REMARKS ABOUT SCHLUMPRECHT SPACE

DENKA KUTZAROVA AND PEI-KEE LIN

(Communicated by Dale Alspach)

ABSTRACT. Let $\mathbf{S}$ denote the Schlumprecht space. We prove that

- (1) ℓ_∞ is finitely disjointly representable in $\mathbf{S}$;
- (2) $\mathbf{S}$ contains an ℓ_1 -spreading model;
- (3) for any sequence (n_k) of natural numbers, $\mathbf{S}$ is isomorphic to the space $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_{\mathbf{S}}$.

Let $(e_i)_{i=1}^\infty$ be the standard basis of the linear space c_{00} , the set of all finitely supported sequences. For $x = \sum_{i=1}^\infty a_i e_i \in c_{00}$, $\text{supp } x$ denotes the set $\{i \in \mathbb{N} : a_i \neq 0\}$. A subset E of $\mathbb{N}$ is said to be an *interval* if there exist a, b such that

$$E = \{c \in \mathbb{N} : a < c < b\}.$$

For finite subsets E, F of $\mathbb{N}$, $E < F$ means $\max E < \min F$ or E is an empty set. For $x = \sum_{i=1}^\infty a_i e_i$ and a subset E of $\mathbb{N}$, Ex denotes the vector $Ex = \sum_{i \in E} a_i e_i$. Let $f : [1, \infty) \rightarrow [1, \infty)$ be the function defined by $f(x) = \log_2(x + 1)$. The Schlumprecht space $\mathbf{S} = (\mathbf{S}, \|\cdot\|)$ is the completion of c_{00} with respect to the norm $\|\cdot\|$ which satisfies the following implicit equation:

$$(1) \quad \|x\| = \max \left\{ \|x\|_\infty, \sup_{E_1 < E_2 < \dots < E_r} \frac{1}{f(r)} \sum_{i=1}^r \|E_i x\| \right\}.$$

Clearly, (e_i) is a 1-unconditional subsymmetric basis of $\mathbf{S}$. It is easy to see that every normalized block basis (y_i) of (e_i) dominates (e_i) , i.e.

$$\left\| \sum_{i=1}^\infty a_i y_i \right\| \geq \left\| \sum_{i=1}^\infty a_i e_i \right\|$$

for all $(a_i) \in c_{00}$.

Recall that a sequence (x_n) in $\mathbf{S}$ is said to be *successive* if $\text{supp } x_1 < \text{supp } x_2 < \dots$. A sequence of finite subspaces (V_n) is said to be *successive* if $x_j \in V_j \setminus \{0\}$, $x_k \in V_k \setminus \{0\}$ and $j < k$ implies that $\text{supp } x_j < \text{supp } x_k$.

In [S1], Th. Schlumprecht introduced the space $\mathbf{S}$ and he showed that $\mathbf{S}$ is arbitrarily distortable. He also showed that $\mathbf{S}$ has neither ℓ_p -spreading model for any $1 < p < \infty$ nor c_0 -spreading model. He asked the following question.

Received by the editors August 3, 1998 and, in revised form, August 27, 1998.

2000 *Mathematics Subject Classification*. Primary 46B20, 46B45.

Part of this paper was done when the second author visited the University of Texas at Austin and was completed when the first author participated in the Workshop in Linear Analysis and Probability Theory at Texas A&M University, 1998. Both authors would like to thank E. Odell and Th. Schlumprecht for their valuable discussions.

Question 1. Does $\mathbf{S}$ contain an ℓ_1 -spreading model?

