COMPACTIFICATIONS AND UNIVERSAL SPACES IN EXTENSION THEORY

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ABSTRACT. We show that for each countable simplicial complex P the following conditions are equivalent:

- $P \in AE(X)$ iff $P \in AE(\beta X)$ for any space X .
- There exists a P -invertible map of a metrizable compactum X with $P \in AE(X)$ onto the Hilbert cube.

1. INTRODUCTION

The following two problems are central in extension theory [6] (see also [4]).

Problem A ([7, Problem 5]). Characterize CW-complexes P such that for any space X with $P \in AE(X)$ there exists a compactification bX of X such that $P \in AE(bX)$.

Problem B ([7, Problems 6, 7]). Characterize CW-complexes P such that the class

$$\{X : X \text{ is a separable metrizable space with } P \in AE(X)\}$$

has a universal space (compactum).

The first example of a space with $K(\mathbb{Z}, n) \in AE(X)$ and $K(\mathbb{Z}, n) \notin AE(\beta X)$ (in other words $\dim_{\mathbb{Z}} X < \dim_{\mathbb{Z}} \beta X$) was constructed by A. N. Dranishnikov [5] (see also [10]). There even exist spaces with $K(\mathbb{Z}, n) \in AE(X)$ and $K(\mathbb{Z}, n) \notin AE(bX)$ for any compactification bX of X [9]. It follows from [7, Corollary 2.5(b) and its proof] that for a finitely dominated complex P the class indicated in problem B has a universal compactum. It is important to emphasize that a universal compactum in this case can be produced (see the proof of [7, Theorem 2.3]) as the domain of a P -invertible map the range of which is the Hilbert cube $\mathbb{I}^{\omega}$. Various results suggest and several authors have observed (see, for instance, [8, Remark], [7, p.1657]) that there seems to be a connection between the existence of “dimension” preserving compactifications and the existence of universal elements in the class of metrizable compacta of a given “dimension”.

Below we consider stronger versions of the above problems.

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Problem A'. Characterize connected locally compact simplicial complexes¹ P such that $P \in AE(X)$ iff $P \in AE(\beta X)$ for any space X .

Problem B'. Characterize connected locally compact simplicial complexes P such that there exists a P -invertible map $f: X \rightarrow \mathbb{I}^\omega$ where X is a metrizable compactum with $P \in AE(X)$.

Below we show (Theorem 2.1) that Problems A' and B' are equivalent.

2. RESULTS

All spaces are assumed to be Tychonov (i.e. completely regular and Hausdorff). All maps are continuous. $\mathbb{I}$ denotes a closed interval. All simplicial complexes are connected and locally compact. If X is a normal space, we say that P is an absolute extensor of X (and write $P \in AE(X)$) if for each closed subspace Y of X any map $f: Y \rightarrow P$ has a continuous extension $\bar{f}: X \rightarrow P$. An extension of this concept for non-normal spaces has been given in [3, Definition 3.1]. A map $f: X \rightarrow Y$ is P -invertible if for any map $g: Z \rightarrow Y$ with $P \in AE(Z)$ there exists a map $h: Z \rightarrow X$ such that $g = fh$.

Theorem 2.1. *Let P be a Polish ANR-space. Then the following statements are equivalent:*

- (a) $P \in AE(\beta X)$ whenever X is a space with $P \in AE(X)$.
- (b) $P \in AE(\beta X)$ whenever X is a normal space with $P \in AE(X)$.
- (c) $P \in AE(\beta(\oplus\{X_t: t \in T\}))$ whenever T is an arbitrary indexing set and X_t , $t \in T$, is a separable metrizable space with $P \in AE(X_t)$.
- (d) $P \in AE(\beta(\oplus\{X_t: t \in T\}))$ whenever T is an arbitrary indexing set and X_t , $t \in T$, is a Polish space with $P \in AE(X_t)$.
- (e) There exists a P -invertible map $f_P: K_P \rightarrow \mathbb{I}^\omega$ where K_P is a metrizable compactum with $P \in AE(K_P)$.

