REFINABLE SUBSPACES OF A REFINABLE SPACE

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ABSTRACT. Local refinable finitely generated shift-invariant spaces play a significant role in many areas of approximation theory and geometric design. In this paper we present a new approach to the construction of such spaces. We begin with a refinable function $\psi : \mathbb{R} \rightarrow \mathbb{R}^m$ which is supported on $[0, 1]$. We are interested in spaces generated by a function $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ built from the shifts of ψ .

1. INTRODUCTION

Local refinable finitely generated shift-invariant spaces naturally arise in the theory of (multi)wavelets, splines, finite-elements, and subdivision schemes. In this paper we introduce and begin to develop a method for constructing and studying such spaces.

Let L^1_{loc} denote the space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which belong to $L^1(\mathbb{R})$ locally; that is, $f \in L^1_{\text{loc}}$ provided that (f is measurable and) $\int_K |f| < \infty$ for every compact $K \subset \mathbb{R}$. This space is topologized by the family of seminorms

$$|f|_N := \int_{[-N, N]} |f|, \quad N \in \mathbb{N}.$$

We refer to a (row) vector $\phi = [\phi_1, \dots, \phi_n]$ of functions in L^1_{loc} as a *generator*.

A generator $\phi = [\phi_1, \dots, \phi_n]$ is said to be *refinable* if there exists a finitely supported sequence $b : \mathbb{Z} \rightarrow \mathbb{R}^{n \times n}$ (called a *mask* for ϕ) for which

$$\phi = \sum_{j \in \mathbb{Z}} \phi(2 \cdot -j) b(j).$$

We begin with a generator $\psi = [\psi_1, \dots, \psi_m]$ supported on $[0, 1]$ that is refinable with a two-term mask:

$$(1.1) \quad \psi = \psi(2 \cdot) a(0) + \psi(2 \cdot - 1) a(1);$$

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and we intend to construct more useful (read “smoother”) refinable generators by using the shifts of ψ . That is, we consider generators of the form

$$\phi = \sum_{j \in \mathbb{Z}} \psi(\cdot - j)c(j),$$

for some sequence $c : \mathbb{Z} \rightarrow \mathbb{R}^{m \times n}$. The motivation for this approach is that it is much easier to study the properties of ψ , since it is supported on $[0, 1]$ (hence the shifts of ψ do not “interfere” with each other). The crux is that ϕ constructed in this way will not, in general, be refinable.

Let V be a subspace of L_{loc}^1 . Then we say V is *shift-invariant* if

$$f \in V \implies f(\cdot \pm 1) \in V;$$

we say V is a *finitely generated shift-invariant (FSI) space* if

$$V = S(\phi) := \text{clos}_{L_{\text{loc}}^1} \text{span}\{\phi_i(\cdot - j) \mid i = 1, \dots, n; j \in \mathbb{Z}\}$$

for some generator ϕ ; and we say an FSI space V is *local* if $V = S(\phi)$ for some compactly supported generator ϕ . Lastly, we say V is *refinable* if

$$f \in V \implies f(\cdot/2) \in V.$$

Evidently, $S(\phi)$ is refinable whenever ϕ is refinable.

The main objective of this paper is: *Given a refinable generator ψ supported on $[0, 1]$ with mask $(a(0), a(1))$, characterize all local refinable FSI subspaces $V \subset S(\psi)$.* Under the condition that $a(0)$ is invertible, we achieve this goal in Corollaries 2.4 and 2.6 and Theorem 2.5.

We point out that every local refinable FSI space is a subspace of $S(\psi)$ for some refinable generator ψ supported on $[0, 1]$. For $f \in L_{\text{loc}}^1$ and $V \subset L_{\text{loc}}^1$, define

$$f|_I := f\chi_{[0,1]} \quad \text{and} \quad V|_I := \{f|_I \mid f \in V\},$$

where χ_A denotes the characteristic function of the set A . Suppose V is a local refinable FSI space. Then $m := \dim V|_I$ is finite. We refer to m as the *local dimension* of V . Let $\psi = [\psi_1, \dots, \psi_m]$ be a basis for $V|_I$. Then ψ is refinable and $V \subset S(\psi)$.