Recall that a vector $x \in S$ is said to have *character* $p \in \mathbb{N}$ if there exist $E_1 < E_2 < \dots < E_p$ such that

$$\frac{1}{f(p)} \sum_{i=1}^p \|E_i x\| \approx \|x\|.$$

So if $x_1, \dots, x_n$ are n unit vectors in $\mathbf{S}$ with a common character $p \gg n$, then

$$\left\| \sum_{i=1}^n x_i \right\| \geq \frac{nf(p)}{f(np)} \approx n.$$

Here $a \approx b$ means that $|\frac{a}{b} - 1| < \varepsilon$ for some $0 < \varepsilon < 1$ and $a \gg b > 10$ means $\frac{f(f(a))}{b} > N$ for some $N \in \mathbb{N}$. Recall that two elements y, z in $\mathbf{S}$ are said to have the *same distribution* if there are two strictly increasing sequences $\{n_k\}, \{m_k\}$ of natural numbers and two sequences $\{a_k\}, \{b_k\}$ such that $|a_k| = |b_k|$ for all $k \in \mathbb{N}$, and $y = \sum_{k=1}^{\infty} a_k e_{n_k}, z = \sum_{k=1}^{\infty} b_k e_{m_k}$. Two subspaces V, U are said to have the *same distribution* if there is an isometry from V onto U such that for any $v \in V$, v and $T(v)$ have the same distribution.

For any $n \in \mathbb{N}$, let $u_n = \frac{f(n)}{n} \sum_{k=1}^n e_k$. It is easy to see that for any $E_1 < E_2 < \dots < E_k$,

$$\sum_{i=1}^k \|E_i u_n\| \leq \frac{f(n)}{f(n/k)}.$$

Hence, for any $n \in \mathbb{N}$, $\|u_n\| = 1$ and if u_n has character p , then $f(p) \approx f(n)$. We will show that for any $\varepsilon > 0$, there is a rapidly increasing sequence $(p_i)_{i=1}^n$ such that for any $n \in \mathbb{N}$, there is a finite sequence $(v_j)_{j=1}^n$ in $\mathbf{S}$ such that

- the support of v_j 's are pairwise disjoint;
- v_j and u_{p_j} have the same distribution;
- $\|\sum_{j=1}^n v_j\| \leq 1 + \varepsilon$.

From that fact, it will be easy (see Theorem 6) to show that the answer to Question 1 is affirmative.

E. Odell and Th. Schlumprecht constructed a space which has neither ℓ_p -spreading model for any $1 \leq p < \infty$ nor c_0 -spreading model [OS]. For background information about spreading models see [BL].

A Banach space X is called *minimal* (a notion defined by H. P. Rosenthal) if every infinite dimensional subspace of X contains a further subspace isomorphic to X , and X is called *complementably minimal* (defined by A. Pełczyński) if every infinite dimensional subspace of X contains a subspace which is isomorphic to X and complemented in X . X is called *prime* [LT] if every complemented infinite dimensional subspace of X is isomorphic to X . A. Pełczyński proved that ℓ_p , $1 \leq p < \infty$, and c_0 are prime, while J. Lindenstrauss showed that ℓ_∞ is prime (see e.g. [LT]) and for a long time these were the only known examples. Recently, W. T. Gowers and B. Maurey [GM2] constructed more prime spaces for which all complemented infinite dimensional subspaces are of finite codimension. In [S2], Th. Schlumprecht showed $\mathbf{S}$ is complementably minimal. As was remarked in [S2], the space $\mathbf{S}$ is either prime or fails the Schroeder-Bernstein property (see [C], [G2]). Recall that a Banach space X is injective if for any Banach space Y containing X

as a subspace, there is a bounded linear projection from Y onto X . It is known that ℓ_∞ is injective. It is natural to ask the following two questions.

Question 2. For any sequence (n_k) of natural numbers, does $\mathbf{S}$ contain a complemented subspace which is isomorphic to $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_S$?

Question 3. If the answer of Question 2 is yes, is $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_S$ isomorphic to $\mathbf{S}$?

