THE FURUTA INEQUALITY IN BANACH *-ALGEBRAS

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ABSTRACT. Let $0 < p, q, r \in \mathbb{R}$ be real numbers with $p + 2r \leq (1 + 2r)q$ and $1 \leq q$. Furuta (1987) proved that if bounded linear operators $A, B \in B(H)$ on a Hilbert space H satisfy $0 \leq B \leq A$, then $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$. This inequality is called the Furuta inequality and has many applications. In this paper, we prove that the Furuta inequality holds in a unital hermitian Banach *-algebra with continuous involution.

Let A, B be bounded linear operators on a Hilbert space H . The celebrated Löwner-Heinz inequality states the following;

Theorem A (Löwner-Heinz inequality [4], [5]). *Let $A, B \in B(H)$ satisfy $0 \leq B \leq A$. If $0 < p < 1$, then $B^p \leq A^p$.*

For an extension of Theorem A, Furuta obtained the following interesting inequality in [1] and one page elementary proof in [2].

Theorem B (Furuta inequality [1], [2]). *Let $0 \leq p, q, r \in \mathbb{R}$ and $A, B \in B(H)$ satisfy $0 \leq B \leq A$. If $p + 2r \leq (1 + 2r)q$ and $1 \leq q$ as shown in the figure, then $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$.*

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One of the authors, Tanahashi [8], proved that the domain drawn for p, q and r in Figure is the best possible one for Theorem B. Many applications of Theorem B have been developed, but we omit citing them.

Let A be a unital Banach $*$ -algebra with unit e and $a, b \in A$. A is called hermitian if $a = a^*$ implies $\sigma(a) \subset \mathbb{R}$ where $\sigma(a)$ denotes the spectrum of a . $a \geq 0$ means that $a = a^*$ and $\sigma(a) \subset [0, \infty)$. $a > 0$ means $a \geq 0$ and $0 \notin \sigma(a)$. $a \geq b$ means $a - b \geq 0$, and $a > b$ means $a - b > 0$. Let $\alpha \in \mathbb{R}$. For $a > 0$, a^α means $\exp(\alpha \log a)$ where $\log$ is the principal branch of the complex logarithm. By the Shirali-Ford theorem [7], $\sigma(a^*a) \subset [0, \infty)$ for all $a \in A$.

Recently, Okayasu [6] proved that the Löwner-Heinz inequality holds in a unital hermitian Banach $*$ -algebra with continuous involution.

Theorem C ([6]). *Let A be a unital hermitian Banach $*$ -algebra with continuous involution. Let $a, b \in A$ and $p \in (0, 1]$. Then $a^p < b^p$ if $0 < a < b$, and $a^p \leq b^p$ if $0 < a \leq b$.*

In this paper, we prove that the Furuta inequality holds in this case.

Theorem 1. *Let A be a unital hermitian Banach $*$ -algebra with continuous involution and $a, b \in A$. Let $0 \leq p, q, r \in \mathbb{R}$ satisfy $p + 2r \leq (1 + 2r)q$ and $1 \leq q$. Then*

$$b^{\frac{p+2r}{q}} < (b^r a^p b^r)^{\frac{1}{q}} \quad \text{if } 0 < b < a,$$

and

$$b^{\frac{p+2r}{q}} \leq (b^r a^p b^r)^{\frac{1}{q}} \quad \text{if } 0 < b \leq a.$$

To prove Theorem 1, we need following lemmas.

Lemma 2. *If $0 < a$, then $0 < a^{-1}$.*

Proof. Let $0 < a$. Then $\sigma(a) \subset (0, \infty)$, and $\sigma(a^{-1}) = \sigma(a)^{-1} \subset (0, \infty)$. This implies $0 < a^{-1}$. $\square$

Lemma 3. *If $0 < a < e$, then $e < a^{-1}$. Also, if $0 < a \leq e$, then $e \leq a^{-1}$.*

Proof. Let $0 < a < e$. Then

$$a^{-1} - e = a^{-\frac{1}{2}}(e - a)a^{-\frac{1}{2}} = \{(e - a)^{\frac{1}{2}}a^{-\frac{1}{2}}\}^* \{(e - a)^{\frac{1}{2}}a^{-\frac{1}{2}}\}.$$

This implies $0 < a^{-1} - e$ and $e < a^{-1}$. The rest of the proof is similar. $\square$

Lemma 4. *If $0 < b < a$, then $0 < a^{-1} < b^{-1}$. Also, if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.*