This has been observed and exploited already by Jia in [J4], [J5], and [J6], where the author studied a given function ϕ via a basis ψ for $S(\phi)|_I$. We point out that ψ is necessarily refinable with a two-term mask, a fact which, though surely known to Jia, was not exploited since his results relate to general shift-invariant spaces as opposed to refinable ones.

The simpler structure due to the small supports of ψ and a in Eq. (1.1) has also been recognized by Micchelli et al. in, for example, [M], [MP1], [MP2], and [MS]. In particular, given a univariate refinable function ϕ with finite mask b , they define $a(0)$ and $a(1)$ by $a(\varepsilon) := [b(\varepsilon + 2j - i)]_{i,j}$ and study ϕ via the refinable function having mask a (also see [DL]). Among other things, this was used to provide necessary and sufficient conditions for the convergence of a given subdivision scheme and, in [MS], to provide a fairly thorough study of regularity for refinable function vectors.

Our approach is different in that the mask a and generator ψ come first. In this paper, we identify all local refinable subspaces $S(\phi)$ of $S(\psi)$. The next steps are to provide further characterizations of the properties of $S(\psi)$ in terms of a ; to determine when these properties are preserved by a subspace $S(\phi)$; and to put these ideas together to construct desirable refinable generators.

2. RESULTS

Throughout this paper, we assume that $(a(0), a(1))$ is a mask for a refinable generator $\psi = [\psi_1, \dots, \psi_m]$ supported on $[0, 1]$ (in particular, each ψ_j is assumed to be in $L^1(\mathbb{R})$). We will show that when $a(0)$ is invertible, each local refinable FSI subspace of $S(\psi)$ corresponds to some $a(0)$ -invariant space (for a matrix $a \in \mathbb{R}^{k \times k}$, a space $\Gamma \subset \mathbb{R}^k$ is *a-invariant* if $a\Gamma \subset \Gamma$). Our specific statements will require a few more definitions.

We use $\mathbb{Z}_+$ to denote the set of non-negative integers and $\mathbb{R}^k$ to denote the set of column vectors of length k . For any set $V \subset L^1_{\text{loc}}$, define

$$V^+ := \{f \in V \mid \text{supp } f \subset [0, \infty)\};$$

and, for $V \subset S(\psi)$, define

$$\Sigma(V) := \{\sigma \in \mathbb{R}^m \mid \psi\sigma \in V^+\}.$$

(By convention, $V^+ := (V^+)_|$.)

Proposition 2.1. *For any refinable subspace V of $S(\psi)$, $\Sigma(V)$ is $a(0)$ -invariant.*

If $\phi = [\phi_1, \dots, \phi_k]$ is a generator supported on $[0, \infty)$, then the sum

$$\phi *' c := \sum_{j=0}^{\infty} \phi(\cdot - j)c(j)$$

is locally finite for any sequence $c : \mathbb{Z}_+ \rightarrow \mathbb{R}^k$. In particular, the set

$$R(\phi) := \{\phi *' c \mid c : \mathbb{Z}_+ \rightarrow \mathbb{R}^k\},$$

spanned by the right shifts of ϕ , is a subset of $S(\phi)^+$.