A positive answer to Question 2 easily follows from our construction of ℓ_∞^n and a theorem of Th. Schlumprecht (see Lemma 2 below). A negative answer to Question 3 would have shown that $\mathbf{S}$ is not prime. Unfortunately, in this article, we will show that its answer is positive. Hence, the question whether $\mathbf{S}$ is prime or not is still open. However, our result shows that c_0 is not the only subsymmetric space X which has the property that it is isomorphic to the X -sum of finite dimensional $\ell_\infty^{n_k}$ spaces for an arbitrary sequence of integers (n_k) .

Recall that a nonzero vector $x \in \mathbf{S}$ is said to be an ℓ_{1+}^n -average with constant C if $\frac{x}{\|x\|} = \sum_{i=1}^n x_i$ for some sequence $x_1 < \dots < x_n$ of nonzero elements of $\mathbf{S}$ such that $\|x_i\| \leq Cn^{-1}$ for all $i \leq n$.

We need the following lemmas.

Lemma 1 (Lemma 4, [GM1]). Let $M, n \in \mathbb{N}$ and $C \geq 1$. There exists an N such that for any ℓ_{1+}^N -vector $x \in \mathbf{S}$ with constant C and any sequence $E_1 < \dots < E_n$ of intervals,

$$\sum_{j=1}^M \|E_j x\| \leq C(1 + 2M/N)\|x\|.$$

Lemma 2 (Theorem 3, [S2]). For any $\varepsilon_k > 0$ with $\sum_{k=1}^\infty \varepsilon_k \leq 1$, there exists a constant $C > 1$ such that for any normalized block basis (y_k) of (e_k) with the following properties: there is a sequence $r_k \uparrow \infty$ in $\mathbb{N}$ so that for all $k \in \mathbb{N}$,

$$(2) \quad \sup_{\substack{r \leq r_{k-1} \\ E_1 < E_2 < \dots < E_r}} \sum_{i=1}^r \|E_i(y_k)\| \leq 1 + \varepsilon_k,$$

$$(3) \quad \text{card}(\text{supp } y_k) \leq \varepsilon_k \cdot f\left(\frac{r_k}{3}\right),$$

we have that (y_k) is C -equivalent to (e_k) .

Theorem 3. For any $\frac{1}{8} > \varepsilon > 0$, there is a rapidly increasing sequence $\{p_k\}$ of natural numbers such that for any n there exist n disjointly supported elements v_j , $1 \leq j \leq n$, such that for any $k \leq n$, v_k and u_{p_k} have the same distribution, and

$$\left\| \sum_{i=k}^n v_j \right\| \leq 1 + \varepsilon.$$

Proof. Let $(p_j)_{j=0}^\infty$ and $(q_j)_{j=0}^\infty$ be two sequences of natural numbers which satisfy the following conditions:

1. $f(q_0) > 20/\varepsilon$ and $p_0 = 1$.
2. If q_j is defined, then $p_{j+1} = p_j q_j^{q_j^2}$.

3. Suppose that p_j is defined. By Lemma 1, there is q_j such that if x is an $\ell_{1+}^{q_j}$ -vector with constant $1 + \varepsilon$, then

$$\sum_{i=1}^p \|E_i x\| \leq (1 + \varepsilon)(1 + 2p/q_j)\|x\| \leq (1 + 2\varepsilon)\|x\|$$

for any $p \leq p_j^{p_j}$ and $E_1 < E_2 < \dots < E_p$.

Let $\{w_{m_1, m_2, \dots, m_j} : j \leq n \text{ and } m_i \leq q_{i=1}^{q_{i-1}^2} \text{ for all } i \leq j\}$ be any sequence of $\mathbb{N}$ such that

$$w_{m_1, m_2, \dots, m_j} < w_{r_1, r_2, \dots, r_\ell}$$

if one of the following conditions holds:

- for some $i \leq \max\{j, \ell\}$, $m_s = r_s$ for all $s < i$ and $m_i < r_i$;
- $j < \ell$ and $m_s = r_s$ for all $s \leq j$.