Proof. Let $0 < b < a$. Then

$$a^{-\frac{1}{2}}ba^{-\frac{1}{2}} = \left(b^{\frac{1}{2}}a^{-\frac{1}{2}}\right)^* \left(b^{\frac{1}{2}}a^{-\frac{1}{2}}\right)$$

and

$$e - a^{-\frac{1}{2}}ba^{-\frac{1}{2}} = a^{\frac{1}{2}}(a - b)a^{-\frac{1}{2}} = \left\{(a - b)^{\frac{1}{2}}a^{-\frac{1}{2}}\right\}^* \left\{(a - b)^{\frac{1}{2}}a^{-\frac{1}{2}}\right\}.$$

This implies $0 < a^{-\frac{1}{2}}ba^{-\frac{1}{2}} < e$. Hence, by Lemma 3,

$$e < \left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-1} = a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}.$$

Then

$$\begin{aligned} b^{-1} - a^{-1} &= a^{-\frac{1}{2}} \left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} - e \right) a^{-\frac{1}{2}} \\ &= \left\{ \left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} - e \right) a^{-\frac{1}{2}} \right\}^* \left\{ \left(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} - e \right) a^{-\frac{1}{2}} \right\}. \end{aligned}$$

This implies $0 < b^{-1} - a^{-1}$. Hence $0 < a^{-1} < b^{-1}$ by Lemma 2.

Let $0 < b \leq a$. Then $0 < a$ by Lemma 2 of [6]. The rest of the proof is similar. $\square$

Lemma 5. *If $0 < b < a$ and $0 < c$, then $cbc < cac$. Also, if $0 < b \leq a$ and $0 < c$, then $cbc \leq cac$.*

Proof. Let $0 < b < a$ and $0 < c$. Then

$$cac - cbc = c(a - b)c = \left\{ (a - b)^{\frac{1}{2}} c \right\}^* \left\{ (a - b)^{\frac{1}{2}} c \right\}.$$

This implies $0 < cac - cbc$ and $cbc < cac$. The rest of the proof is similar. $\square$

Lemma 6. *If $0 < a$, then $(a^2)^{\frac{1}{2}} = a$.*

Proof. Since $\sigma(a^2) = \{\sigma(a)\}^2$, there exists a positively oriented circle Λ containing $\sigma(a^2)$ with $0 < \Re \lambda$ for all $\lambda \in \Lambda$. Let

$$\lambda = |\lambda|e^{i\theta} \in \Lambda \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

and

$$\Gamma = \{\mu = |\lambda|^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \mid \lambda = |\lambda|e^{i\theta} \in \Lambda\}.$$

Then

$$\begin{aligned} \log a^2 &= \int_{\Lambda} (\log \lambda) (\lambda - a^2)^{-1} d\lambda \\ &= \int_{\Gamma} (\log \mu^2) (\mu^2 - a^2)^{-1} 2\mu d\mu \\ &= \int_{\Gamma} 2(\log \mu) (\mu + a)^{-1} d\mu + \int_{\Gamma} 2(\log \mu) (\mu - a)^{-1} d\mu \\ &= 0 + 2 \log a = 2 \log a. \end{aligned}$$

Thus

$$(a^2)^{\frac{1}{2}} = e^{\frac{1}{2} \log a^2} = e^{\log a} = a.$$

$\square$

Lemma 7. *If $0 < a, b$ and λ is a real number, then*

$$(bab)^{\lambda} = ba^{\frac{1}{2}} \left(a^{\frac{1}{2}} b^2 a^{\frac{1}{2}} \right)^{\lambda-1} a^{\frac{1}{2}} b.$$

Proof. Let

$$F(\lambda) = (bab)^{\lambda} \quad \text{and} \quad G(\lambda) = ba^{\frac{1}{2}} \left(a^{\frac{1}{2}} b^2 a^{\frac{1}{2}} \right)^{\lambda-1} a^{\frac{1}{2}} b.$$

Since $bab = \left(a^{\frac{1}{2}}b\right)^* \left(a^{\frac{1}{2}}b\right)$, we have $0 < F(\lambda), G(\lambda)$. By the definitions, we have $F(1) = G(1)$. Also,

$$\begin{aligned} \{G(2^{-1})\}^2 &= \left\{ba^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b\right\} \left\{ba^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b\right\} \\ &= ba^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b^2a^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b \\ &= bab = \{F(2^{-1})\}^2. \end{aligned}$$

Hence $F(2^{-1}) = G(2^{-1})$ by Lemma 6. Similarly, we have

$$F(2^{-n}) = G(2^{-n}) \quad \text{for } n = 2, 3, \dots$$

Since $F(\lambda), G(\lambda)$ are analytic on the real line $\mathbb{R}$ and $2^{-n} \rightarrow 0 (n \rightarrow \infty)$, we have

$$F(\lambda) = G(\lambda) \quad \text{for } \lambda \in \mathbb{R}.$$

□

Remark. We remark that Lemma 7 has been obtained in [3] for operator case on Hilbert space, but our method in Banach $*$ -algebra case is completely different from one in [3].

