ON THE DIVERGENCE OF THE $(C, 1)$ MEANS OF DOUBLE WALSH-FOURIER SERIES

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(Communicated by Christopher D. Sogge)

ABSTRACT. In 1992, Móricz, Schipp and Wade proved the a.e. convergence of the double $(C, 1)$ means of the Walsh-Fourier series $\sigma_n f \rightarrow f$ ($\min(n_1, n_2) \rightarrow \infty, n = (n_1, n_2) \in \mathbb{N}^2$) for functions in $L \log^+ L(I^2)$ (I^2 is the unit square). This paper aims to demonstrate the sharpness of this result. Namely, we prove that for all measurable function $\delta : [0, +\infty) \rightarrow [0, +\infty)$, $\lim_{t \rightarrow \infty} \delta(t) = 0$ we have a function f such as $f \in L \log^+ L \delta(L)$ and $\sigma_n f$ does not converge to f a.e. (in the Pringsheim sense).

INTRODUCTION

This paper is devoted to the problem of a.e. divergence of the $(C, 1)$ means of integrable functions with respect to the two-dimensional Walsh-Paley system. The problem of a.e. Cesàro summability is “quite delicate” in any local field setting (Taibleson, [6, p.114]). The dyadic case is no exception (see Fine [1], [2]). For double Walsh-Fourier series, Móricz, Schipp and Wade proved [4] that $\sigma_{(n_1, n_2)} f$ converges to f a.e. in the Pringsheim sense (that is, no restriction on the indices n_1, n_2 other than $\min(n_1, n_2) \rightarrow \infty$) for all function f in $L \log^+ L$. Since for the compact set I^2 (the unit square) the space $L \log^+ L$ is a proper subset of $L^1(I^2)$, then it is interesting to ask whether this theorem holds for $L^1(I^2)$ functions also? We give a negative answer. More precisely, we prove that the theorem of Móricz, Schipp and Wade cannot be sharpened. Namely, let $\delta : [0, +\infty) \rightarrow [0, +\infty)$ be a measurable function with property $\lim_{t \rightarrow \infty} \delta(t) = 0$. We prove the existence of a function $f \in L^1(I^2)$ such as $f \in L \log^+ L \delta(L)$ (i.e., $|f(x)| \log^+(|f(x)|) \delta(|f(x)|) \in L^1(I^2)$) and $\sigma_{(n_1, n_2)} f$ does not converge to f a.e. as $\min(n_1, n_2) \rightarrow \infty$.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $I := [0, 1)$. For any set E let E^2 be the cartesian product $E \times E$. Thus $\mathbb{N}^2$ is the set of integral lattice points in the first quadrant and I^2 is the unit square. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j -dimensional Lebesgue measure (μ) of any set $E \subset I^j$ by $\mu(E)$. Denote the $L^p(I^j)$ norm of any function f by $\|f\|_p$ ($1 \leq p \leq \infty$).

Denote the dyadic expansion of $n \in \mathbb{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \frac{k}{2^m}$ $k, m \in \mathbb{N}$ choose the expansion which

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Received by the editors July 13, 1998.

1991 *Mathematics Subject Classification.* Primary 42C10; Secondary 43A75, 40G05, 42B08.

Key words and phrases. Walsh group, double $(C, 1)$ means, divergence.

Research supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. F020334 and by the Hungarian “Művelődési és Köznevelési Minisztérium”, grant no. FKFP 0710/1997.

terminates in zeros). n_i, x_i are the i -th coordinates of n, x , respectively. Set $e_i := 1/2^{i+1} \in I$, the i th coordinate of e_i is 1, the rest are zeros ($i \in \mathbb{N}$). Define the dyadic addition $+$ as

$$x + y = \sum_{j=0}^{\infty} (x_j + y_j \bmod 2) 2^{-j-1}.$$

The sets $I_n(x) := \{y \in I : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbb{P}$ and $I_0(x) := I$ are the dyadic intervals of I . Denote $\mathcal{I} := \{I_n(x) : x \in I, n \in \mathbb{N}\}$; the elements of $\mathcal{I}$ are called the dyadic intervals on I . Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ ($x \in I$) and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ ($n \in \mathbb{N}$) ($f \in L^1$). For $t = (t^1, t^2) \in I^2$, $b = (b_1, b_2) \in \mathbb{N}^2$ set the two-dimensional dyadic interval

