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ON A CONJECTURE OF DUKE-IMAMOGLU

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ABSTRACT. In this note we present some theoretical results and numerical calculations on a recent conjecture of W. Duke and Ö. Imamoğlu.

INTRODUCTION

The purpose of this note is to present some remarks and numerical calculations on a recent conjecture of W. Duke and Ö. Imamoğlu which can be considered as a generalization of the Saito–Kurokawa correspondence.

In the first section we present the conjecture saying that for any elliptic Hecke eigenform f of weight 2k - g > 0 where g and k are positive even integers there is a Siegel eigenform F of weight k and degree g such that the standard L-series of F essentially is the product of shifted Hecke L-series of f. We proceed in giving a local version of this conjecture as well as equivalent relations between the eigenvalues in the case g = 4.

The second section deals with the examination of non-cusp forms with respect to the conjecture. We prove that for a pair (F, f) satisfying the conjecture F is cuspidal if and only if f is cuspidal. Furthermore we show that Siegel Eisenstein series fulfill the conjecture.[†]

As an example for g = 4 we show in the third section that the Schottky form and the Delta function satisfy the local conjecture for small primes. Tables of some Fourier coefficients and eigenvalues of the Schottky form are given in two appendices.

In the last section we prove that the conjecture – formulated as in the first section – does not hold if g = k = 12 using recent results of Borcherds, Freitag and Weissauer.

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Notation. Let $M_k(\Gamma_g)$, resp. $S_k(\Gamma_g)$, be the space of Siegel modular forms, resp. cusp forms, of weight $k \in \mathbb{Z}$ and degree $g \in \mathbb{N}$ for the Siegel modular group $\Gamma_g :=$ Sp $(g; \mathbb{Z})$. For a Siegel modular form F we denote by a(S) (S half-integral and symmetric) the Fourier coefficients of F. For simplicity of notation we put a(S) = 0 if S is not half-integral.

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 $^{^\}dagger {\rm The}$ referee kindly informed the authors that Ö. Imamoğlu also has proved the conjecture for Siegel Eisenstein series.

Let T(n) $(n \in \mathbb{N})$ be the Hecke operators acting on $M_k(\Gamma_g)$, resp. $S_k(\Gamma_g)$, in the usual way (cf. [Ma], [An1]). Furthermore, we denote by $T_{i,j}(p^2)$ $(i, j \in \mathbb{N}_0, i+j=g, p$ prime) the Hecke operator corresponding to the double coset $\Gamma_g \operatorname{diag}(I^{(i)}, pI^{(j)}, p^2I^{(i)}, pI^{(j)})\Gamma_g$ where $I^{(m)}$ is the *m*-rowed identity matrix.

As usual, for matrices A and B of appropriate size we put $A[B] := A^t B A$ where A^t is the transpose of A.

1. The conjecture

The following conjecture is due to W. Duke and Ö. Imamoglu.

Conjecture (DIC). Let k and g be positive even integers with 2k - g > 0, $f \in M_{2k-g}(\Gamma_1)$ a Hecke eigenform. Then there exists a Hecke eigenform $F \in M_k(\Gamma_g)$ such that for $s \in \mathbb{C}$, Res $\gg 0$

(1)
$$L^{st}(F,s) = \zeta(s) \prod_{\substack{j=1\\j \text{ odd}}}^{g-1} \widetilde{L}^H(f,s-\frac{j}{2})\widetilde{L}^H(f,s+\frac{j}{2})$$

with $L^{st}(F,s)$ denoting the standard L-function of F, $\widetilde{L}^{H}(f,s)$ denoting the Hecke L-function of f normalized such that its functional equation is with respect to $s \mapsto 1-s$, and $\zeta(s)$ is the Riemann zeta function.