Proof. Implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are trivial. Proof of implication (d) $\Rightarrow$ (e) follows the proof of [3, Proposition 5.3]. Let $\mathcal{A}$ denote the set of all maps $\{r_t: t \in T\}$ such that domain $\text{Dom}(r_t)$ is a Polish subspace of $\mathbb{I}^\omega$, $P \in AE(\text{Dom}(r_t))$ and $\text{Ran}(r_t) \subseteq \mathbb{I}^\omega$. Let $Y = \bigoplus\{\text{Dom}(r_t): t \in T\}$. Clearly, $P \in AE(Y)$. Consider also the map $r: Y \rightarrow \mathbb{I}^\omega$ which coincides with r_t on $\text{Dom}(r_t)$ for each $t \in T$. Let $\bar{r}: \beta Y \rightarrow \mathbb{I}^\omega$ be the unique continuous extension of r to the Stone-Ćech compactification βY of Y . By (d), $P \in AE(\beta Y)$. By [3, Theorem 4.4] and by the compactness of βY , the latter is the limit space of a Polish spectrum $\mathcal{S} = \{Y_\alpha, q_\alpha^\beta, A\}$ consisting of metrizable compacta Y_α (compactness of Y_α follows from the fact that $q_\alpha(\beta Y)$ is dense in Y_α , according to assumption made in [3, p. 201]) with $P \in AE(Y_\alpha)$. Write $\mathbb{I}^\omega = \prod\{\mathbb{I}_n: n \in \omega\}$ where $\mathbb{I}_n$, $n \in \omega$, denotes a copy of $\mathbb{I}$. Also let $\pi_n: \mathbb{I}^\omega \rightarrow \mathbb{I}_n$, $n \in \omega$, denote the corresponding projection. Since the spectrum $\mathcal{S}$ is factorizing, for each $n \in \omega$ there exist an index $\alpha_n \in A$ and a map $s_n: Y_{\alpha_n} \rightarrow \mathbb{I}_n$ such that $\pi_n \bar{r} = s_n q_{\alpha_n}$, where $q_{\alpha_n}: \beta Y \rightarrow Y_{\alpha_n}$ is the α_n -th limit projection of the spectrum $\mathcal{S}$. Since $\mathcal{S}$ is a Polish spectrum (see [3, page 201]) there exists an index $\alpha \in A$ such that $\alpha \geq \alpha_n$ for each $n \in \omega$. Next consider the map

$$s = \triangle\{s_n q_{\alpha_n}^\alpha: n \in \omega\}: Y_\alpha \rightarrow \prod\{\mathbb{I}_n: n \in \omega\},$$

¹We prefer to work with connected locally compact simplicial complexes because they are Polish ANR's and consequently results of [2], [3] apply.

where $q_{\alpha_n}^\alpha : Y_\alpha \rightarrow Y_{\alpha_n}$, $n \in \omega$, denotes the corresponding projection of the spectrum $\mathcal{S}$. It is easy to see that $\bar{r} = sq_\alpha$, where $q_\alpha : \beta Y \rightarrow Y_\alpha$ is the α -th limit projection of the spectrum $\mathcal{S}$. It now suffices to let $K_P = Y_\alpha$ and $f_P = s$. Let us show that $s : Y_\alpha \rightarrow \mathbb{I}^\omega$ is indeed P -invertible. Since the spaces Y_α and $\mathbb{I}^\omega$ are Polish (even compact and metrizable), it suffices (according to [3, Proposition 5.2]) to consider only Polish spaces Z in the definition of P -invertibility given above. Indeed, let $g : Z \rightarrow \mathbb{I}^\omega$ be a map defined on a Polish space Z with $P \in AE(Z)$. We may as well assume that $Z \subseteq \mathbb{I}^\omega$. By the definition of $\mathcal{A}$, there is an index $t \in T$ such that $r_t = g$. Let $i_t : \text{Dom}(r_t) \rightarrow Y$ denote the corresponding embedding. Clearly, $r_t = \bar{r}i_t$. Then the composition $h = q_\alpha i_t : Z \rightarrow Y_\alpha$ lifts the map g , i.e. $sh = g$.

(e) $\implies$ (a). As in the proof of [3, Theorem 5.13] (see also [2, Section 6.2] where the case $P = S^n$ is considered) one shows that for any uncountable cardinal number τ there exists a P -invertible map $f = f_{P,\tau} : K_{P,\tau} \rightarrow \mathbb{I}^\tau$, where $K_{P,\tau}$ is a compactum of weight τ such that $P \in AE(K_{P,\tau})$ and $\mathbb{I}^\tau$ denotes the Tychonov cube of weight τ (for $\tau = \omega$ the existence of such a map is guaranteed by condition (e); note also that an $AE(0)$ -space of countable weight is Polish [2, Corollary 6.4.5]). Consider now a space X with $P \in AE(X)$ and choose τ large enough so that βX can be identified with a subspace of $\mathbb{I}^\tau$. Since the map f is P -invertible there exists a map $g : X \rightarrow K_{P,\tau}$ such that $fg = \text{id}_X$. Since $K_{P,\tau}$ is compact, the map g admits a continuous extension $\bar{g} : \beta X \rightarrow K_{P,\tau}$. Since $fg = \text{id}_X$ and since $\bar{g}|_X = g$, it follows that $f\bar{g} = \text{id}_{\beta X}$. In this situation it can easily be seen that $\bar{g}$ is an embedding. In other words, $\bar{g}(\beta X)$ is a topological copy of βX . Finally, since $P \in AE(K_{P,\tau})$ it follows that $P \in AE(\beta X)$. $\square$

Corollary 2.2. *Let P be a finitely dominated connected locally compact complex. Then the following conditions are equivalent for any space X :*

- (1) $P \in AE(X)$.
- (2) $P \in AE(\beta X)$.