$$I_b^2(t) := I_{b_1}(t^1) \times I_{b_2}(t^2).$$

If $b \in \mathbb{N}$, then $I_b^2(t) := I_b(t^1) \times I_b(t^2)$. For $n = (n_1, n_2) \in \mathbb{N}^2$ denote by $E_n = E_{(n_1, n_2)}$ the two-dimensional expectation operator with respect to $\mathcal{A}_n = \mathcal{A}_{(n_1, n_2)} = \mathcal{A}_{n_1} \times \mathcal{A}_{n_2}$. For $n \in \mathbb{N}$ denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. The Rademacher functions are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in I, n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in I, n \in \mathbb{N}).$$

That is, $\omega := (\omega_n, n \in \mathbb{N})$. Let us consider the Dirichlet and Fejér kernel functions:

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n := \frac{1}{n} \sum_{k=1}^n D_k, \quad D_0, K_0 := 0.$$

The Fourier coefficients, the n -th partial sum of the Fourier series, the n -th $(C, 1)$ mean of $f \in L^1(I)$:

$$\hat{f}(n) := \int_I f(x) \omega_n(x) d\mu(x) \quad (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k) \omega_k(y) = \int_I f(x+y) D_n(x) d\mu(x) \quad (n \in \mathbb{P}, S_0 f = 0),$$

$$\sigma_n(y) := \frac{1}{n} \sum_{k=1}^n S_k(y) = \int_I f(x+y) K_n(x) d\mu(x) \quad (n \in \mathbb{P}, \sigma_0 f = 0, y \in I).$$

Define the two-dimensional Dirichlet and Fejér kernel functions as

$$D_n(x) := \sum_{0 \leq j_1 < n_1, 0 \leq j_2 < n_2} \omega_{j_1}(x^1) \omega_{j_2}(x^2) = D_{n_1}(x^1) D_{n_2}(x^2),$$

$$K_n(x) := \frac{1}{n_1 n_2} \sum_{1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2} D_n(x) = K_{n_1}(x^1) K_{n_2}(x^2),$$

where $x = (x^1, x^2) \in I^2$, $n = (n_1, n_2) \in \mathbb{P}^2$ (for $n_1 n_2 = 0$ set $D_n = K_n = 0$). Also define the Fourier coefficients, the $n \in \mathbb{N}^2$ -th partial sum of the Fourier series, the $n \in \mathbb{N}^2$ -th $(C, 1)$ mean of $f \in L^1(I^2)$:

$$\hat{f}((n_1, n_2)) := \int_{I^2} f(x^1, x^2) \omega_{n_1}(x^1) \omega_{n_2}(x^2) d\mu(x^1, x^2) \quad (n \in \mathbb{N}^2),$$

$$\begin{aligned} S_{(n_1, n_2)} f(y) &:= \sum_{k_1 < n_1, k_2 < n_2} \hat{f}((k_1, k_2)) \omega_{k_1}(y^1) \omega_{k_2}(y^2) \\ &= \int_{I^2} f(x + y) D_n(x) d\mu(x) \end{aligned}$$

($n \in \mathbb{P}^2$, $S_n f = 0$ for $n_1 n_2 = 0$),

$$\sigma_{(n_1, n_2)}(y) := \frac{1}{n_1 n_2} \sum_{1 \leq k_1 \leq n_1, 1 \leq k_2 \leq n_2} S_k(y) = \int_{I^2} f(x + y) K_n(x) d\mu(x)$$

($n \in \mathbb{P}^2$, $\sigma_n f = 0$ for $n_1 n_2 = 0$, $y \in I^2$).

THE CONSTRUCTION

Define a subset of the set of the two-dimensional intervalls $\mathcal{I} \times \mathcal{I}$:

$$\mathcal{I}_{n,a}(x) := \{I_{n+j}(x^1) \times I_{n+a-j}(x^2) : j = 0, 1, \dots, a\} \quad (x \in I^2, a, n \in \mathbb{N}).$$

It is easy to have

$$\bigcap \mathcal{I}_{n,a}(x) = I_{n+a}(x^1) \times I_{n+a}(x^2), \quad \mu\left(\bigcap \mathcal{I}_{n,a}(x)\right) = 2^{-2n-2a};$$

$F \in \mathcal{I}_{n,a}(x)$ implies $\mu(F) = 2^{-2n-a}$. Next we prove

Lemma 1. $\mu(\bigcup \mathcal{I}_{n,a}(x)) = (1 + a/2)2^{-2n-a}$.