For g = 2 the DIC reduces to the Saito–Kurokawa correspondence (see e.g. [EZ]). Since the weight of f is 2k - g, one has $\tilde{L}^{H}(f, s) = L^{H}(f, s + \frac{2k-g-1}{2})$ with $L^{H}(f, s)$ denoting the usual Hecke *L*-function of f (with functional equation with respect to $s \mapsto 2k - g - s$) and we can reformulate (1) to

(2)
$$L^{st}(F,s) = \zeta(s) \prod_{j=1}^{\frac{q}{2}} L^{H}(f,s+k-\frac{q}{2}-j)L^{H}(f,s+k-\frac{q}{2}+j-1) = \zeta(s) \prod_{j=1}^{q} L^{H}(f,s+k-j).$$

It is not difficult to see that this factorization is equivalent to a formal factorization of the corresponding local p-factors where p is an arbitrary prime, i.e. (3)

$$\begin{split} L_p^{st}(F,X) &:= \left((1-X) \prod_{i=1}^g (1-\alpha_{p,i}X) (1-\alpha_{p,i}^{-1}X) \right)^{-1} \\ &= \left((1-X) \prod_{j=1}^{\frac{g}{2}} (1-\lambda(p) p^{\frac{g}{2}-k+j}X + p^{2j-1}X^2) \right) \\ &\times (1-\lambda(p) p^{\frac{g}{2}-k-j+1}X + p^{1-2j}X^2) \end{split}$$

Here, $\alpha_{p,i}$ (i = 1, ..., g) are Satake parameters of F, $\lambda(p)$ is the eigenvalue of f with respect to T(p) and X is an indeterminant.

In the whole paper we use the normalization of Andrianov ([An1]) for Hecke operators and Satake parameters.

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Now fix a prime p, choose $a, b \in \mathbb{C}^{\times}$ such that

(4)
$$a+b=\lambda(p), \quad ab=p^{2k-g-1}$$

and define $a_j := p^{\frac{q}{2}-k+j}a$, $b_j := p^{\frac{q}{2}-k+j}b$ $(j = 1, \dots, \frac{q}{2})$. Then (3) can be rewritten as

(5)

$$L_p^{st}(F,X) = \left((1-X) \prod_{j=1}^{\frac{q}{2}} (1-a_j X)(1-b_j X)(1-a_j^{-1} X)(1-b_j^{-1} X) \right)^{-1}$$

so that we can regard (a_j, b_j) $(j = 1, \dots, \frac{g}{2})$ as Satake parameters of F since these parameters are determined up to the action of the Weyl group only.

Since the Satake parameters $\alpha_{p,i}$ $(i = 0, \ldots, g)$ satisfy the relation

$$\alpha_{p,0}^2 \alpha_{p,1} \cdot \ldots \cdot \alpha_{p,g} = p^{gk - \frac{g(g+1)}{2}},$$

we are led to the following

Local Version of the DIC. Let k and g be positive even integers with 2k-g > 0, $f \in M_{2k-g}(\Gamma_1)$ a Hecke eigenform with Satake parameters $\beta_{p,0}, \beta_{p,1}$ for every prime p. Then there exists a Hecke eigenform $F \in M_k(\Gamma_g)$ such that Satake parameters $\alpha_{p,0}, \ldots, \alpha_{p,g}$ of F for every prime p are given by

(6)
$$\begin{aligned} \alpha_{p,0} &= \pm p^{\frac{4gk-3g^2-2g}{8}}, \\ \alpha_{p,j} &= p^{\frac{q}{2}-k+j}\beta_{p,0} \quad (j=1,\ldots,\frac{q}{2}), \\ \alpha_{p,\frac{q}{2}+j} &= p^{\frac{q}{2}-k+j}\beta_{p,0}\beta_{p,1} \quad (j=1,\ldots,\frac{q}{2}). \end{aligned}$$

To bring this local version into a more convenient form for our purposes let us introduce some notation. Let $E_i(X_1, \ldots, X_g)$ $(i = 0, \ldots, g)$ be the *i*-th elementary symmetric polynomial in the variables X_1, \ldots, X_g . For $0 \le \nu \le g$ define

$$R^{g}_{\nu}(X_{1},\ldots,X_{g}) := \sum_{\substack{(r_{1},\ldots,r_{g}) \in \{0,1,-1\}^{g} \\ |r_{1}|+\cdots+|r_{g}|=\nu}} X_{1}^{r_{1}}\ldots X_{g}^{r_{g}}.$$