Proof. The implication (2) $\implies$ (1) follows from [3, Proposition 6.8]. Let us prove the implication (1) $\implies$ (2). According to Theorem 2.1 it suffices to construct a P -invertible map $f_P : K_P \rightarrow \mathbb{I}^\omega$ where K_P is a metrizable compactum with $P \in AE(K_P)$. Since P is finitely dominated, there exist a finite complex L and two maps $u : P \rightarrow L$ and $d : L \rightarrow P$ such that $du \simeq \text{id}_P$. By [7, Theorem 2.3] there exist a compactum K_P and a map $f_P : K_P \rightarrow \mathbb{I}^\omega$ with the following properties:

- (a) For each map $g : Z \rightarrow \mathbb{I}^\omega$, defined on a separable metrizable space with $P \in AE(Z)$, there exists a map $h : Z \rightarrow K_P$ such that $f_P h = g$.
- (b) For each map $\varphi : C \rightarrow P$, where C is a closed subset of K_P , there exists a map $\varphi' : K_P \rightarrow L$ such that $\varphi'|_C \simeq u\varphi$.

Observe that, by (b), $d\varphi'|_C \simeq du\varphi \simeq \varphi$. Consequently, by the Homotopy Extension Theorem, the map φ has a continuous extension over K_P . This in turn means that $P \in AE(K_P)$. By (a) and [3, Proposition 5.2], the map f_P is P -invertible. $\square$

Proposition 2.3. *Let P be a connected locally compact simplicial complex of finite type with a finite fundamental group. Then the following conditions are equivalent for any space X and any integer $n \geq 2$:*

- (a) $P \vee S^n \in AE(X)$.
- (b) $P \vee S^n \in AE(\beta X)$.

Proof. The implication (b) $\implies$ (a) follows from [3, Proposition 6.8].

In order to prove the implication (a) $\implies$ (b) it suffices to show that $P \vee S^n \in AE(\beta X)$ for each normal space X with $P \vee S^n \in AE(X)$ (see Theorem 2.1(b)). Let $\varphi: F \rightarrow P \vee S^n$ be a map defined on a closed subset F of βX . Since $P \vee S^n$ is an *ANR*, there exists an extension $\psi: \text{cl}_{\beta X} V \rightarrow P \vee S^n$ of φ , where V is an open neighborhood of F in βX . Clearly $V \cap X \neq \emptyset$ and $\text{cl}_{\beta X}(\text{cl}_X(V \cap X)) = \text{cl}_{\beta X} V$. Since $P \vee S^n \in AE(X)$, there exists a map $f: X \rightarrow P \vee S^n$ such that $f|_{\text{cl}_X(V \cap X)} = \psi|_{\text{cl}_X(V \cap X)}$. An argument similar to [1, Proof of Lemma 4.1] shows that f is homotopic to a map $g: X \rightarrow P \vee S^n$ such that $\text{cl}(g(X))$ is compact. Consequently g has a continuous extension $\bar{g}: \beta X \rightarrow P \vee S^n$ onto the whole βX . Now consider the two maps ψ and $\bar{g}|_{\text{cl}_{\beta X} V}$. Their restrictions $\psi|_{\text{cl}_X(V \cap X)}$ and $\bar{g}|_{\text{cl}_X(V \cap X)}$ are homotopic. Since X is normal, it follows that $\text{cl}_{\beta X} V = \text{cl}_{\beta X}(\text{cl}_X(V \cap X)) = \beta(\text{cl}_X(V \cap X))$. By [1, Theorem 4.2], the restriction operator provides a bijection of homotopy classes $[\beta(\text{cl}_X(V \cap X)), P \vee S^n]$ and $[\text{cl}_X(V \cap X), P \vee S^n]$. Consequently, $\psi \simeq \bar{g}|_{\text{cl}_{\beta X} V}$. By the homotopy extension theorem (recall that $\bar{g}|_{\text{cl}_{\beta X} V}$ has an extension $\bar{g}$ onto βX and that $P \vee S^n$ is an *ANR*), ψ also has an extension onto βX which serves as an extension of the originally given map φ . $\square$

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