Finally, let

$$v_j = \frac{f(p_j)}{p_j} \sum_{i_1=1}^{q_0^2} \sum_{i_2=1}^{q_1^2} \dots \sum_{i_j=1}^{q_{j-1}^2} e_{w_{i_1, i_2, \dots, i_j}}.$$

We claim that for any $k \leq n$ and any interval E either $E(v_k) = 0$ or

$$(4) \quad \left\| E \left(\sum_{j=k}^n v_j \right) \right\| \leq (1 + \varepsilon/4 + \varepsilon/2^{k+2}) \|v_k\|.$$

Clearly, (4) implies our theorem and it holds if $k = n$. Assume that (4) holds for all $k \geq m > 1$. We will show that (4) also holds for $k = m - 1$. We note that:

1. there is a vector y_m such that

$$(a) \quad \|y_m\| = \frac{f(p_{m-1} q_{m-1}^2)}{p_{m-1} f(q_{m-1}^2)} \leq \frac{1 + \varepsilon/4}{p_{m-1}};$$

- (b) v_m is the sum of p_{m-1} successive vectors which have the same distribution as y_m .

2. $E(v_{m-1}) = \frac{f(p_{m-1})}{p_{m-1}} \sum_{i=1}^j e_{\ell_i}$, for some j and $\ell_1 < \ell_2 < \dots < \ell_j$.

We shall use induction (on j above) to prove (4). First, we summarize some facts that we shall use in our proof.

Fact 1. If $E(v_{m-1}) = 0$, then by the induction hypothesis, we have

$$\left\| E \left(\sum_{k=m}^n v_k \right) \right\| \leq \left(1 + \frac{\varepsilon}{2} \right) \|E(v_m)\| \leq \left(1 + \frac{\varepsilon}{2} \right) \|y_m\| \leq \frac{\sqrt{2}}{p_{m-1}}.$$

If $\|E(v_{m-1})\| = \frac{f(p_{m-1})}{p_{m-1}}$, then

$$\left\| E \left(\sum_{k=m}^n v_k \right) \right\| \leq (1 + \varepsilon) \|E(v_m)\| \leq 2(1 + \varepsilon) \|y_m\| \leq \frac{4}{p_{m-1}}.$$

Fact 2. We claim that $\|(v_{m-1})\| = \frac{f(p_{m-1})}{p_{m-1}}$ implies

$$\left\| E \left(\sum_{k=m-1}^n v_k \right) \right\| = \frac{f(p_{m-1})}{p_{m-1}} = \|E(v_{m-1})\|.$$

For any $r \geq 2$, and $E_1 < E_2 < \cdots < E_r$

$$\begin{aligned} & \frac{1}{f(r)} \sum_{i=1}^r \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \frac{1}{f(r)} \left(\frac{f(p_{m-1}) + 4}{p_{m-1}} + \sum_{E_i \circ E(v_{m-1})=0} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \right) \\ & \leq \frac{1}{f(r)} \left(\frac{f(p_{m-1}) + 4}{p_{m-1}} + \frac{4f(r)}{p_{m-1}} \right) < \frac{f(p_{m-1})}{p_{m-1}}. \end{aligned}$$

Therefore,

$$\left\| E \left(\sum_{k=m-1}^n v_k \right) \right\| = \left\| E \left(\sum_{k=m-1}^n v_k \right) \right\|_{\infty} = \frac{f(p_{m-1})}{p_{m-1}} = \|E(v_{m-1})\|.$$

Fact 3. Suppose that $\|E(v_{m-1})\| > \frac{f(p_{m-1})}{p_{m-1}}$. It is known that for any $r \geq q_{m-1}^{2m-1}$, u_r is an $\ell_{1+}^{q_{m-1}-1}$ -vector with constant $1 + \varepsilon/2$. This implies that $E(v_m)$ is an $\ell_{1+}^{q_{m-1}-1}$ -vector with constant $1 + \varepsilon/2$ and $E(\sum_{k=m}^n v_k)$ is an $\ell_{1+}^{q_{m-1}-1}$ -vector with constant $1 + \varepsilon$.