If $F \in M_k(\Gamma_g)$ is a Hecke eigenform with local Satake parameters $\alpha_{p,0}, \ldots, \alpha_{p,g}$ and eigenvalues t(p) (resp. $t_{i,j}(p^2)$) under the Hecke operators T(p) (resp. $T_{i,j}(p^2)$) for a prime p, then (cf. [Fr], note the different normalization)

(7)
$$t(p) = \alpha_{p,0} \sum_{i=0}^{g} E_i(\alpha_{p,1}, \dots, \alpha_{p,g}),$$
$$t_{i,j}(p^2) = p^{gk - \frac{3g(g+1)}{2}} \sum_{\nu=0}^{i} c_{\nu}(i,j) R_{\nu}^g(\alpha_{p,1}, \dots, \alpha_{p,g})$$

with certain constants $c_{\nu}(i, j)$ which were computed in [Kr].

Especially in the case g = 4 we have a quite explicit version of the DIC:

Remark. Let $F \in M_k(\Gamma_4)$ and $f \in M_{2k-4}(\Gamma_1)$ be eigenforms with eigenvalues t(p), $t_{i,j}(p^2)$, resp. $\lambda(p)$, for every prime p. Then F and f satisfy the DIC if and only if

$$\begin{split} \pm t(p) &= \lambda(p)^2 + p^{k-4}(p+1)(p^2+1)\lambda(p) + p^{2k-7}(p^3+1)(p+1), \\ t_{1,3}(p^2) &= p^{3k-15}(p+1)(p^2+1)\lambda(p) + p^{4k-20}(p^4-1), \\ (8) \quad t_{2,2}(p^2) &= p^{2k-10}(p^2+p+1)(p^2+1)\lambda(p)^2 \\ &\quad + p^{3k-15}(p^4-1)(p^2+p+1)\lambda(p) + p^{4k-18}(p^6-1)(p^2+1), \\ t_{3,1}(p^2) &= p^{k-5}(p+1)(p^2+1)\lambda(p)^3 + p^{2k-10}(p^4-1)(p^2+p+1)\lambda(p)^2 \\ &\quad + p^{3k-13}(p^4-1)(p+1)(p^3+p^2+1)\lambda(p) \\ &\quad + p^{4k-18}(p^6-1)(p^4-1). \end{split}$$

Indeed, assume the DIC holds. Putting (6) into (7) yields the formulas (8).

Conversely, let the formulas in (8) hold. Define parameters $\alpha_{p,0}, \ldots, \alpha_{p,g}$ by (6) and consider the homomorphism $L_p^g \to \mathbb{C}$, $T \mapsto t(F,T)$ where L_p^g is the local Hecke algebra and t(F,T) is the eigenvalue of F under the Hecke operator T (cf. [An2], p. 165 ff.). The above calculation shows that this homomorphism is parameterized by $\alpha_{p,0}, \ldots, \alpha_{p,g}$ whence this in fact defines Satake parameters of F ([An2], p. 168).

2. Non-cusp forms

For $g \in \mathbb{N}$ and $k \in \mathbb{Z}$ even, k > g+1 we denote by E_k^g the Siegel Eisenstein series of weight k with respect to Γ_g . Our aim in this section is to prove the following

Theorem. a) Let $F \in M_k(\Gamma_g)$ and $f \in M_{2k-g}(\Gamma_1)$ be as in the DIC.

(i) If f is a cusp form, then so is F.

(ii) If f is not cuspidal, then $f \in \mathbb{C}E_{2k-g}^1$ and local Satake parameters $\alpha_{p,i}$ (p prime, $i = 0, \ldots, g$) of F are given by

(9)
$$\alpha_{p,0} = 1, \quad \alpha_{p,i} = p^{k-i} \quad (i = 1, \dots, g).$$

In particular, F is not cuspidal.

b) Suppose k > g+1 even. Local Satake parameters of an eigenform $F \in M_k(\Gamma_g)$ for every prime p are given by (9) if and only if $F \in \mathbb{C}E_k^g$.

Corollary. If g and k are positive even integers with k > g+1, then the Eisenstein series E_k^g and E_{2k-q}^1 satisfy the DIC.

