CLASP-PASS MOVE AND VASSILIEV INVARIANTS OF TYPE THREE FOR KNOTS

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ABSTRACT. Recently it has been proved that if and only if two knots K_1 and K_2 have the same value for the Vassiliev invariant of type two, then K_1 can be deformed into K_2 by a finite sequence of clasp-pass moves. In this paper, we determine the difference of the values of the Vassiliev invariant of type three between two knots which can be deformed into each other by a clasp-pass move.

1. INTRODUCTION

In [4], K. Habiro defined the C_k -move and showed the following theorem giving an answer to the question of when two knots take the same value for all Vassiliev invariants of type k , where $k (\geq 2)$ is a fixed non-negative integer. We will study the case of $k = 1, 2$ in this paper.

Theorem 1.1 (Habiro, [4]). *Let $k \geq 1$ be an integer and let K_1 and K_2 be two knots. Then, the following conditions are equivalent.*

- (i) K_1 can be transformed into K_2 by a finite sequence of C_{k+1} -moves.
- (ii) K_1 and K_2 have the same value for any Vassiliev invariant of type k .

We remark here that condition (i) is equivalent to the following condition.

(i') K_1 can be transformed into K_2 by a finite sequence of X_{k+1} -moves, where the X_k -move is the following local move defined by Habiro in [3]:

Here we describe terminology used in this paper. For more details we refer the reader to [1], [2]. For a (complex-valued) knot invariant v_d , we regard it as an

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invariant for spatial 4-valent knotted graphs (or singular knots) as follows:

$$v_d(\text{cross with dot}) = v_d(\text{cross}) - v_d(\text{cross}).$$

If the graph invariant defined above vanishes for all graphs with more than d vertices, then we call v_d a *Vassiliev invariant of type d* . This is equivalent to saying that v_d is constant on all graphs with the same $[d]$ -configuration. Here a $[d]$ -configuration is the cyclically ordered collection of d pairs of $2d$ points on a circle with dotted lines indicating the pairing. For example,

is a $[2]$ -configuration. A singular knot G with d vertices respects a $[d]$ -configuration if each set of paired points forms a vertex in G . The following singular knot respects the $[2]$ -configuration above:

The following equality, called the *4-term relation*, holds for any Vassiliev invariant v .

$$v(\text{diagram 1}) - v(\text{diagram 2}) = v(\text{diagram 3}) - v(\text{diagram 4}).$$

Now we define the local move and the equivalence of local moves.

Definition. Let K_1 and K_2 be two knots. Let t_1 and t_2 be two 1-manifolds properly embedded in the unit 3-ball $\mathbb{B}^3$ with $\partial t_1 = \partial t_2$. We say that we can obtain K_2 from K_1 by the *local move of type $(t_1 \leftrightarrow t_2)$* if there is an embedding $h : \mathbb{B}^3 \rightarrow S^3$ such that $h(\mathbb{B}^3) \cap K_1 = h(t_i)$ and $K_2 \approx (K_1 - h(\mathbb{B}^3)) \cup h(t_j)$, where $(i, j) = (1, 2)$ or $(2, 1)$ and $\approx$ means an ambient isotopy of S^3 .

Definition. We say that two local moves of type $(t_1 \leftrightarrow t_2)$ and type $(t_3 \leftrightarrow t_4)$ are *equivalent* if there is a homeomorphism $\varphi : \mathbb{B}^3 \rightarrow \mathbb{B}^3$ such that $\varphi(t_i) = t_3$ and $\varphi(t_j) = t_4$ rel. $\partial \mathbb{B}^3$, where $(i, j) = (1, 2)$ or $(2, 1)$.

We mention here that, for two equivalent local moves of type $(t_1 \leftrightarrow t_2)$ and type $(t_3 \leftrightarrow t_4)$, the following conditions are equivalent.

- (1) K_2 is obtained from K_1 by the local move of type $(t_1 \leftrightarrow t_2)$.
- (2) K_2 is obtained from K_1 by the local move of type $(t_3 \leftrightarrow t_4)$.

2. Δ -UNKNOTTING OPERATION AND VASSILIEV INVARIANTS OF TYPE TWO

In this section, we study the X_2 -move and Vassiliev invariants of type two. The X_2 -move is equivalent to the Δ -unknotting operation (Figure 2.1).

FIGURE 2.1.

The Δ -unknotting operation was defined by H. Murakami and Y. Nakanishi in [8]. They have already proved that every knot can be transformed into a trivial knot by a finite sequence of Δ -unknotting operations, which is a corollary of Theorem 1.1 since the Vassiliev invariant of type one is a constant. The following theorem implies the theorem showed by M. Okada in [9].