Fact 4. If $\|E(\sum_{k=m-1}^n v_k)\| = \|E(\sum_{k=m-1}^n v_k)\|_{\infty}$, then (4) holds. So we assume that there are $p \geq 2$ and $E_1 < E_2 < \cdots < E_p$ such that

$$\left\| E \left(\sum_{k=m-1}^n v_k \right) \right\| = \frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\|.$$

We claim that either $E_i \circ E(v_{m-1}) \neq 0$ for all $i \leq p$ or $\|E_i \circ E(v_{m-1})\| \leq \frac{f(p_{m-1})}{p_{m-1}}$ for all $i \leq p$.

First, we note that by the proof of Fact 2, if $E_i \circ E(v_{m-1}) = E(v_{m-1}) \neq 0$, then

$$\frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| < \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\|,$$

a contradiction. Hence, $\|E_i \circ E(v_{m-1})\| < \|E(v_{m-1})\|$ for all $i \leq p$. Suppose that our claim is not true. Without loss of generality, we assume that $E_1 \circ E(v_{m-1}) = 0$ and $\|E_2 \circ E(v_{m-1})\| > \frac{f(p_{m-1})}{p_{m-1}}$. Let $E_{2,1} < E_{2,2} < \cdots < E_{2,p_2}$ be a successive sequence of subintervals of E_2 such that

$$\left\| E_2 \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| = \frac{1}{f(p_2)} \sum_{j=1}^{p_2} \left\| E_{2,j} \circ E \left(\sum_{k=m-1}^n v_k \right) \right\|.$$

Let ℓ be the smallest number such that $E_{2,\ell} \circ E(v_{m-1}) \neq 0$ and $E'_2 = \bigcup_{j=\ell+1}^{p_2} E_{2,j}$. The proof of Fact 2 shows that

$$\begin{aligned} & \frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & < \frac{1}{f(p)} \left(\left\| E_{2,\ell} \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| + \left\| E'_2 \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \right. \\ & \quad \left. + \sum_{i=3}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \right), \end{aligned}$$

a contradiction.

Fact 5. For any $k \leq p_{m-1}$, $\frac{f(p_{m-1}q_{m-1}^2)}{f(kq_{m-1}^2)} \leq \frac{f(p_{m-1})}{f(k)}$. Therefore, if $E(v_{m-1}) \neq 0$, then

$$\|E(v_m)\| \leq \|E(v_{m-1})\| + \|y_m\| \leq \|E(v_{m-1})\| + \frac{1 + \varepsilon/4}{p_{m-1}}.$$

Fact 2 shows that (4) holds when $j = 1$. So we may assume that $j > 1$. The proof consists of four cases.

Case 1. $1 < p \leq j$. By Fact 4, for all $i \geq p$,

$$E_i \circ E(v_{m-1}) \neq 0 \quad \text{and} \quad \|E_i \circ E(v_{m-1})\| < \frac{jf(p_{m-1})}{p_{m-1}f(j)}.$$

Using the induction hypothesis, we have

$$\begin{aligned} & \frac{1}{f(p)} \sum_{i=1}^p \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \frac{1 + \varepsilon/4 + \varepsilon/2^{2m+1}}{f(p)} \sum_{i=1}^p \|E_i \circ E(v_{m-1})\| \\ & \leq (1 + \varepsilon/4 + \varepsilon/2^{m+1}) \|E(v_{m-1})\|. \end{aligned}$$