Proof of the theorem. a) (i) Assume $\Phi F \neq 0$ where Φ is the Siegel operator. The Žarkovskaja relations ([Za]) imply

$$L^{st}(F,s) = \zeta(s-k+g)\zeta(s+k-g)L^{st}(\Phi F,s) \quad (\text{Re } s \gg 0).$$

This means that for the local components we have

(10)
$$L_p^{st}(F,X) = \left((1-p^{k-g}X)(1-p^{g-k}X)\right)^{-1} L_p^{st}(\Phi F,X)$$

where p is a prime and X is an indeterminant. In the notation of (4) and (5) we obtain from (10) the existence of $j \in \{1, \ldots, \frac{g}{2}\}$ such that

$$p^{k-g} = p^{\frac{g}{2}-k+j}a,$$

resp.

$$p^{k-g} = p^{-\frac{g}{2}+k-j}a^{-1}$$

(or with b instead of a). In any case, by the condition $a + b = \lambda(p)$ we have

(11)
$$p^{2k-g-j} + p^{j-1} = \lambda(p)$$

for a $j \in \{1, \ldots, g\}$. Since f is a cusp form, the famous theorem of Deligne ([De]) implies that

$$|\lambda(p)| \le 2p^{k-\frac{q}{2}-\frac{1}{2}}.$$

Together with (11) we obtain

$$p^{k-\frac{g}{2}-j} + p^{-k+\frac{g}{2}+j-1} \le \frac{2}{\sqrt{p}}$$

with $j \in \{1, \ldots, g\}$ and this is easily seen to be impossible. Hence we obtain a contradiction which proves (i).

(ii) It is well-known that if f is a non-cuspidal Hecke eigenform in $M_{2k-g}(\Gamma_1)$, then f has to be a constant multiple of E_{2k-g}^1 . Hence, in the notation of (i) we have $\lambda(p) = 1 + p^{2k-g-1}$ for every prime p so (3) implies that

(12)
$$L^{st}(F,s) = \zeta(s) \prod_{i=1}^{g} \zeta(s+k-i)\zeta(s-k+i)$$

for Re $s \gg 0$. By [We] this is only possible if $\Phi^g F \neq 0$. Since $\Phi^{g-1}F$ is an eigenform of the Hecke algebra (cf. e.g. [Fr]), we must have $\Phi^{g-1}F \in \mathbb{C}^{\times}E_k^1$ by the arguments at the beginning of (ii). That means that local Satake parameters of $\Phi^{g-1}F$ are given by

 $1, p^{k-1}$

and the Žarkovskaja relations prove what we want.

b) That the Eisenstein series E_k^g has the Satake parameters (9) is obvious. Let us prove the converse. By the arguments above we must have $\Phi^g F \neq 0$ and $\Phi^{g-1} = cE_k^1$ with a suitably chosen $c \in \mathbb{C}^{\times}$. This means that $\Phi^{g-2}F - cE_k^2$ is zero or a cuspidal Hecke eigenform with local Satake parameters

$$1, p^{k-1}, p^{k-2}$$

By [We] the latter is impossible, so $\Phi^{g-2}F = cE_k^2$. Inductively we obtain $F = cE_k^g$ and the theorem is proved.

3. An example:

SCHOTTKY FORM VERSUS DELTA FUNCTION

In this section we prove that the Schottky form J, the (up to a scalar) unique cusp form in $M_8(\Gamma_4)$ (for a nice proof of this cf. [DI]), and the Delta function Δ , the (up to a scalar) unique cusp form in $M_{12}(\Gamma_1)$, satisfy the local DIC for small primes. To construct the Schottky form in a numerically nice way, we essentially proceed as in [Mi].

