$, then $\mathfrak{C} \succ \mathfrak{A}$.

Remark 3. One should also note that if $\mathfrak{B} \succ \mathfrak{A}$, and $\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}}$ are enveloping algebras of $\mathfrak{A}, \mathfrak{B}$ correspondingly, then $\mathfrak{B} \succ \tilde{\mathfrak{A}}, \tilde{\mathfrak{B}} \succ \mathfrak{A}, \tilde{\mathfrak{B}} \succ \tilde{\mathfrak{A}}$, and $M_n(\tilde{\mathfrak{A}})$ is an enveloping algebra of the algebra $M_n(\mathfrak{A})$ with the natural embedding of the algebra $M_n(\mathfrak{A})$ into $M_n(\tilde{\mathfrak{A}})$.

Theorem 1. $\mathfrak{S}_2 \succ \mathfrak{S}_m$ for any $m = 1, 2, \dots$.

Proof. For the algebra $\tilde{\mathfrak{S}}_m$, take the algebra

$$\mathfrak{S}_m = \mathbb{C}\langle b_1, \dots, b_m \mid b_i = b_i^*, i = 1, \dots, m \rangle$$

itself, $n = m + 2$. Define a homomorphism $\psi: \mathfrak{S}_2 \rightarrow M_{m+2}(\mathfrak{S}_m)$ as follows:

$$\psi(a_1) = \begin{bmatrix} e & & & & & & \\ & \frac{1}{2}e & & & & & \\ & & \frac{1}{3}e & & & & \\ & & & \ddots & & & \\ & & & & \frac{1}{m}e & & \\ & 0 & & & \frac{1}{m+1}e & & \\ & & & & & \frac{1}{m+2}e & \\ & & & & & & 0 \end{bmatrix},$$

$$\psi(a_2) = \begin{bmatrix} 0 & e & b_1 & & & & \\ e & 0 & e & b_2 & & & \\ b_1 & e & 0 & e & \ddots & & \\ & b_2 & e & \ddots & & b_{m-1} & \\ & & \ddots & & 0 & e & b_m \\ 0 & & & b_{m-1} & e & 0 & e \\ & & & & b_m & e & 0 \end{bmatrix}.$$

One can directly check that the functor F_ψ is full and faithful. $\square$

Theorem 1 permits us to say that the problem of unitary classification of pairs of self-adjoint operators contains, as a subproblem, the problem of unitary classification of representation of any *-algebra with a countable number of generators (because it is always possible to choose these generators to be self-adjoint).

Corollary 1. $\mathfrak{S}_2 \succ \mathfrak{U}_m$ for any $m = 1, 2, 3, \dots$

Theorem 2. $\mathfrak{U}_2 \succ \mathfrak{S}_2$.

Proof. Choose the enveloping algebra $\tilde{\mathfrak{S}}_2$ to be the quotient algebra of the algebra $\mathfrak{S}_2$ with respect to the set

$$\Sigma = \{a - ie, a + ie, b - ie, b + ie\}, \quad n = 1,$$

$$\psi(u_1) = (a - ie)(a + ie)^{-1}, \quad \psi(u_2) = (b - ie)(b + ie)^{-1}$$

(the Cayley transformation). The rest is obvious. $\square$

Theorems 1 and 2 allow us, as a model of complexity for problems of unitary classification of representations of *-algebras, to choose the problem of unitary classification of representations of the algebra $\mathfrak{U}_2$ or, which is the same thing, its enveloping C^* -algebra $C^*(\mathcal{F}_2)$, where $\mathcal{F}_2$ is a free group with two generators.

Definition 3. A *-algebra $\mathfrak{A}$ is called *-wild if $\mathfrak{A} \succ C^*(\mathcal{F}_2)$.

Remark 4. A C^* -algebra $\mathfrak{A}$ is *-wild if and only if there exist $n \in \mathbb{N} \cup \{\infty\}$ and a C^* -ideal $\mathfrak{I} \subset \mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{I} \approx M_n(C^*(\mathcal{F}_2))$.