Proof. Denote (for the sake of this proof, only)

$$\mu_k := \mu\left(\bigcup_{j=0}^k (I_{n+j}(x^1) \times I_{n+a-j}(x^2))\right)$$

for $k = 0, 1, \dots, a$. Then, $\mu_0 = 2^{-2n-a}$ and for $k > 0$ we have

$$\begin{aligned} \mu_k &= \mu_{k-1} + \mu(I_{n+k}(x^1) \times I_{n+a-k}(x^2)) \\ &\quad - \mu\left(\bigcup_{j=0}^{k-1} (I_{n+j}(x^1) \times I_{n+a-j}(x^2)) \cap (I_{n+k}(x^1) \times I_{n+a-k}(x^2))\right) \\ &= \mu_{k-1} + \frac{1}{2^{2n+a}} - \mu\left(\bigcup_{j=0}^{k-1} (I_{n+k}(x^1) \times I_{n+a-j}(x^2))\right) \\ &= \mu_{k-1} + \frac{1}{2^{2n+a}} - \mu(I_{n+k}(x^1) \times I_{n+a-k+1}(x^2)) \\ &= \mu_{k-1} + \frac{1}{2^{2n+a}} - \frac{1}{2^{2n+a+1}} = \mu_{k-1} + \frac{1}{2^{2n+a+1}}. \end{aligned}$$

This gives

$$\begin{aligned}\mu\left(\bigcup \mathcal{I}_{n,a}(x)\right) &= \mu\left(\bigcup_{j=0}^a (I_{n+j}(x^1) \times I_{n+a-j}(x^2))\right) = \mu_a \\ &= \mu_0 + a \frac{1}{2^{2n+a+1}} = \frac{1}{2^{2n+a}} + a \frac{1}{2^{2n+a+1}} = \frac{1+a/2}{2^{2n+a}}.\end{aligned}$$

This completes the proof of Lemma 1. $\square$

For $t \in I^2$, $a, b, k \in \mathbb{N}$ define the sets $J_{a,b}^k(t)$, $\Omega_{a,b}^k(t)$ recursively:

$$J_{a,b}^0(t) := \{t\}, \quad \Omega_{a,b}^0(t) := \bigcup \mathcal{I}_{b,a}(t).$$

Suppose that the sets $J_{a,b}^j(t)$, $\Omega_{a,b}^j(t)$ are defined for $j < k$. Then decompose

$$I_b^2(t) \setminus \bigcup_{j=0}^{k-1} \Omega_{a,b}^j(t)$$

as the disjoint union of dyadic squares of the form $I_{b+ka}^2(x)$. Take from each dyadic rectangle an element to represent. The set of x 's corresponding to these squares is $J_{a,b}^k(t)$. That is,

$$I_b^2(t) \setminus \bigcup_{j=0}^{k-1} \Omega_{a,b}^j(t) = \bigcup_{x \in J_{a,b}^k(t)} I_{b+ka}^2(x).$$

Then, set

$$\Omega_{a,b}^k(t) := \bigcup_{x \in J_{a,b}^k(t)} \bigcup \mathcal{I}_{b+ka,a}(x).$$

This gives the a.e. equality $I_b^2(t) = \bigcup_{j=0}^{\infty} \Omega_{a,b}^j(t)$. By this we get

$$I^2 = \bigcup_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \bigcup_{j=0}^{\infty} \Omega_{a,b}^j(t),$$

where $t = (t^1, t^2) = (t_0^1 e_0 + \dots + t_{b-1}^1 e_{b-1}, t_0^2 e_0 + \dots + t_{b-1}^2 e_{b-1}) \in I^2$. Set for $a, b, d \in \mathbb{N}$ ($b \geq 2$) the functions ($b^\circ := [b/2] - 1$ ($[x]$ denotes the integer part of x)) $f_{a,b}^d : I^2 \rightarrow \mathbb{R}$ as follows:

$$f_{a,b}^d(x) := \begin{cases} (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} 2^a, & \text{if there exists } t \in I^2, k \leq d, y \in J_{a,b}^k(t) \\ & \text{for which } x \in \bigcap \mathcal{I}_{b+ka,a}(y), \\ 0, & \text{otherwise.} \end{cases}$$