To be precise, let $Q = (I^{(4)}, iI^{(4)}) \in M(4, 8; \mathbb{C})$ and $\mathcal{E}_8 := \{(x_1, \ldots, x_8) \in \frac{1}{2}\mathbb{Z} : x_{\nu} - x_{\mu} \in \mathbb{Z} \text{ for all } \nu \text{ and } \mu, x_1 + \cdots + x_8 \in 2\mathbb{Z} \}$ be the (up to equivalence) unique even unimodular lattice of rank 8 and consider the Theta series

(13)
$$\Theta(Z) := \sum_{v_1, \dots, v_4 \in \mathcal{E}_8} \det(Q(v_1, \dots, v_4))^4 \exp(\pi i \operatorname{tr}((\langle v_i, v_j \rangle) Z)).$$

Here, tr is the trace of a square matrix, $\langle \cdot, \cdot \rangle$ is the standard scalar product on \mathbb{C}^8 and Z is an element of Siegel's upper half-space $\mathbb{H}_4 := \{X + iY \in \text{Sym}(4; \mathbb{C}) : Y \text{ positive definite}\}$. By [Fr], $\Theta \in S_8(\Gamma_4)$ and as the calculation of some Fourier coefficients shows, $\Theta \neq 0$. The Schottky form J is a constant multiple of Θ normalized such that its Fourier coefficient $a\left(\frac{1}{2}\begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&1&2\\1&1&1&2\end{pmatrix}\right)$ equals 1.

We proceed to calculate some eigenvalues t(p) and $t_{i,j}(p^2)$ of J for small primes p. For $F \in M_k(\Gamma_g)$ a short calculation shows that the S-th Fourier coefficient a(S, T(p)) of F|T(p), resp. $a(S, T_{i,j}(p^2))$ of $F|T_{i,j}(p)$, is given by the formula (cf. [Ma], [Fr])

$$\begin{array}{ll} (14) & a(S,T(p)) & = \displaystyle\sum_{\substack{0 \le r \le g \\ U \in G(r,g-r,0)}} p^{\frac{r(r-2g+2k-1)}{2}} a(\frac{1}{p}S[UD(r,g-r,0)]), \\ (15) & a(S,T_{i,j}(p^2)) & = \displaystyle\sum_{\substack{0 \le r,s \le g \\ r+s \le g \\ U \in G(r,s,g-r-s)}} p^{(2r+s)(k-g+r-1)-r(r-1)} e(S,U) \\ & \times a(\frac{1}{p^2}S[UD(r,s,g-r-s)]) \end{array}$$

Here, the following notations are applied: $D(r, s, t) := \operatorname{diag}(I^{(r)}, pI^{(s)}, p^2I^{(t)}),$ $G(r, s, t) := \operatorname{GL}_g(\mathbb{Z})/\operatorname{GL}_g(\mathbb{Z}) \cap D(r, s, t)\operatorname{GL}_g(\mathbb{Z})D(r, s, t)^{-1}$ (g = r + s + t) and

(16)
$$e(S,U) := \sum_{\substack{M \mod p \\ \mathrm{rk}_p(M) = s - j}} \exp(\frac{2\pi i}{p} \mathrm{tr}(S[U]\mathrm{diag}(0^{(r)}, M^{(s)}, 0^{(g-r-s)}))),$$

where $\operatorname{rk}_p(M)$ denotes the rank of M considered as an element of $\operatorname{M}(q; \mathbb{Z}/p\mathbb{Z})$.

By means of Siegel's formula for the number of solutions of matrix congruences mod p (cf. [Si]), it is possible to express the exponential sum e(S, U) for p > 2 in terms of lower determinants of S[U]. Since we are only dealing with small primes and e(S, U) is integral (replace M in (16) by λM with $\lambda \in (\mathbb{Z}/p\mathbb{Z})^{\times}$), it is easier to calculate e(S, U) using a C++ program.

By straightforward induction one can show that a set of representatives for G(r, s, t) can be chosen in the form

$$\left\{ \begin{pmatrix} x & 1 \\ U & 0 \end{pmatrix} \middle| U^{(g-1)} \in G(r, s, t-1), \ x = (pa, 0), \ a^{(1,r)} \mod p \right\} \\ \dot{\cup} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix} \middle| U^{(g-1)} \in G(r-1, s, t), \ x = (0, a, b)^t, \ a^{(1,s)} \mod p, \ b^{(1,t)} \mod p^2 \right\} \\ \dot{\cup} \left\{ \begin{pmatrix} 0 & 1 & 0 \\ U_1 & 0 & U_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & x & I \end{pmatrix} \middle| (U_1^{(g-1,r)}, U_2^{(g-1,g-r-1)}) \in G(r, s-1, t), \\ x = (0, a)^t, \ a^{(1,t)} \mod p \right\},$$

where g = r + s + t and $G(r, s, t) = \emptyset$ if r < 0 or s < 0 or t < 0.