We say a subspace Λ of $\mathbb{R}^m$ is *preserved by $a(0)$* if $a(0)\Lambda = \Lambda$, and a matrix $\lambda \in \mathbb{R}^{m \times n}$ is *preserved by $a(0)$* if its columns form a basis for a space that is preserved by $a(0)$. Note that a matrix $\lambda \in \mathbb{R}^{m \times n}$ is preserved by $a(0)$ if and only if $a(0)\lambda = \lambda\beta_\lambda$ for a unique invertible $\beta_\lambda \in \mathbb{R}^{n \times n}$. Suppose that $\lambda \in \mathbb{R}^{m \times n}$ is preserved by $a(0)$. Set

$$(2.1) \quad \ell(0) := \lambda \quad \text{and} \quad \ell(2j + \varepsilon) := a(\varepsilon)\ell(j)\beta_\lambda^{-1} \quad \text{for } \varepsilon \in \{0, 1\}, 2j + \varepsilon > 0.$$

We define the *generalized truncated power* e_λ by

$$e_\lambda := \psi *' \ell.$$

Proposition 2.2. *Suppose that $\lambda \in \mathbb{R}^{m \times n}$ is preserved by $a(0)$. Then*

- (i) $e_\lambda = e_\lambda(2\cdot)\beta_\lambda$;
- (ii) if $\lambda' \in \mathbb{R}^{m \times n}$ has the same column space as λ , then $S(e_\lambda) = S(e_{\lambda'})$; and
- (iii) $S(e_\lambda)$ is a local refinable FSI subspace of $S(\psi)$.

Property (2.2.ii) allows us to unambiguously define, for any Λ preserved by $a(0)$, the space $S_\Lambda := S(e_\lambda)$ where the columns of λ form a basis for Λ .

Property (2.2.iii) ensures that $S(e_\lambda) = S(\phi)$ for some compactly supported generator ϕ . A procedure for constructing such ϕ can be based on the proofs in [BD].

Theorem 2.3. *Suppose V is a local refinable FSI subspace of $S(\psi)$. If $\Sigma(V)$ is preserved by $a(0)$, then $V = S_{\Sigma(V)}$.*

If $a(0)$ is invertible, every $a(0)$ -invariant subspace is, in fact, preserved by $a(0)$ and we have the following corollary—one of the main results of this paper.

Corollary 2.4. *Suppose $a(0)$ is invertible. Then V is a local refinable FSI subspace of $S(\psi)$ if and only if $V = S_\Lambda$ for some $a(0)$ -invariant Λ .*

In other words, when $a(0)$ is invertible, every local refinable FSI subspace of $S(\psi)$ is of the form S_Λ for some $a(0)$ -invariant space Λ . The $a(0)$ -invariant spaces are easily identified from the Jordan-Canonical form of $a(0)$. By Theorem 2.3, $S_\Lambda = S_{\Sigma(S_\Lambda)}$. So, if $a(0)$ is invertible and ψ is linearly independent (meaning the entries of ψ are linearly independent), the local refinable subspaces of $S(\psi)$ are in one-to-one correspondence with the $a(0)$ -invariant spaces Λ satisfying $\Lambda = \Sigma(S_\Lambda)$. By the definitions of S_Λ and $\Sigma(V)$, Λ is always a subset of $\Sigma(S_\Lambda)$; but, as Example 4.2 illustrates, it is often a proper subset. Our next result offers a means for determining $\Sigma(S_\Lambda)$ from Λ .

First, define

$$A_0 := \begin{bmatrix} a(1) & 0 \\ 0 & a(0) \end{bmatrix} \quad \text{and} \quad A_1 := \begin{bmatrix} 0 & a(0) \\ 0 & a(1) \end{bmatrix}.$$

Let $\mathcal{H}_\Lambda$ be the minimal subspace of $\mathbb{R}^{2m}$ that contains

$$\begin{bmatrix} 0 \\ \Lambda \end{bmatrix} := \left\{ \begin{bmatrix} 0 \\ v \end{bmatrix} \mid v \in \Lambda \right\}$$

and is $\{A_0, A_1\}$ -invariant, i.e., A_ε -invariant for $\varepsilon = 0, 1$. And define

$$\mathcal{H}_\Lambda^0 := \{v \in \mathbb{R}^m \mid \begin{bmatrix} 0 \\ v \end{bmatrix} \in \mathcal{H}_\Lambda\}.$$

Theorem 2.5. *Suppose Λ is an $a(0)$ -invariant subspace of $\mathbb{R}^m$. If $a(0)$ is invertible and ψ is linearly independent, then $\Sigma(S_\Lambda) = \mathcal{H}_\Lambda^0$.*

We have the following corollary.