Denoting by 1_B the characteristic set of any set $B \subset I^2$ we have

$$f_{a,b}^d(x) = (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} 2^a \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^d \sum_{y \in J_{a,b}^k(t)} 1_{I_{b+ka}^2}(y)(x).$$

Lemma 2. For all $a, b, d \in \mathbb{N}$ we have $\int_{I^2} |f_{a,b}^d| \log^+ |f_{a,b}^d| \leq 2$.

Proof.

$$\begin{aligned}
 & \int_{I^2} |f_{a,b}^d(x)| \log^+(|f_{a,b}^d(x)|) d\mu(x) \\
 &= 2^a \log(2^a) \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^d \sum_{y \in J_{a,b}^k(t)} \mu(1_{I_{b+ka(a+1)}^2(y)}(x) = 1) \\
 &= 2^a \log(2^a) \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^d \sum_{y \in J_{a,b}^k(t)} \mu(\bigcap \mathcal{I}_{b+ka,a}(y)) \\
 &= 2^a \log(2^a) \sum_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^d \sum_{y \in J_{a,b}^k(t)} \frac{\mu(\bigcup \mathcal{I}_{b+ka,a}(y))}{2^a(1+a/2)} \\
 &\leq \frac{\log(2^a)}{1+a/2} \mu(I^2) \leq 2.
 \end{aligned}$$

□

By the definition of the functions $f_{a,b}^d$ we have that

$$f_{a,b}^d(\cdot + e_{b^\circ}, \cdot) = f_{a,b}^d(\cdot, \cdot + e_{b^\circ}) = -f_{a,b}^d(\cdot, \cdot).$$

It is known [5, p. 46] that for all $n \in \mathbb{P}$, $f \in L^1(I)$

$$\sigma_{2^n} f(x) = (2^n + 1/2) \int_{I_n(x)} f + \sum_{j=0}^{n-1} 2^{j-1} \int_{I_n(x)} f(\cdot + e_j),$$

by which we have for $n = (n_1, n_2) \in \mathbb{P}^2$, $x = (x^1, x^2) \in I^2$, $(I_n^2(x) = I_{n_1}(x^1) \times I_{n_2}(x^2))$, $f \in L^1(I^2)$

$$\begin{aligned}
 \sigma_{(2^{n_1}, 2^{n_2})} f(x) &= (2^{n_1} + 1/2)(2^{n_2} + 1/2) \int_{I_n^2(x)} f(\cdot) \\
 &+ (2^{n_1} + 1/2) \sum_{j_2=0}^{n_2-1} 2^{j_2-1} \int_{I_n^2(x)} f(\cdot, \cdot + e_{j_2}) \\
 &+ (2^{n_2} + 1/2) \sum_{j_1=0}^{n_1-1} 2^{j_1-1} \int_{I_n^2(x)} f(\cdot + e_{j_1}, \cdot) \\
 &+ \sum_{j_2=0}^{n_2-1} \sum_{j_1=0}^{n_1-1} 2^{j_1+j_2-2} \int_{I_n^2(x)} f(\cdot + e_{j_1}, \cdot + e_{j_2}).
 \end{aligned} \tag{1}$$

Lemma 3. Let $a, b, d \in \mathbb{P}$, $a - b^\circ < 0$ ($b^\circ = [b/2] - 1$), $b \geq 4$,

$$x \in \bigcup_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b-1}} \bigcup_{k=0}^d \Omega_{a,b}^k(t);$$

then there exists a unique $k \leq d$ for which

$$(2) \quad x \in I_{b+ka+j}(y^1) \times I_{b+(k+1)a-j}(y^2),$$

$y \in J_{a,b}^k(t)$. Setting $n := (b + ka + j, b + (k + 1)a - j) \in \mathbb{P}^2$ we have

$$(3) \quad |\sigma_{(2^{n_1}, 2^{n_2})} f_{a,b}^d(x)| \geq \frac{1}{4}.$$

Proof. (2) is a straightforward consequence of the definition of $\Omega_{a,b}^k(t)$. Next, verify