To calculate the Fourier coefficients a(S, T(p)), $a(S, T_{i,j}(p^2))$ from (14) and (15) we used a C++ program. The program also Minkowski-reduces (cf. [Fr]) the matrices belonging to the Fourier coefficients contributing to the right-hand side of (14), resp. (15).

For p = 2 we get the following results:

By calculating the corresponding Fourier coefficients (cf. Appendix A) we find $t(2) = 8.640, t_{13}(2) = -122.880, t_{22}(2) = 5.160.960$ and $t_{31}(2) = -11.059.200$ in accordance with (8) (for eigenvalues of the Delta function see e.g. [Le]). Further examples of eigenvalues proving the local DIC for the pair (J, Δ) for $p \in \{2, 3, 5, 7\}$ are given in Appendix B.

4. A COUNTEREXAMPLE IN A SMALL WEIGHT

The aim of this section is to show that the DIC – in the form presented in the first section – does not hold in the special case g = k = 12.

So let $f := \Delta \in S_{12}(\Gamma_1)$ be the Delta function as above and assume that there is a $F \in M_{12}(\Gamma_{12})$ such that (2) holds. By the theorem of the second section F has to be cuspidal. Since $L^H(\Delta, 6) \neq 0$, $L^{st}(F, s)$ has a pole at s = 1 which implies that F is a linear combination of the Theta series attached to the 24 classes of even unimodular positive definite matrices of size 24 (see [Bö]). On the other hand, in [BFW] Borcherds, Freitag and Weissauer prove that the vector space of *cusp* forms spanned by these Theta series has dimension one. Furthermore, they construct a cuspidal eigenform F_0 in this space.

To have a counterexample it hence suffices to show that the pair (F_0, Δ) does not satisfy the DIC.

By [BFW] the eigenvalue t(2) of F_0 (in the notation of the first section) is given by

$$2^{\star} \cdot 3^{11} \cdot 5 \cdot 17 \cdot 901.141$$

where the exact exponent of the prime 2 depends on the normalization of the Hecke operator T(2). On the other hand, if F_0 and Δ fulfilled the DIC, local Satake parameters of F_0 would be given by (cf. (6))

(17)
$$\begin{aligned} \alpha_{2,0} &= \pm 2^{15}, \\ \alpha_{2,i} &= 2^{i-6}\beta_{2,0} \quad (i = 1, \dots, 6), \\ \alpha_{2,6+i} &= 2^{i-6}\beta_{2,0}\beta_{2,1} \quad (i = 1, \dots, 6) \end{aligned}$$

where

$$\beta_{2,0}(1+\beta_{2,1}) = \lambda(2) = -24.$$

Substituting the terms on the right-hand sides of (17) into (7) yields

$$t(2) = \pm 2^{13} \cdot 3^{10} \cdot 5^4 \cdot 41 \cdot 167;$$

thus we obtain a contradiction.

Appendix A. Some Fourier coefficients of the Schottky form

The main difficulty in calculating Fourier coefficients a(S) with $S = (s_{ij})$ of the Schottky form by means of (13) is to produce all quadruples $(v_1, v_2, v_3, v_4) \in \mathcal{E}_8^4$ satisfying $(\langle v_i, v_j \rangle)_{1 \le i,j \le 4} = S$ very fast.

First of all it is convenient to have a matrix S with small diagonal entries, so we Minkowski-reduce S (cf. [Fr]). Then for every diagonal element s_{ii} of S we create a list L_i of vectors v in \mathcal{E}_8 satisfying $\langle v, v \rangle = s_{ii}$. Now we let v_1 run in L_1 . For fixed v_1 we create sublists L_{i1} of vectors in L_i satisfying $\langle v_1, v_i \rangle = s_{1i}$. Then we let v_2 run in L_{21} and create sublists of L_{31} and L_{41} and so on. A lot of computing time can be saved if symmetries in v_1 , v_2 , v_3 , v_4 , their components and their signs are considered.