Nuclear C^* -algebras have only hyperfinite factor representations and, consequently, they cannot be *-wild. There also exist non-nuclear C^* -algebras which are not wild. For example, the group C^* -algebra $C^*(B(m, 2))$ of the Burnside group with two generators and sufficiently large odd n is not nuclear ([11] and [12]), and it is also not wild [13].

3. A UNITARY DESCRIPTION OF ORTHOGONAL PROJECTIONS

Following the general ideology of representation theory for $*$ -algebras, we will try to solve the problem of unitary description of $*$ -representations of the algebra $\mathfrak{A}$ by using a description of its irreducible representations and of all its representations as an integral of irreducible ones; a representation π of a $*$ -algebra $\mathfrak{A}$ is irreducible if there is no nontrivial subspace in H , invariant with respect to the operators $\pi(x)$ for all $x \in \mathfrak{A}$.

For representations of the $*$ -algebra $\mathfrak{P}_2 = \mathbb{C}\langle p_1, p_2 \mid p_1^* = p_1 = p_1^2, p_2^* = p_2 = p_2^2 \rangle$ (a pair of orthogonal projections P_1, P_2), there is a structure theorem (see, for example, [14] and others) that gives a decomposition of representations into a direct sum (or integral) of irreducible representations which are either one-dimensional or two-dimensional, and, up to a unitary equivalence, they coincide with one of the following:

- 1) four one-dimensional: $\pi_{(0,0)}(p_1) = \pi_{(0,0)}(p_2) = 0$; $\pi_{(0,1)}(p_1) = 0$, $\pi_{(0,1)}(p_2) = 1$; $\pi_{(1,0)}(p_1) = 1$, $\pi_{(1,0)}(p_2) = 0$; $\pi_{(1,1)}(p_1) = \pi_{(1,1)}(p_2) = 1$;
- 2) the family, parametrized by $\phi \in (0, \pi/2)$, of two-dimensional

$$\pi_\phi(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_\phi(p_2) = \begin{pmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{pmatrix}.$$

One possible way to prove this theorem is to directly verify that

$$\mathfrak{P}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*; \{a, b\} = ab + ba = 0; a^2 + b^2 = e \rangle,$$

where $a = p_1 - p_2$, $b = e - p_1 - p_2$, and then to apply results from [15] about the structure of a pair of anticommuting self-adjoint operators.

The problem is to describe, up to a unitary equivalence, a family of orthogonal projections $P_1, P_2, \dots, P_n$ for $n \geq 3$ that is $*$ -wild. We give a fairly simple proof that this problem is $*$ -wild for $n = 3$.

Theorem 3. *Let*

$$\mathcal{P} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^2 = p_i^* = p_i \ (i = 1, 2, 3) \rangle.$$

Then $\mathcal{P}_3 \succ C^(\mathcal{F}_2)$, i.e. $\mathcal{P}_3$ is $*$ -wild.*

Proof. Let us define the homomorphism $\psi: \mathcal{P}_3 \rightarrow M_4(\mathcal{F}_2)$ as follows:

$$\begin{aligned} \psi(p_1) &= \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \psi(p_2) &= \begin{pmatrix} \frac{1}{2}e & 0 & \frac{1}{2}e & 0 \\ 0 & \frac{1}{2}e & 0 & \frac{1}{2}e \\ \frac{1}{2}e & 0 & \frac{1}{2}e & 0 \\ 0 & \frac{1}{2}e & 0 & \frac{1}{2}e \end{pmatrix}, \\ \psi(p_3) &= \begin{pmatrix} \frac{3}{8}e & \frac{\sqrt{3}}{8}uv^8 & \frac{\sqrt{3}}{8}u & \frac{3}{8}e \\ \frac{\sqrt{3}}{8}vu^* & \frac{5}{8}e & \frac{3}{8}v & -\frac{\sqrt{3}}{8}vu^* \\ \frac{\sqrt{3}}{8}u^* & \frac{3}{8}v^* & \frac{1}{4}e & 0 \\ \frac{3}{8}e & -\frac{\sqrt{3}}{8}uv^* & 0 & \frac{3}{4}e \end{pmatrix}. \end{aligned}$$