(3). Define $\tilde{f}_{a,b}^d$ by $f_{a,b}^d = (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} 2^a \tilde{f}_{a,b}^d$. Then $\tilde{f}_{a,b}^d$ is either 0 or 1. Recall that

$$f_{a,b}^d(x^1, x^2 + e_{j_2}) = (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} 2^a \tilde{f}_{a,b}^d(x^1, x^2 + e_{j_2}),$$

$$f_{a,b}^d(x^1 + e_{j_1}, x^2) = (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} 2^a \tilde{f}_{a,b}^d(x^1 + e_{j_1}, x^2)$$

for $j_1, j_2 \neq e_{b^\circ}$ and

$$f_{a,b}^d(\cdot + e_{b^\circ}, \cdot) = f_{a,b}^d(\cdot, \cdot + e_{b^\circ}) = -f_{a,b}^d(\cdot, \cdot).$$

By (1) we have

$$\begin{aligned} \sigma_{(2^{n_1}, 2^{n_2})} f_{a,b}^d(x) &= 2^a (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} \left((2^{n_1} + 1/2)(2^{n_2} + 1/2) \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot) \right. \\ &\quad + (2^{n_1} + 1/2) \sum_{j_2=0, j_2 \neq b^\circ}^{n_2-1} 2^{j_2-1} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot, \cdot + e_{j_2}) \\ &\quad + (2^{n_2} + 1/2) \sum_{j_1=0, j_1 \neq b^\circ}^{n_1-1} 2^{j_1-1} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot + e_{j_1}, \cdot) \\ &\quad + \sum_{j_2=0, j_2 \neq b^\circ}^{n_2-1} \sum_{j_1=0, j_1 \neq b^\circ}^{n_1-1} 2^{j_1+j_2-2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot + e_{j_1}, \cdot + e_{j_2}) \Big) \\ &\quad - 2^a (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} \left((2^{n_1} + 1/2)(2^{b^\circ-1}) \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot) \right. \\ &\quad + (2^{n_2} + 1/2)(2^{b^\circ-1}) \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot) \\ &\quad - 2^a (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} \left(\sum_{j_2=0, j_2 \neq b^\circ}^{n_2-1} 2^{j_2+b^\circ-2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot, \cdot + e_{j_2}) \right. \\ &\quad + \sum_{j_1=0, j_1 \neq b^\circ}^{n_1-1} 2^{j_1+b^\circ-2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot + e_{j_1}, \cdot) \\ &\quad \left. \left. + 2^a (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2} \left(2^{b^\circ+b^\circ-2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot, \cdot) \right) \right) \right). \end{aligned}$$

From the above it follows that

$$\begin{aligned}
& |\sigma_{(2^{n_1}, 2^{n_2})} f_{a,b}^d(x)| \\
& \geq 2^{a+n_1+n_2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot) - 2^a(2^{n_1} + 1/2)(2^{b^\circ-1}) \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot) \\
& \quad - 2^a(2^{n_2} + 1/2)(2^{b^\circ-1}) \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot) - 2^a \sum_{j_2=0, j_2 \neq b^\circ}^{n_2-1} 2^{j_2+b^\circ-2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot, \cdot + e_{j_2}) \\
& \quad - 2^a \sum_{j_1=0, j_1 \neq b^\circ}^{n_1-1} 2^{j_1+b^\circ-2} \int_{I_n^2(x)} \tilde{f}_{a,b}^d(\cdot + e_{j_1}, \cdot) \\
& \geq 2^a \left(2^{n_1+n_2} \mu(I_{b+(k+1)a}(y^1) \times I_{b+(k+1)a}(y^2)) \right. \\
& \quad - ((2^{n_1} + 1/2)(2^{b^\circ-1}) + (2^{n_2} + 1/2)(2^{b^\circ-1})) \mu(I_{b+(k+1)a}(y^1) \times I_{b+(k+1)a}(y^2)) \\
& \quad \left. - \left(\sum_{j_2=0, j_2 \neq b^\circ}^{n_2-1} 2^{j_2+b^\circ-2} + \sum_{j_1=0, j_1 \neq b^\circ}^{n_1-1} 2^{j_1+b^\circ-2} \right) \mu(I_n^2(x)) \right) \\
& \geq 2^a \left(2^{b+ka+j+b+(k+1)a-j} 2^{-2b-2(k+1)a} - 2^{b+(k+1)a+b^\circ+1} 2^{-2b-2(k+1)a} \right. \\
& \quad \left. - 2^{b^\circ-2} 2^{-2b-ka-(k+1)a} \left(\sum_{j_1=0}^{n_1-1} 2^{j_1} + \sum_{j_2=0}^{n_2-1} 2^{j_2} \right) \right) \geq \frac{1}{2}.
\end{aligned}$$