Theorem 2.1. *Let K and K' be two knots that differ by a single Δ -unknotting operation. Then, we have*

$$v_2(K) - v_2(K') = \pm v_2\left(\text{diagram of a circle with two intersecting dashed lines forming an 'X' shape}\right),$$

where v_2 is a Vassiliev invariant of type two.

Corollary 2.2 (Okada, [9]). *Let K and K' be two knots that differ by a single Δ -unknotting operation. Then, we have $a_2(K) - a_2(K') = \pm 1$, where a_2 is the second coefficient of the Conway polynomial.*

Proof of Theorem 2.1. We only show the following since the cases of other orientations can be treated similarly.

$$\begin{aligned} v_2(K) - v_2(K') &= v_2\left(\text{diagram 1}\right) - v_2\left(\text{diagram 2}\right) \\ &= v_2\left(\text{diagram 3}\right) + v_2\left(\text{diagram 4}\right) - v_2\left(\text{diagram 5}\right) - v_2\left(\text{diagram 6}\right) \\ &= v_2\left(\text{diagram 7}\right) - v_2\left(\text{diagram 8}\right) \\ &= v_2\left(\text{diagram 9}\right) + v_2\left(\text{diagram 10}\right) - v_2\left(\text{diagram 11}\right) - v_2\left(\text{diagram 12}\right) \\ &= v_2\left(\text{diagram 13}\right) - v_2\left(\text{diagram 14}\right). \end{aligned}$$

Now we consider the values of their [2]-configurations. Since K is a knot, we may consider two cases as indicated in Figure 2.2 (where dotted lines denote the connecting relations). Thus,

$$\begin{aligned}
 \text{(i)} \quad v_2(K) - v_2(K') &= v_2\left(\begin{array}{c} \text{circle with two vertical dotted lines} \end{array}\right) - v_2\left(\begin{array}{c} \text{circle with two diagonal dotted lines} \end{array}\right) = -v_2\left(\begin{array}{c} \text{circle with two diagonal dotted lines} \end{array}\right), \\
 \text{(ii)} \quad v_2(K) - v_2(K') &= v_2\left(\begin{array}{c} \text{circle with two diagonal dotted lines} \end{array}\right) - v_2\left(\begin{array}{c} \text{circle with two vertical dotted lines} \end{array}\right) = v_2\left(\begin{array}{c} \text{circle with two diagonal dotted lines} \end{array}\right). \quad \square
 \end{aligned}$$

FIGURE 2.2.

3. CLASP-PASS MOVE AND VASSILIEV INVARIANTS OF TYPE THREE

In this section, we study the X_3 -move and Vassiliev invariants of type three. The X_3 -move is equivalent to the move in Figure 3.1, the clasp-pass move, introduced by Habiro. Figure 3.2 shows the equivalence.

By Theorem 1.1, if K and K' have the same value of a_2 , we can define the CP -Gordian distance from K to K' that is the minimum number of clasp-pass moves which are necessary to deform K into K' . The following is our main theorem which is useful to determine the distance.

FIGURE 3.1.

FIGURE 3.2.

Theorem 3.1. *Let K and K' be two knots that differ by a single clasp-pass move. Then, we have*

$$v_3(K) - v_3(K') = 0 \quad \text{or} \quad \pm v_3\left(\text{diagram of a circle with a cross inside}\right),$$

where v_3 is a Vassiliev invariant of type three.

Corollary 3.2. *Let K and K' be two knots that differ by a single clasp-pass move. Then, we have $d^3/dt^3 V_K(1) - d^3/dt^3 V_{K'}(1) = 0$ or ± 36 , where $V_K(t)$ is the Jones polynomial of K .*

Proof. Here we denote $d^3/dt^3 V_K(1)$ simply by $v(K)$. Since $v(K)$ is constant on all singular knots with the same [3]-configuration, we have

$$\begin{aligned} v(K)\left(\text{diagram of a circle with a cross inside}\right) &= v(K)\left(\text{diagram of a knot with two crossings}\right) = -v(K)\left(\text{diagram of a knot with two crossings}\right) \\ &= -v(K)\left(\text{diagram of a knot with one crossing}\right) + v(K)\left(\text{diagram of a knot with one crossing}\right) \\ &= -v(K)\left(\text{diagram of a knot with one crossing}\right) - v(K)\left(\text{diagram of a knot with one crossing}\right). \end{aligned}$$

Since $V_{3_1}(t) = t + t^3 - t^4$ and $V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$, $v(K)\left(\text{diagram of a circle with a cross inside}\right) = 18 + 18 = 36$. □

FIGURE 3.3.