The computations were done on a dual Pentium-II 300 MHz machine using the C++ programming language. The "smaller" coefficients were calculated in a few seconds while the "largest" one we have computed —

$$a\left(\frac{1}{2}\begin{pmatrix}62&31&31&31\\31&62&31&31\\31&31&62&31\\31&31&62&31\\31&31&31&62\end{pmatrix}\right) = 22.942.589.386.402.160.704$$

— took about 60 hours.

Some Fourier coefficients of the Schottky form are listed in the Table 1. For a longer table please contact the authors.

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$\det(2S)$	2S	a(S)	$\det(2S)$	2S	a(S)
5	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$	1	80	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 6 & 2 \\ 1 & 0 & 2 & 6 \end{pmatrix}$	1.344
8	$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$	-40	80	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$	2.368
20	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 6 \end{pmatrix}$	-56	80	$\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 & 4 \end{pmatrix}$	2.880
20	$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$	8	256	$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$	-96.768
20	$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}$	72	320	$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 2 & 2 \\ 0 & 2 & 6 & 2 \\ 0 & 2 & 2 & 6 \end{pmatrix}$	-159.232
80	$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 6 \end{pmatrix}$	1.856	320	$\begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 4 & 2 \\ 2 & 2 & 2 & 8 \end{pmatrix}$	143.872

TABLE 1.

Appendix B.	Some	EIGENVALUES	OF THE	Schottky	FORM
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The following eigenvalues were calculated using the method indicated in the third section.

t(2) =	8.640	t(17) =	7.526.180.295.259.920
t(3) =	3.084.480	t(19) =	54.445.724.082.232.000
t(5) =	1.970.816.400	t(23) =	592.703.957.533.674.240
t(7) =	95.252.550.400	t(29) =	12.925.469.303.112.042.000
t(11) =	49.434.346.166.976	t(31) =	23.706.533.697.033.220.096
t(13) =	287.383.591.359.760		
(02)	100 000	. (-2)	1 400 004 055 000
$t_{13}(2^{-}) =$	-122.880	$t_{13}(5^2) =$	1.623.984.375.000
$t_{13}(2^2) \equiv t_{22}(2^2) \equiv$	-122.880 5.160.960	$\begin{array}{l}t_{13}(5^2) = \\t_{22}(5^2) = \end{array}$	1.623.984.375.000 $2.955.676.303.125.000$
$ t_{13}(2^2) = t_{22}(2^2) = t_{31}(2^2) = $	-122.880 5.160.960 -11.059.200	$t_{13}(5^2) = t_{22}(5^2) = t_{31}(5^2) = $	$\begin{array}{c} 1.623.984.375.000\\ 2.955.676.303.125.000\\ 202.084.501.846.500.000\end{array}$
$t_{13}(2^2) = t_{22}(2^2) = t_{31}(2^2) = t_{13}(3^2) = $	-122.880 5.160.960 -11.059.200 240.919.920	$t_{13}(5^2) = t_{22}(5^2) = t_{31}(5^2) = t_{13}(7^2) = $	$\begin{array}{r} 1.623.984.375.000\\ 2.955.676.303.125.000\\ 202.084.501.846.500.000\\ -237.053.228.960.800\end{array}$
$t_{13}(2^2) = t_{22}(2^2) = t_{31}(2^2) = t_{13}(3^2) = t_{22}(3^2) = $	-122.880 5.160.960 -11.059.200 240.919.920 45.996.809.040	$t_{13}(5^2) = t_{22}(5^2) = t_{31}(5^2) = t_{13}(7^2) = t_{22}(7^2) = $	$\begin{array}{r} 1.623.984.375.000\\ 2.955.676.303.125.000\\ 202.084.501.846.500.000\\ -237.053.228.960.800\\ 3.991.151.395.782.045.600\end{array}$

NOTES ADDED IN PROOF

1) It seems that T. Ikeda very recently has proved the DIC in general. Unfortunately the authors are not aware of any details.

2) E. Freitag kindly informed the authors that the calculations he and his coauthors made in [BFW] and which the authors used to provide a counterexample in section 4 might be incorrect. Fortunately this would coincide with 1).

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