Recall that $a < b^\circ = [b/2] - 1$. This completes the proof of Lemma 3. $\square$

THE CONSTRUCTION OF THE FUNCTION

Define the sequences $(\beta_n), (\delta_n), (a_n), (b_n), (d_n)$ recursively in the following way: $(\beta_0) = (\delta_0) = (a_0) = (b_0) = (d_0) := 4$. For $n \in \mathbb{P}$

$$\beta_n := 72 \max(n, \sum_{k=0}^{n-1} \beta_k 2^{a_k}),$$

$$\delta_n := [\sup\{t \in \mathbb{R} : \delta(t) > \frac{1}{2^n \beta_n}\}] + 1 \text{ (if } \{t : \delta(t) > 1/(2^n \beta_n)\} = \emptyset, \text{ then let } \delta_n = 4),$$

$$a_n := \max(2^{n+1}, \delta_n, \log^+(2\beta_n)),$$

$$b_n := \max(b_{n-1} + (d_{n-1} + 1)a_{n-1} + 2, 2a_n).$$

By construction it is easy to see that the union of the disjoint sets $\Omega_{a,b}^k(t)$ ($k \in \mathbb{N}$) is equal (neglecting a set of measure zero) to $I_b^2(t)$; hence there exists a d_n for which

$$(4) \quad \frac{1}{2^{2b}} \left(1 - \frac{1}{2^n}\right) \leq \mu\left(\bigcup_{k \leq d_n} \Omega_{a_n, b_n}^k(t)\right) \leq \frac{1}{2^{2b}}.$$

Set

$$G_{a_n, b_n, d_n} := \bigcup_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\dots,b_n-1}} \bigcup_{j=0}^{d_n} \Omega_{a_n, b_n}^j(t)$$

for all $n \in \mathbb{N}$ and $G := \liminf G_{a_n, b_n, d_n} = \bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} G_{a_n, b_n, d_n}$. That is, $x \in I^2$ is an element of G if and only if x is an element of all but a finite number of

G_{a_n, b_n, d_n} 's. By (4) we have $1 - \frac{1}{2^n} \leq \mu(G_{a_n, b_n, d_n}) \leq 1$, which easily gives that $\mu(I^2 \setminus G) = 0$.

Define the function f as $f = \sum_{n=0}^{\infty} \beta_n f_n := \sum_{n=0}^{\infty} \beta_n f_{a_n, b_n}^{d_n}$. At first we prove

Lemma 4. $\int_{I^2} |f(x)| \log^+(|f(x)|) \delta(|f(x)|) d\mu(x) < \infty$.

Proof. Set

$$H_n := \{x \in I^2 : f_n(x) \neq 0, f_{n+j}(x) = 0 (j \in \mathbb{P})\} \quad (n \in \mathbb{N})$$

and $H_{-1} := \{x \in I^2 : f_j(x) = 0 (j \in \mathbb{P})\}$. The definition of $f_{a,b}^d$, (a_n) gives

$$\begin{aligned} \mu(H_n) &\geq 1 - \mu\left(\bigcup_{k>n} \{x \in I^2 : f_k(x) \neq 0\}\right) \geq 1 - \sum_{k>n} \mu(\{x \in I^2 : f_k(x) \neq 0\}) \\ &\geq 1 - \sum_{k>n} \frac{1}{2^{a_k}(a_k/2 + 1)} \geq 1 - \sum_{k>n} \frac{1}{2^{2^k}}. \end{aligned}$$