Case 2. $j \leq p \leq q_0$. By Fact 4, for any $i \leq p$ either $E_i \circ E(v_{m-1}) = 0$ or $E_i \circ E(v_{m-1}) = \frac{f(p_{m-1})}{p_{m-1}} e_\ell$. Note that $2p \leq 2q_0 \leq \frac{f(p_1)}{q_0} \leq \frac{\varepsilon f(p_1)}{10}$. Hence,

$$\begin{aligned} & \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \|E(v_{m-1})\| + \sum_{E_i \circ E(v_{m-1})=0} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \|E(v_{m-1})\| + \frac{2p}{p_{m-1}} \\ & \leq (1 + \varepsilon/4) \|E(v_{m-1})\|. \end{aligned}$$

Case 3. $q_0 \leq p \leq p_{m-1}^{p_{m-1}}$. By Fact 3, $E(\sum_{k=m}^n v_k)$ is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon$. Therefore,

$$\begin{aligned} & \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_{m-1})\| + \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m}^n v_k \right) \right\| \\ & \leq \|E(v_{m-1})\| + (1 + \varepsilon) \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_m)\| \\ & \leq \|E(v_{m-1})\| + \frac{(1 + 2\varepsilon)(1 + \varepsilon) \|E(v_m)\|}{f(p)} \\ & \leq \left(1 + \frac{2}{f(p)} \right) \|E(v_{m-1})\| \leq (1 + \varepsilon/4) \|E(v_{m-1})\|. \end{aligned}$$

Case 4. $p_{m-1}^{p_{m-1}} < p$. In this case, we have $\|E(v_{m-1})\|_1 / f(p) \leq \frac{\varepsilon \|v_{m-1}\|_\infty}{4^{m+2}}$.

$$\begin{aligned} & \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m-1}^n v_k \right) \right\| \\ & \leq \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_{m-1})\| + \sum_{i=1}^p \frac{1}{f(p)} \left\| E_i \circ E \left(\sum_{k=m}^n v_k \right) \right\| \\ & \leq \frac{\varepsilon \|v_{m-1}\|_\infty}{4^{m+2}} + \left(1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{m+2}} \right) \sum_{i=1}^p \frac{1}{f(p)} \|E_i \circ E(v_m)\| \\ & \leq \frac{\varepsilon \|v_{m-1}\|_\infty}{4^{m+2}} + \left(1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{m+2}} \right) \|E(v_m)\| \\ & \leq \left(1 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{m+1}} \right) \|E(v_{m-1})\|. \end{aligned}$$

The proof is complete. $\square$

Remark 4. The above theorem was first proved by the second author and independently by the first author with techniques which were later used in [ADKM] for some asymptotic ℓ_1 spaces. Recently, A. Manoussakis has the same result for a more general class of spaces, containing S .

Remark 5. Let the v_j 's be the vectors in the proof of Theorem 3. We have proved that for any $m \leq n$ and $x \in \text{span}\{v_m, \dots, v_n\}$, x is an $\ell_{1+}^{q_{m-1}}$ -vector with constant $1 + \varepsilon$.

By Lemma 1, for any $N, r, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists an m -dimensional space V such that

- $N < V$ and V is $1 + \varepsilon$ equivalent to ℓ_∞^m ;
- For any $x \in V$ and any $E_1 < E_2 < \dots < E_r$, we have

$$\sum_{i=1}^r \|E_i x\| \leq (1 + \varepsilon) \|x\|.$$

Theorem 6. $\mathbf{S}$ has an ℓ_1 -spreading model.

Proof. By Theorem 3, there exists a successive sequence $\{y_k\}$ in the unit ball of $\mathbf{S}$ such that for any k , there exist k disjointly supported elements $\{v_{k,j} : j \leq k\}$ so that for each $j \leq k$, $v_{k,j}, u_{p_j}/2$ have the same distribution and $y_k = \sum_{j=1}^k v_{k,j}$. Hence, if $N < n_1 < n_2 < \dots < n_j$, then

$$\left\| \sum_{i=1}^j y_{n_i} \right\| \geq \left\| \sum_{i=1}^j v_{n_i, N} \right\| \approx \frac{jf(p_N)}{2f(jp_N)}.$$

