AN INDUCTIVE EXPLICIT CONSTRUCTION OF $*$ -PRODUCTS ON SOME POISSON MANIFOLDS

SANTOS ASIN LARES

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ABSTRACT. We extend the Cahen Gutt coboundary construction on cotangent bundles of n -dimensional parallelisable manifolds to manifolds which admit n global vector fields defining a parallelisation on a dense open set. This result is used to give an inductive explicit construction of $*$ -products on certain Poisson manifolds.

INTRODUCTION

The theory of formal deformation quantization of Poisson manifolds was introduced by Bayen et al. in [2]. The main notion of this theory is the concept of a $*$ -product. The general question of the existence of such a product for symplectic manifolds has been completely solved by several authors, using various techniques [5],[14],[11]. Recently, M. Kontsevitch has proved the existence of $*$ -products on arbitrary finite-dimensional Poisson manifolds [9]. Nevertheless, since Kontsevitch's result is not given by a simple geometrical construction, it has increased the interest of having a simple geometrical of $*$ -products on non-regular Poisson manifolds.

Since every Poisson manifold splits into a collection of symplectic submanifolds, known as the *leaves of the symplectic foliation*, one naturally asks whether a $*$ -product on a Poisson manifold restricts to give a $*$ -product on the symplectic leaves. Lately, in [4],[13] it has been proved that such $*$ -products do not always exist. When they exist we called them *tangential*. In particular, the dual of the so-called “book algebra” with the *Lie Poisson structure* admits a tangential $*$ -product [1]. Furthermore, this example provides the basic idea to construct explicit $*$ -products on some other Poisson manifolds (see Theorem 1).

The goal of this paper will be to give such a construction. In order to do this, we first generalize a coboundary construction due to Cahen and Gutt [3], and use this result to construct explicitly the $*$ -product.

Using a different method a similar result has been obtained in [7]. However, the approach used there does not provide an explicit construction of a $*$ -product.

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*-PRODUCTS

A *Poisson structure* on a manifold M is a Lie algebra structure $\{\cdot, \cdot\}$ on $C^\infty(M)$ which satisfies the derivation property

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \forall f, g, h \in C^\infty(M).$$

The operation $\{\cdot, \cdot\}$ determines a contravariant skew-symmetric 2-tensor Λ such that $\{f, g\} = \Lambda(df, dg)$. A Poisson structure may also be defined by such a tensor (the Poisson tensor); the Jacobi identity for the Poisson structure is equivalent to the vanishing of the so-called *Schouten bracket* $[\Lambda, \Lambda]_s = 0$ (see [2]).

If (M, Λ) is a Poisson manifold we set $N = C^\infty(M)$. Let $N[[\lambda]]$ be the space of formal power series in a parameter λ , with coefficients in N .

Definition 1 ([2]). A $*$ -product on (M, Λ) is a bilinear map $N^2 \longrightarrow N[[\lambda]]$ defined by

$$(1) \quad (f, g) \longrightarrow f * g = \sum_{n=0}^{\infty} \lambda^n C_n(f, g),$$

where the so-called *cochains* C_n , are bilinear maps with values in N and satisfy the following axioms:

1. $C_0(f, g) = fg$, $C_1(f, g) = \{f, g\}$, $\forall f, g \in C^\infty(M)$;
2. $C_n(f, g) = (-1)^n C_n(g, f)$, $\forall f, g \in C^\infty(M)$, $\forall n \geq 1$;
3. $C_n(f, k) = 0$, $\forall f \in C^\infty(M)$, $\forall k \in \mathbb{R}$, $\forall n \geq 1$;
4. $\sum_{r+s=k} C_r(C_s(f, g), h) = \sum_{r+s=k} C_r(f, C_s(g, h))$, $k \geq 0$.

The theory of *deformations* in the sense of [6] relates the deformations of an associative algebra to the corresponding *Hochschild cohomology*.