It follows that $\bigcup_{n=-1}^{\infty} H_n = I^2$ (neglecting a set of measure zero). Corresponding to this argument if $x \in H_n$ ($n \in \mathbb{N}$), then

$$\begin{aligned} |f(x)| &\leq \sum_{k=0}^{n-1} \beta_k 2^{a_k} + \beta_n 2^{a_n} \\ &\leq \beta_n + \beta_n 2^{a_n} \leq 2\beta_n 2^{a_n} = |\beta_n 2 f_{a_n, b_n}^{d_n}(x)|, \\ |f(x)| &\geq \beta_n 2^{a_n} - \sum_{k=0}^{n-1} \beta_k 2^{a_k} \\ &\geq \beta_n 2^{a_n} - \frac{1}{2} \beta_n \geq \frac{1}{2} \beta_n 2^{a_n} \\ &= \frac{1}{2} |\beta_n f_{a_n, b_n}^{d_n}(x)|. \end{aligned}$$

Moreover, for $x \in H_n$ we have $|f(x)| \geq \frac{1}{2} \beta_n 2^{a_n} \geq 2^{a_n} > \delta_n$, which gives $\delta(|f(x)|) \leq \frac{1}{2^n \beta_n}$. Consequently, by Lemma 2

$$\begin{aligned} &\int_{H_n} |f(x)| \log^+(|f(x)|) \delta(|f(x)|) d\mu(x) \\ &\leq \int_{H_n} 2 |\beta_n f_{a_n, b_n}^{d_n}(x)| \log^+(2 |f_{a_n, b_n}^{d_n}(x)|) \frac{1}{2^n \beta_n} \\ &\leq \int_{H_n} 2 |\beta_n f_{a_n, b_n}^{d_n}(x)| \log^+(|f_{a_n, b_n}^{d_n}(x)|^2) \frac{1}{2^n \beta_n} \\ &\leq \frac{4}{2^n} \int_{I^2} |f_{a_n, b_n}^{d_n}(x)| \log^+(|f_{a_n, b_n}^{d_n}(x)|) d\mu(x) \leq \frac{8}{2^n}. \end{aligned}$$

Since for $x \in H_{-1}$ we have $f(x) = 0$, then we get

$$\begin{aligned} &\int_{I^2} |f(x)| \log^+(|f(x)|) \delta(|f(x)|) d\mu(x) \\ &\leq \sum_{n \in \mathbb{N}} \int_{H_n} |f(x)| \log^+(|f(x)|) \delta(|f(x)|) d\mu(x) \leq 16. \end{aligned}$$

□

Now, we are ready to prove

Theorem. *For all measurable function $\delta : [0, +\infty) \rightarrow [0, +\infty)$, $\lim_{t \rightarrow \infty} \delta(t) = 0$ we have a function f such that $f \in L \log^+ L \delta(L)$ and $\limsup \sigma_{(2^{n_1}, 2^{n_2})} f(x) = +\infty$ ($\min(n_1, n_2) \rightarrow \infty$), i.e. the two-dimensional $(C, 1)$ means does not converge to f a.e. (in the Pringsheim sense).*

It is not possible to prove more (e.g. to prove that $\lim \sigma_{(2^{n_1}, 2^{n_2})} f(x) = +\infty$ ($\min(n_1, n_2) \rightarrow \infty$)), whereas Móricz, Schipp and Wade proved [4] for functions in $L^1(I^2)$ the a.e. convergence $\sigma_{(2^{n_1}, 2^{n_2})} f \rightarrow f$ as the indices $n_1, n_2 \rightarrow \infty$ ($|n_1 - n_2|$ is bounded). The author in a different way proved [3] even more. Namely, for functions in $L^1(I^2)$ the a.e. convergence $\sigma_{(n_1, n_2)} f \rightarrow f$ holds as the indices $n_1, n_2 \rightarrow \infty$ restricted as $\beta^{-1} \leq n_1/n_2 \leq \beta$ for some $\beta > 1$ (compare with [7], [8], [9]).