So $\{y_k : k \in \mathbb{N}\}$ generates an ℓ_1 -spreading model. $\square$

In the sequel we shall use a variant of the refinement of Johnson's argument for a space with a subsymmetric basis ([BCLT], Proposition 6.3). We shall work with finite dimensional decompositions instead of block-vectors. Fix a sequence (n_k) of natural numbers. Represent any k in the form $k = 2^i(2j-1)$. Let $m_k = n_j$, i.e. we repeat each n_j infinitely many times. In the following, X , Y and V denote the spaces

$$X = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{n_k-1} \right)_{\mathbf{S}}, \quad Y = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{\ell_k} \right)_{\mathbf{S}}, \quad V = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_k} \right)_{\mathbf{S}}.$$

Then

$$Y = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_{2k-1}} \right)_{\mathbf{S}}, \quad V = \left(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{m_{2k}} \right)_{\mathbf{S}}.$$

So we have

$$V \approx Y \oplus V \quad \text{and} \quad Y = \mathbf{S} \oplus X.$$

Proposition 7. *There exists a complemented subspace U of $\mathbf{S}$ which is 2C-isomorphic to Y .*

Proof. By Theorem 3 and Remark 5, we can construct a sequence (r_k) of natural numbers and a successive sequence (V_k) of the finite dimensional subspaces of $\mathbf{S}$ which satisfy the following conditions:

1. For any $k \in \mathbb{N}$, V_k is a subspace of $\mathbf{S}$ which is $(1 + \varepsilon)$ isomorphic to $\ell_{\infty}^{n_k}$.
2. $V_1 < V_2 < \dots$.
3. For any $k \in \mathbb{N}$, $p \leq r_{k-1}$ and normalized vector $y_k \in V_k$,

$$\sup_{\substack{r \leq r_{k-1} \\ E_1 < E_2 < \dots < E_r}} \sum_{i=1}^r \|E_i(y_k)\| \leq 1 + \varepsilon_k.$$

4. For any $y_k \in V_k$,

$$\text{card}(\text{supp } y_k) \leq \varepsilon_k \cdot f\left(\frac{r_k}{3}\right).$$

Let U be the space generated by $\bigcup_{k=1}^{\infty} V_k$. By (3), (4) and Lemma 2, for any $y_k \in V_k$ with $\sum_{k=1}^{\infty} y_k \in \mathbf{S}$, we have

$$\left\| \sum_{k=1}^{\infty} \|y_k\| e_k \right\| \leq \left\| \sum_{k=1}^{\infty} y_k \right\| \leq C \left\| \sum_{k=1}^{\infty} \|y_k\| e_k \right\|.$$

Hence U is isomorphic to Y . Now, let F_k be the smallest interval of $\mathbb{N}$ which contains the support of V_k . Since V_k is 2-equivalent to $\ell_{\infty}^{n_k}$ and ℓ_{∞} is an injective

space, there is a projection P_k from $F_k(\mathbf{S})$ onto V_k such that $\|P\| \leq 2$. Define $P : \mathbf{S} \rightarrow U$ by

$$P(x) = \sum_{k=1}^{\infty} P_k(F_k(x)).$$

Since

$$\left\| \sum_{k=1}^{\infty} P_k(F_k(x)) \right\| \leq C \left\| \sum_{k=1}^{\infty} \|P_k(F_k(x))\| e_k \right\| \leq 2C\|x\|,$$

P is a bounded projection from $\mathbf{S}$ onto U . $\square$

Proposition 8. *The space Y is isomorphic to $\mathbf{S}$.*

Proof. We use Pełczyński's decomposition technique (see [LT] or [BCLT]). By Proposition 7, we may assume that V and Y are complemented subspaces of $\mathbf{S}$. Let W be the space such that

$$\mathbf{S} \approx V \oplus W.$$

Then

$$Y \oplus \mathbf{S} \approx Y \oplus V \oplus W \approx V \oplus W \approx \mathbf{S}.$$