It is easy to check that the corresponding functor $F_\psi: \text{Rep}C^*(\mathcal{F}_2) \rightarrow \text{Rep}\mathcal{P}_3$ is full and faithful. $\square$

Moreover, the following theorem holds ([1]).

Theorem 4. *Let*

$$\mathfrak{C} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^* = p_i, p_i^2 = p_i, p_1 p_2 = p_2 p_1 = 0 \rangle.$$

Then $\mathfrak{C} \succ C^(\mathcal{F}_2)$, i.e. $\mathfrak{C}$ is *-wild.*

Proof. The proof given here is simpler than the proof in [1].

Let

$$E_k = \begin{bmatrix} e & & 0 \\ & \ddots & \\ 0 & \underbrace{\hspace{1cm}}_{k \text{ times}} & e \end{bmatrix}, \quad e \text{ is the identity in the algebra } C^*(\mathcal{F}_2),$$

$$J_1 = \begin{bmatrix} E_4 \\ 0_{3 \times 4} \\ 0_{5 \times 4} \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0_{4 \times 3} \\ E_3 \\ 0_{5 \times 3} \end{bmatrix}, \quad J_3 = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix},$$

where

$$A_1 = \frac{1}{N} \begin{bmatrix} e & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & 2e & 0 & 0 \\ 0 & 0 & 0 & 3e & 0 \end{bmatrix}, \quad A_2 = \frac{1}{N} \begin{bmatrix} e & 0 & e & e & e \\ 0 & 2e & e & u_1 & 0 \\ 0 & 0 & e & 0 & u_2 \end{bmatrix},$$

u_1, u_2 are generators of the algebra $C^*(\mathcal{F}_2)$, $A_3 = \sqrt{E_5 - A_1^* A_1 - A_2^* A_2}$. N is chosen so that $\|A_1^* A_1 + A_2^* A_2\| < 1$ in $M_5(C^*(\mathcal{F}_2))$. Then $J_1^* J_1 = E_4$, $J_2^* J_2 = E_3$, and $J_3^* J_3 = E_4$. This implies that $(J_i J_i^*)^2 = J_i J_i^*$, $i = 1, 2, 3$. Moreover, since $J_1^* J_2 = 0$ and $J_2 J_1^* = 0$, we see that

$$(J_1 J_1^*)(J_2 J_2^*) = (J_2 J_2^*)(J_1 J_1^*) = 0.$$

Set $\psi(p_i) = J_i J_i^*$; ψ defines a homomorphism of the *-algebra $\mathfrak{C}$ into $M_{14}(C^*(\mathcal{F}_2))$. One can directly check that the functor $F_\psi: \text{Rep}(\mathfrak{C}) \rightarrow \text{Rep}(C^*(\mathcal{F}_2))$ is full and faithful. $\square$

Corollary 2. *The problem of a unitary classification of “all but one” orthogonal projections $p, p_1, \dots, p_n$ ($p_i p_j = 0$ for $i \neq j$) is *-wild if $n \geq 2$.*

Corollary 3. *The problem of unitary classification of quadruples of orthogonal projections p_1, p_2, p_3, p_4 such that*

$$\alpha(p_1 + p_2 + p_3 + p_4) = I, \quad 0 < \alpha < 1,$$

for a fixed $\alpha \neq \frac{1}{2}$, has only a finite number of irreducible representations, the dimension of which depends on the parameter α ([16]). If $\alpha = \frac{1}{2}$, then irreducible representations are only in dimensions one and two.