Definition 2. A (Hochschild) p -cochain is a p -linear map $N^p \rightarrow N$. The *Hochschild coboundary* of a p -cochain is the $(p+1)$ -cochain ∂C given by

$$\begin{aligned} \partial C(u_0, \dots, u_p) &= u_0 C(u_1, \dots, u_p) - C(u_0 u_1, u_2, \dots, u_p) + C(u_0, u_1 u_2, \dots, u_p) \\ &+ \dots + (-1)^p C(u_0, u_1, \dots, u_{p-1} u_p) + (-1)^{p+1} C(u_0, \dots, u_{p-1}) u_p. \end{aligned}$$

A cochain C is called differential if it is defined by multi-differential operators in each argument. A $*$ -product is called differential if all its cochains are differential. In [12] it has been proved that if E is a p -cocycle (differential and null on the constants), then there exist a skew-symmetric contravariant smooth p -tensor A and a $(p-1)$ -cochain C such that

$$(2) \quad E(f_1, \dots, f_p) = \partial C(f_1, \dots, f_p) + A(df_1, \dots, df_p), \quad f_i \in C^\infty(M).$$

A bilinear map (1) is said to be an *associative formal deformation* up to the order k if

$$(3) \quad (f * g) * h - f * (g * h) = 0, \quad f, g, h \in C^\infty(M),$$

is satisfied modulo λ^{k+1} . By developing (3) into powers of λ , the coefficients of λ^t will vanish if

$$(4) \quad E_t(f, g, h) := \sum_{r+s=t, \ r, s \geq 1} C_r(C_s(f, g), h) - C_r(f, C_s(g, h)) = \partial C_t(f, g, h).$$

Thus, an associative formal deformation up to the order k can be extended to one of order $k+1$ provided that the cocycle E_{k+1} is a 3-coboundary.

TANGENTIAL *-PRODUCTS

Let (M, Λ) be a Poisson manifold, and let O be a symplectic leaf.

Definition 3. Let $x \in O$. A differential operator D on M is *tangential* to O at x , if there exist a neighbourhood V of x in O and a neighbourhood U of V in M , such that when $\varphi_1, \varphi_2 \in C^\infty(U)$ with $\varphi_1|_V = \varphi_2|_V$, then

$$D(\varphi_1)|_V = D(\varphi_2)|_V.$$

A bi-differential operator C on M is said to be *tangential* to O , if for any function $f \in C^\infty(M)$, the differential operators $C(f, \cdot)$ and $C(\cdot, f)$ are tangential to O , at x for all $x \in O$.

Definition 4. A differential *-product is called tangential to O , if all its cochains C_n , $n \geq 1$, are tangential.

A COBOUNDARY CONSTRUCTION

In what follows, we shall use the summation convention on pairs of upper and lower indices. Let (M, Λ) be a Poisson manifold of dimension n , and let $T^1, \dots, T^n$ be smooth vector fields on M such that they are pointwise linearly independent on a dense open set of M . The following proposition is a simple generalization of Proposition 2 in [3]. The argument given in [3] is combinatorial, and is based on 3 lemmas which in fact only require independence of the vector fields T^i on a dense open set.

Proposition 1 ([3]). Let E be a differential 3-cocycle (null on the constants), of the form

$$E(f, g, h) = \sum_{0 < a, b, c \leq K} E_{i_1 \dots i_a, j_1 \dots j_b, k_1 \dots k_c} T^{i_1} \dots T^{i_a} f T^{j_1} \dots T^{j_b} g T^{k_1} \dots T^{k_c} h,$$

where $f, g, h \in C^\infty(M)$, and $E_{i_1 \dots i_a, j_1 \dots j_b, k_1 \dots k_c}$ are smooth functions on M symmetric in the i 's, in the j 's and in the k 's. Then, there is a 2-cochain C completely determined by E of the form

$$(5) \quad C(f, g) = \sum_{0 < p, q \leq K} C_{i_1 \dots i_p, j_1 \dots j_q} T^{i_1} \dots T^{i_p} f T^{j_1} \dots T^{j_q} g,$$

such that $E = \partial C + A$, where A is the completely antisymmetric part of E , i.e., a 3-contravariant smooth tensor. Moreover, the coefficients $C_{i_1 \dots i_p, j_1 \dots j_q}$ are constant (rational) linear combinations of the coefficients $E_{k_1 \dots k_a, l_1 \dots l_b, m_1 \dots m_c}$ of E .

Remark 1. Note that if the C_r 's ($r \leq k$) satisfy the symmetry properties of Definition 1, then the cocycle E_{k+1} satisfies $E_{k+1}(f, g, h) = (-1)^k E_{k+1}(h, g, f)$. Thus, if $E_{k+1} = \partial C_{k+1}$ we can always assume that $C_{k+1}(f, g) = (-1)^{k+1} C_{k+1}(g, f)$ just by replacing C_{k+1} by its symmetrization or antisymmetrization.