Proof of Theorem 1. Let $0 < b < a$. We show first the case $0 < p \leq 1$. In this case, $b^p < a^p$ by Theorem C, and $b^{p+2r} < b^r a^p b^r$ by Lemma 5. Hence $b^{\frac{p+2r}{q}} < (b^r a^p b^r)^{\frac{1}{q}}$ by Theorem C. We consider the case $1 < p$. We may assume $q = \frac{p+2r}{1+2r}$ by Theorem C. Let $0 < r \leq \frac{1}{2}$. Then $0 < b^{2r} < a^{2r}$ by Theorem C. Hence

$$\begin{aligned} b^{-r} (b^r a^p b^r)^{\frac{1}{q}} b^{-r} &= b^{-r} b^r a^{\frac{p}{2}} \left(a^{\frac{p}{2}} b^{2r} a^{\frac{p}{2}}\right)^{\frac{1}{q}-1} a^{\frac{p}{2}} b^r b^{-r} \\ &= a^{\frac{p}{2}} \left(a^{\frac{p}{2}} b^{2r} a^{\frac{p}{2}}\right)^{\frac{1-p}{p+2r}} a^{\frac{p}{2}} \end{aligned}$$

by Lemma 7. Since $a^{\frac{p}{2}} b^{2r} a^{\frac{p}{2}} < a^{p+2r}$ by Lemma 5 and $-1 < \frac{1-p}{p+2r} < 0$, we have

$$b^{-r} (b^r a^p b^r)^{\frac{1}{q}} b^{-r} > a^{\frac{p}{2}} (a^{p+2r})^{\frac{1-p}{p+2r}} a^{\frac{p}{2}} = a > b$$

by Lemma 4 and Theorem C. Hence

$$(1) \quad (b^r a^p b^r)^{\frac{1}{q}} > b^{1+2r} \quad \text{for } q = \frac{p+2r}{1+2r}$$

by Lemma 5. The rest of the proof is the same as in [1], but we cite its proof for the sake of importance as follows.

Put $a_1 = (b^r a^p b^r)^{\frac{1}{q}}$ and $b_1 = b^{1+2r}$ in (1). Repeating (1) again for $a_1 > b_1 > 0, 0 < r_1 \leq \frac{1}{2}$ and $1 < p_1$,

$$(b_1^{r_1} a_1^{p_1} b_1^{r_1})^{\frac{1}{q_1}} > b_1^{1+2r_1} \quad \text{for } q_1 = \frac{p_1+2r_1}{1+2r_1}.$$

Put $p_1 = q \geq 1$ and $r_1 = \frac{1}{2}$; then

$$(b^{2r+\frac{1}{2}} a^p b^{2r+\frac{1}{2}})^{\frac{1}{q_1}} > b^{2(1+2r)}.$$

Put $s = 2r + \frac{1}{2}$. Then $q_1 = \frac{p_1 + 2r_1}{1 + 2r_1} = \frac{p + 2s}{1 + 2s}$ since $p_1 = q$ and $2(1 + 2r) = 1 + 2s$. Hence

$$(2) \quad (b^s a^p b^s)^{\frac{1}{q_1}} > b^{1+2s} \quad \text{for } q_1 = \frac{p + 2s}{1 + 2s}.$$

Since $r \in (0, \frac{1}{2}]$ in (1), $s = 2r + \frac{1}{2} \in (\frac{1}{2}, \frac{3}{2}]$ in (2). Consequently (1) holds for $r \in (0, \frac{3}{2}]$ and $q = \frac{p+2r}{1+2r}$. Repeating this method, (1) holds for each $r > 0$ and $q = \frac{p+2r}{1+2r}$.

If $0 < b \leq a$, then a is invertible by Lemma 2 of [6] and $0 < a^{-1} \leq b^{-1}$. The rest of the proof is similar to the case $0 < b < a$. $\square$

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REFERENCES

- [1] T. Furuta, $A \geq B \geq O$ assures $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq (p + 2r)$, Proc. Amer. Math. Soc., **101** (1987), 85–88. MR **89b**:47028
- [2] T. Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126. MR **90g**:47029
- [3] T. Furuta, *Two operator functions with monotone property*, Proc. Japan Acad., **111** (1991), 511–516. MR **91f**:47023
- [4] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann., **123** (1951), 415–438. MR **13**:471f
- [5] K. Löwner, Über monotone Matrixfunktionen, Math. Z., **38** (1934), 177–216.
- [6] T. Okayasu, Heinz's inequality in Banach *-algebras, (preprint).
- [7] S. Shiralì and J. W. M. Ford, *Symmetry in complex involutory Banach algebras II*, Duke Math. J., **37** (1970), 275–280. MR **41**:5977
- [8] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146. MR **96d**:47025

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