Proof. We apply Lemmas 2,3 and 4. The rest is to prove the “divergence”. Namely, we verify that for a.e. $x \in I^2$ $\limsup \sigma_{(2^{n_1}, 2^{n_2})} f(x) = +\infty$ ($\min(n_1, n_2) \rightarrow \infty$). This is the same as $\sup_{n_1, n_2 \in \mathbb{N}} \sigma_{(2^{n_1}, 2^{n_2})} f(x) = +\infty$. Let $x \in G$ (recall that $\mu(I^2 \setminus G) = 0$). Then there is an infinite number $n \in \mathbb{N}$ for which (even for all, but a finite number) $x \in G_{a_n, b_n, d_n}$. Then Lemma 3 gives that there exist a $t \in I^2$, $k \leq d_n$ for which $x \in \Omega_{a_n, b_n}^k(t)$, whereby, there are a $y \in J_{a_n, b_n}^k(t)$, a unique $j \in \{0, 1, \dots, a_n\}$ for which $x \in I_{b_n + ka_n + j}(y^1) \times I_{b_n + (k+1)a_n - j}(y^2)$. Set

$$N := (2^{N_1}, 2^{N_2}) := (b_n + ka_n + j, b_n + (k+1)a_n - j) \in \mathbb{P}^2.$$

By Lemma 3 we have

$$|\sigma_{(2^{N_1}, 2^{N_2})} f_{a_n, b_n}^{d_n}(x)| \geq \frac{1}{2}.$$

In [5, p. 46] one can find that $\|K_n\|_1 \leq 3$ for $n \in \mathbb{N}$ which in the standard way (see e.g. [5]) gives $\|\sigma_n f\|_\infty \leq 9\|f\|_\infty$ ($f \in L^1(I^2)$, $n \in \mathbb{P}^2$). It follows for $i \in \mathbb{P}$ that

$$\|\sigma_{(2^{N_1}, 2^{N_2})} f_{a_{n-i}, b_{n-i}}^{d_{n-i}}\|_\infty \leq 9\|f_{a_{n-i}, b_{n-i}}^{d_{n-i}}\|_\infty \leq 9 \cdot 2^{a_{n-i}}.$$

The definition of function $f_{a,b}^d$ gives that $E_{(b^\circ, b^\circ)} f_{a,b}^d = 0$ and also that for $j_1, j_2 < b^\circ$ natural numbers $E_{(b^\circ, b^\circ)} f_{a,b}^d(\cdot + e_{j_1}, \cdot) = E_{(b^\circ, b^\circ)} f_{a,b}^d(\cdot, \cdot + e_{j_2}) = 0$. Hence, for $k \in \mathbb{P}$ it follows that

$$\begin{aligned} & 2^{N_1 N_2} \int_{I_N^2(x)} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(\cdot + e_{j_1}, \cdot) \\ &= E_{(N_1, N_2)} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(\cdot + e_{j_1}, \cdot)(x) \\ &= E_{(N_1, N_2)} \left(E_{(b_{n+k}^\circ, b_{n+k}^\circ)} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(\cdot + e_{j_1}, \cdot) \right)(x) = 0 \end{aligned}$$

and

$$\begin{aligned} & 2^{N_1 N_2} \int_{I_N^2(x)} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(\cdot, \cdot + e_{j_2}) \\ &= E_{(N_1, N_2)} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(\cdot, \cdot + e_{j_2})(x) \\ &= E_{(N_1, N_2)} \left(E_{(b_{n+k}^\circ, b_{n+k}^\circ)} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(\cdot, \cdot + e_{j_2}) \right)(x) = 0. \end{aligned}$$

This gives that for all $k \in \mathbb{P}$

$$\sigma_{(2^{N_1}, 2^{N_2})} f_{a_{n+k}, b_{n+k}}^{d_{n+k}}(x) = 0.$$

Consequently,

$$\begin{aligned}
 & |\sigma_{(2^{N_1}, 2^{N_2})} f(x)| \\
 & \geq |\sigma_{(2^{N_1}, 2^{N_2})} \beta_n f_{a_n, b_n}^{d_n}(x)| - \sum_{i=1}^{n-1} |\sigma_{(2^{N_1}, 2^{N_2})} \beta_i f_{a_i, b_i}^{d_i}(x)| \\
 & \geq \frac{1}{2} \beta_n - 9 \sum_{i=0}^{n-1} \beta_i \|f_{a_i, b_i}^{d_i}\|_{\infty} \\
 & \geq \frac{1}{2} \beta_n - 9 \sum_{i=0}^{n-1} \beta_i 2^{a_i} \geq \frac{1}{4} \beta_n \geq 18n.
 \end{aligned}$$

This completes the proof of the theorem. $\square$

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