Theorem 1. Let (M, Λ) be a Poisson manifold, and let us assume that there exist T^1, T^2 smooth vector fields on M such that they are pointwise linearly independent on a dense open set of M , and such that Λ can be written as $\Lambda = T^1 \wedge T^2$. Then, there is a *-product on (M, Λ) with 2-cochains C_r of the form (5).

Proof. By assumption the Poisson structure C_1 on M admits the expression $C_1(f, g) = T^1 f T^2 g - T^2 f T^1 g$. Let us assume that there exist k ($k \geq 1$) 2-cochains $C_1, \dots, C_k$ constructed recursively (using Proposition 1) from the equations

$$E_t = \partial C_t \quad t = 1, \dots, k,$$

defining a deformation (up to the order k) on M so that

$$C_t(f, g) = \sum_{0 < p, q \leq K_t} C_{i_1 \dots i_p, j_1 \dots j_q} T^{i_1} \dots T^{i_p} f T^{j_1} \dots T^{j_q} g,$$

where all the T^i 's and T^j 's are T^1 or T^2 .

Since $[T^1 \wedge T^2, T^1 \wedge T^2]_s = 0$, it follows that $[T^1, T^2]_s = f_1 T^1 + f_2 T^2$ ($f_1, f_2 \in C^\infty(M)$), and so that E_{k+1} expressed as in Proposition 1 only includes T^1 's and T^2 's. Let C_{k+1} and A_{k+1} be the 2-cochains constructed by means of Proposition 1 such that $E_{k+1} = \partial C_{k+1} + A_{k+1}$. Then, since A_{k+1} is a 3-contravariant (skew-symmetric) tensor only including T^1 and T^2 , it follows that A_{k+1} vanishes, i.e., $E_{k+1} = \partial C_{k+1}$. Thus, the theorem follows by induction on k . $\square$

Remark 2. The $*$ -product constructed is tangential to the 2-dimensional symplectic leaves.

EXAMPLES

Example 1. Let $\mathfrak{g}$ be the Lie algebra (*book algebra*) with basis (e_1, e_2, e_3) , such that $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$. Let (x_1, x_2, x_3) be a coordinate system on $\mathfrak{g}^*$ determined by the dual basis (e^1, e^2, e^3) . The Lie-Poisson structure Λ can be expressed in terms of the above global coordinate system as

$$\Lambda = \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \wedge \frac{\partial}{\partial x_3}.$$

Therefore, using Theorem 1 we get an explicit $*$ -product (in fact, the *Gutt $*$ -product* [8]).

Example 2. Let us consider the Lie group $SU(2)$, and let us choose

$$e_2 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

as a basis of its Lie algebra $\mathfrak{su}(2)$.

Let $Sp(1)$ be the group of unitary quaternions. We identify $Sp(1)$ and $SU(2)$ as Lie groups by means of

$$\psi : Sp(1) \longrightarrow SU(2),$$

$$(x_1, x_2, x_3, x_4) \longrightarrow \begin{pmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{pmatrix}.$$

As usual, we denote by R_g (L_g) the right translation (left translation) map. Let $\mathbf{r} = e_3 \wedge e_4 \in \wedge^2 \mathfrak{su}(2)$; then the *Iwasawa-Poisson-Lie structure* π on $SU(2)$ is defined by [10]

$$\pi(g) := dR_g \mathbf{r} - dL_g \mathbf{r}, \quad g \in SU(2).$$

The *linearization* (at the identity) of this Poisson structure is *isomorphic* to the book algebra, and therefore this Poisson structure can be considered as the non-linear version of that in Example 1. Let X_i ($i = 2, 3, 4$) be the right invariant vector fields on $SU(2)$ corresponding to e_i . We define two vector fields on $SU(2)$ by setting

$$T^1 := x_2 X_2 + x_3 X_3 + x_4 X_4,$$

$$T^2 := (-2x_1)X_2 + (-2x_4)X_3 + (2x_3)X_4.$$

A straightforward computation shows that the Poisson structure π on $SU(2)$ can be written as $\pi = T^1 \wedge T^2$. Hence, using Theorem 1 one can construct an explicit *-product on this Poisson manifold (the non-linear version of the one constructed in the previous example).

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MATHEMATICAL INSTITUTE, UNIVERSITY OF WARWICK, CV4-7AL, UNITED KINGDOM
E-mail address: asin@maths.warwick.ac.uk