Evidently, $\mathbf{S} = \mathbf{S} \oplus \mathbf{S}$. Therefore,

$$Y \approx \mathbf{S} \oplus X \approx \mathbf{S} \oplus \mathbf{S} \oplus X \approx \mathbf{S} \oplus Y \approx \mathbf{S}.$$

The proof is complete. $\square$

Let (x_k) be a sequence of successive vectors in $\mathbf{S}$ such that for each $n \in \mathbb{N}$, x_k has the same distribution as u_{n_k} for some $n_k \in \mathbb{N}$. Let Y_1 be the space spanned by $\{x_k : k \in \mathbb{N}\}$. It is known that Y_1 is a complemented subspace of $\mathbf{S}$ ([LT], Theorem 3.a.4) and $Y_1 \oplus \mathbf{S} \approx \mathbf{S}$ ([BCLT], Proposition 6.3). Since $\mathbf{S}$ is complementably minimal, the proof of Proposition 8 shows that $Y_1 \approx \mathbf{S}$. It is also known that if $(x_k)_{k=1}^n$ is a sequence of successive vectors such that for any $k \leq n$, x_k has the same distribution as u_{n_k} , then $(x_k)_{k=1}^n$ is $1 + \frac{3}{n}$ -equivalent to the unit vector basis of ℓ_1^n and for each x in the space of $\{x_k : k \leq n\}$, x is an ℓ_1^n -vector with constant $1 + \frac{3}{n}$. By the proof of Proposition 7, we have the following.

Proposition 9. *Schlumprecht space $\mathbf{S}$ is isomorphic to $(\sum_k \oplus \ell_1^{n_k})_{\mathbf{S}}$ for any sequence of integers (n_k) .*

A Banach space with an unconditional basis is said to have a *unique conditional basis* if any two normalized unconditional bases are equivalent after a permutation; see [BCLT] and for recent results see [CK]. It was remarked in [CK] that the example of Gowers in [G1] has a unique unconditional basis. Proposition 8 (or Proposition 9) implies that S has no unique unconditional basis. This also immediately follows if one uses that S is complementably minimal and combines Proposition 6.3 and Proposition 6.4 of [BCLT].

Clearly, we have by duality that for any sequence (n_k) of natural numbers, $\mathbf{S}^*$ is isomorphic to $(\sum_{k=1}^{\infty} \oplus \ell_1^{n_k})_{\mathbf{S}^*}$ and $(\sum_{k=1}^{\infty} \oplus \ell_{\infty}^{n_k})_{\mathbf{S}^*}$. P. Casazza pointed out the following immediate consequence of our observations (in fact, not using their full generality). Recall that the same property for the classical spaces ℓ_p for $p > 2$, $1 < p < 2$ and $p = 1$ was shown in [R], [BDGJN] and [B] respectively.

Corollary 10. *Schlumprecht space $\mathbf{S}$ (resp. its dual $\mathbf{S}^*$) has a subspace isomorphic to the whole space and not complemented in $\mathbf{S}$ (resp. $\mathbf{S}^*$).*

Remark 11. Since $\mathbf{S}$ is complementably minimal, most of the above results are valid hereditarily, i.e. in every subspace of $\mathbf{S}$.

Remark 12. Let Z be a complemented subspace of $\mathbf{S}$ which has a subsymmetric basis. Since Z is isomorphic to its square, then by the decomposition technique we have that Z is isomorphic to $\mathbf{S}$.

We do not know the answer to the following question:

Question 4. *Does $\mathbf{S}$ have a unique subsymmetric basis?*

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INSTITUTE OF MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, 1113 SOFIA, BULGARIA
Current address: Department of Mathematics, University of South Carolina, Columbia, South Carolina

E-mail address: denka@math.sc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MEMPHIS, MEMPHIS, TENNESSEE 38152

E-mail address: linpk@msci.memphis.edu