*It directly follows from Theorem 4 that the problem of unitary classification of five orthogonal projections p_1, p_2, p_3, p_4, p_5 such that $p_1 + p_2 + p_3 + p_4 + p_5 = 2I$ is *-wild.*

4. A UNITARY DESCRIPTION OF IDEMPOTENTS

For a single idempotent, the situation is similar to the situation for two orthogonal projections. Consider the $*$ -algebra $\mathfrak{Q}_1$ generated by a pair of an idempotent and its adjoint q_1, q_1^* . Let $q_1 = a_1 + ib_1$, $q_1^* = a_1 - ib_1$, where $a_1^* = a_1$, $b_1^* = b_1$. The $*$ -algebra $\mathfrak{Q}_1$ coincides with the algebra

$$\mathbb{C}\langle a, b \mid a = a^*, b = b^*; \{a, b\} = ab + ba = 0, a^2 - b^2 = e \rangle,$$

where $a = 2(a_1 - \frac{1}{2}e)$, $b = 2b_1$.

Irreducible representations of the algebra $\mathfrak{Q}_1$, up to a unitary equivalence, coincide with one of the following:

- 1) two one-dimensional representations given by $\pi_0(q_1) = 0$ and $\pi_1(q_1) = 1$;
- 2) a family, depending on a parameter $y > 0$, of two-dimensional representations:

$$\pi_{(x,y)}(q_1) = Q_1 = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}.$$

By decomposing a representation of the algebra $\mathfrak{Q}_1$ into the direct sum of irreducible representations on a finite-dimensional space H , we obtain a structure theorem (see [17] and [18]) for the unitary description of idempotents in the finite-dimensional case.

There is a structure theorem that gives a description of any bounded idempotent on any separable Hilbert space in the form of an integral of irreducible representations [4].

Consider the problem of a unitary description of pairs of idempotents Q_1, Q_2 ($Q_1^2 = Q_1, Q_2^2 = Q_2$). The fact that the problem of a unitary description of pairs of idempotents is difficult is just mathematical folklore. We will prove a corresponding theorem and show that, even if an additional restriction of self-adjointness is imposed on one of the idempotents (one of the idempotents is an orthogonal projection), the problem does not become easier.

Theorem 5. *Let $\mathfrak{Q}_2 = \mathbb{C}\langle q_1, q_2 \mid q_1^2 = q_1, q_2^2 = q_2 \rangle$, $\mathfrak{D} = \mathbb{C}\langle q, p \mid q^2 = q, p^2 = p = p^* \rangle$, $\mathfrak{S}_2 = \mathbb{C}\langle a_1, a_2 \mid a_1 = a_1^*, a_2 = a_2^* \rangle$. Then $\mathfrak{Q}_2 \succ \mathfrak{D} \succ \mathfrak{S}_2$, so that the $*$ -algebras $\mathfrak{Q}_2, \mathfrak{D}$ are $*$ -wild.*

Proof. Because $\mathfrak{D}$ is a factor algebra of the algebra $\mathfrak{Q}_2$, we have that $\mathfrak{Q}_2 \succ \mathfrak{D}$ (we choose an enveloping algebra for $\mathfrak{D}$ to be the algebra $\mathfrak{D}$ itself, $n = 1$, $\psi: \mathfrak{Q}_2 \rightarrow \mathfrak{D}$ is the natural epimorphism of the algebra onto the factor algebra).