Remark. Although a single Δ -unknotting operation always changes a knot type, there is a single clasp-pass move which does not change the knot type for any knot. Figure 3.3 shows that there is such a single clasp-pass move for the trivial knot. The case of non-trivial knot follows this example since any knot is a connected sum of itself and the trivial knot.

Proof of Theorem 3.1. We only show the following since the case of the other orientation can be treated similarly.

$$\begin{aligned}
 v_3(K) - v_3(K') &= v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &= v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &= v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &= -v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &= -v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &= -v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &\quad + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) \\
 &= v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) + v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right) - v_3\left(\begin{array}{c} \uparrow \\ \text{---} \text{---} \text{---} \\ \downarrow \end{array}\right).
 \end{aligned}$$

FIGURE 3.4.

Now we consider the value of their $[3]$ -configurations. Since K is a knot, we may consider six cases as indicated in Figure 3.4. Thus,

(i)

$$\begin{aligned} v_3(K) - v_3(K') &= v_3\left(\text{circle with 4 vertical dotted lines}\right) - v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right) + v_3\left(\text{circle with 4 horizontal dotted lines}\right) - v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right) \\ &= v_3\left(\text{circle with 4 horizontal dotted lines}\right), \end{aligned}$$

(iii)

$$\begin{aligned} v_3(K) - v_3(K') &= v_3\left(\text{circle with 4 horizontal dotted lines}\right) - v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right) + v_3\left(\text{circle with 4 horizontal dotted lines}\right) - v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right) \\ &= 2v_3\left(\text{circle with 4 horizontal dotted lines}\right) - v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right). \end{aligned}$$

From the 4-term relation, $2v_3\left(\text{circle with 4 horizontal dotted lines}\right) = v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right)$. Thus, $v_3(K) - v_3(K') = 0$.

(iv)

$$\begin{aligned} v_3(K) - v_3(K') &= v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right) - v_3\left(\text{circle with 4 arcs (C)}\right) + v_3\left(\text{circle with 4 diagonal dotted lines (X)}\right) - v_3\left(\text{circle with 4 horizontal dotted lines}\right) \\ &= -v_3\left(\text{circle with 4 horizontal dotted lines}\right). \end{aligned}$$

Since the cases (ii), (v) and (vi) can be treated similarly as above, we omit these cases. Thus, we complete the proof. $\square$

4. APPLICATIONS

In this section, we consider knots with a_2 vanishing.

Example 4.1. Let K be the Kinoshita-Terasaka knot ([6]) and let 0_1 be the trivial knot. Since $a_2(K) = 0$, we can transform K into 0_1 by a series of clasp-pass moves. In fact, we can accomplish that by carrying out the moves twice (Figure 4.1). Thus, the CP -Gordian distance from K to 0_1 is less than or equal to 2. Since $d^3/dt^3 V_K(1) = 72$, we can conclude that the distance is exactly equal to 2 from Corollary 3.2.

FIGURE 4.1.

Next we show that there are two knots which have the same value of the Vassiliev invariant of type three and can be transformed into each other by a single clasp-pass move.

Example 4.2. Since $a_2(8_{14}) = 0$, we can transform 8_{14} into 0_1 by a series of clasp-pass moves. Moreover, $d^3/dt^3 V_{8_{14}}(1) = 0$ and we can transform 8_{14} into 0_1 by a single clasp-pass move (Figure 4.2).

FIGURE 4.2.

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REFERENCES

1. J. S. Birman, *New point of view in knot theory*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), 253–287. MR **94b**:57007
2. J. S. Birman and X. -S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270. MR **94d**:57010
3. K. Habiro, Master thesis of University of Tokyo (1994).
4. K. Habiro, *Claspers and the Vassiliev skein modules*, preprint, University of Tokyo (1997).
5. K. Habiro, *Clasp-pass-moves on knots*, preprint, University of Tokyo (1997).
6. S. Kinoshita and H. Terasaka, *On unions of knots*, Osaka Math. J. **9** (1957), 131–153. MR **20**:4846
7. H. Murakami, *Some metrics on classical knots*, Math. Ann. **270** (1985), 35–45. MR **86g**:57007
8. H. Murakami and Y. Nakanishi, *On a certain move generating link-homology*, Math. Ann. **284** (1989), 75–89. MR **90f**:57007
9. M. Okada, *Delta-unknotted operation and the second coefficient of the Conway polynomial*, J. Math. Soc. Japan **42** (1990), 713–717. MR **91i**:57002

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