Let us show that $\mathfrak{D} \succ \mathfrak{S}_2$. Construct the homomorphism $\psi: \mathfrak{D} \rightarrow M_2(\mathfrak{S}_2)$:

$$\psi(q) = \begin{pmatrix} e & a_1 + ia_2 \\ 0 & 0 \end{pmatrix}, \quad \psi(p) = \frac{1}{2} \begin{pmatrix} e & e \\ e & e \end{pmatrix}.$$

It is easy to check that the corresponding functor $F_\psi: \text{Rep } \mathfrak{S}_2 \rightarrow \text{Rep } \mathfrak{D}$ is full and faithful. $\square$

Corollary 4. *The algebra $\mathfrak{Q}_n$, for $n \geq 2$ (the problem of unitary description of n idempotents if $n \geq 2$) is $*$ -wild.*

Finally, we show that the $*$ -algebra $\mathfrak{Q}_{n,\perp}$ (the problem of unitary classification of a family of pairwise orthogonal idempotents $Q_1, Q_2, \dots, Q_n$, $Q_i Q_j = 0$ for $i \neq j$) is $*$ -wild for $n \geq 2$.

Theorem 6. *Let*

$$\mathfrak{Q}_{2,\perp} = \mathbb{C}\langle q_1, q_2 \mid q_1^2 = q_1, q_2^2 = q_2, q_1 q_2 = q_2 q_1 = 0 \rangle.$$

*Then $\mathfrak{Q}_{2,\perp} \succ \mathfrak{S}_2$, i.e. $\mathfrak{Q}_{2,\perp}$ is a wild *-algebra.*

Proof. Let us define a homomorphism $\psi: \mathfrak{Q}_{2,\perp} \rightarrow M_3(\mathfrak{S}_2)$ as follows:

$$\psi(q_1) = \begin{bmatrix} e & e & a_1 + ia_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \psi(q_2) = \begin{bmatrix} 0 & -e & -e \\ 0 & e & e \\ 0 & 0 & 0 \end{bmatrix}.$$

One can directly check that $[\psi(q_k)]^2 = \psi(q_k)$, $k = 1, 2$, $\psi(q_1)\psi(q_2) = \psi(q_2)\psi(q_1) = 0$, and that the functor $F_\psi: \text{Rep } \mathfrak{S}_2 \rightarrow \text{Rep } \mathfrak{Q}_{2,\perp}$ is full and faithful. $\square$

Corollary 5. *The problem of unitary classification of pairs of commuting idempotents is *-wild.*

Corollary 6. *The *-algebra $\mathfrak{Q}_{n,\perp} = \mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i, i = 1, \dots, n; q_i q_j = 0 \text{ for } i \neq j \rangle$ (the problem of unitary classification of n pairwise orthogonal idempotents) is *-wild for $n \geq 2$.*

Corollary 7. *The *-algebra $\mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i, i = 1, \dots, n; q_1 + q_2 + \dots + q_n = e \rangle$ (the problem of unitary classification of n idempotents $Q_1, \dots, Q_n$ such that $Q_1 + \dots + Q_n = I$) is *-wild for $n \geq 3$.*

Proof. If $m = 3$, the condition $q_1 + q_2 + q_3 = e$ implies that the idempotents q_1, q_2, q_3 are pairwise orthogonal. Then the algebra under consideration coincides with the algebra $\mathfrak{Q}_{2,\perp}$. $\square$

Corollary 8. *Let $\mathfrak{A}_{R_3} = \mathbb{C}\langle x \mid R_3(x) \stackrel{\text{def}}{=} (x - \alpha_1 e)(x - \alpha_2 e)(x - \alpha_3 e) = 0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}, \alpha_k \neq \alpha_l \text{ for } k \neq l \rangle$. Then $\mathfrak{A}_{R_3} \succ \mathfrak{Q}_{2,\perp}$, and consequently, the *-algebra $\mathfrak{A}_{R_3}$ is *-wild.*

Proof. Define the homomorphism $\psi: \mathfrak{A}_{R_3} \rightarrow \mathfrak{Q}_{2,\perp}$ as follows:

$$\psi(a) = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 (e - q_1 - q_2).$$

It is easy to check that the functor F_ψ is full and faithful. $\square$

Remark 5. Corollary 8 is given in [19]. The proof in [19] actually uses the fact that the problem of unitary classification of two orthogonal idempotents is *-wild, and implicitly contains this proof.

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