Corollary 2.6. *Suppose Λ is an $a(0)$ -invariant subspace of $\mathbb{R}^m$. Let the columns of $\lambda \in \mathbb{R}^{m \times n}$ form a basis for Λ . If $a(0)$ is invertible and ψ is linearly independent, then the following are equivalent.*

- (i) $\Lambda = \Sigma(V)$ for some local refinable FSI subspace $V \subset S(\psi)$.
- (ii) $\Lambda = \Sigma(S_\Lambda)$.
- (iii) $\Lambda = \mathcal{H}_\Lambda^0$.
- (iv) $S_\Lambda^+ = R(e_\lambda)$.

It is clear that $S_{\Lambda|}$ is always a subset of $\text{span } \psi := \text{span}\{\psi_1, \dots, \psi_m\}$. We now give a characterization of when these sets are actually equal.

Theorem 2.7. *Suppose ψ is linearly independent. Suppose $\Lambda \subset \mathbb{R}^m$ is preserved by $a(0)$. Define $\mathcal{L}_\Lambda$ to be the minimal $\{a(0), a(1)\}$ -invariant subspace of $\mathbb{R}^m$ containing Λ . Then $S_{\Lambda|} = \text{span } \psi$ if and only if $\mathcal{L}_\Lambda = \mathbb{R}^m$.*

The preceding few results assume that ψ is linearly independent. Corollary 2.9 below characterizes this property in terms of the mask a . Corollary 2.9 is an immediate consequence of Theorem 2.8, which identifies all dependency relations among the entries of ψ and provides the mask for a basis for $\text{span } \psi$ (also in terms of a).

To begin with, define $\mathcal{W}$ to be the smallest subspace of $\mathbb{R}^{1 \times m}$ satisfying

$$(2.2) \quad \hat{\psi}(0) \in \mathcal{W}, \quad \mathcal{W}a(0) \subset \mathcal{W}, \quad \text{and} \quad \mathcal{W}a(1) \subset \mathcal{W}.$$

Let the rows of $W \in \mathbb{R}^{k \times m}$ form a basis for $\mathcal{W}$. Then there exists a unique $\tilde{v} \in \mathbb{R}^{1 \times k}$ such that $\hat{\psi}(0) = \tilde{v}W$, and unique $\tilde{a}(0)$ and $\tilde{a}(1)$ such that $Wa(\varepsilon) = \tilde{a}(\varepsilon)W$.

Now, define

$$T := a(0) + a(1).$$

Then two necessary conditions for the generator ψ to be linearly independent are:

- N1) 2 is a simple eigenvalue of the matrix T with left eigenvector $\hat{\psi}(0)$; and
  - N2) all other eigenvalues have modulus strictly less than 2
- (cf., e.g., [DM], [H], [JS]).

Theorem 2.8. *Suppose T satisfies N1 and N2. Then*

- (i) $\psi\sigma = 0 \iff \sigma \in \mathcal{W}^\perp := \{v \in \mathbb{R}^m \mid wv = 0 \text{ for all } w \in \mathcal{W}\}$;
- (ii) the refinement equation with mask $\tilde{a}$ has a unique solution $\tilde{\psi}$ with $\hat{\tilde{\psi}}(0) = \tilde{v}$;
- (iii) $\tilde{\psi}$ is a basis for $\text{span } \psi$, i.e., $\tilde{\psi}$ is independent and $\text{span } \psi = \text{span } \tilde{\psi}$.

So, one may assume that ψ is linearly independent without loss of generality. Note, if $M \in \mathbb{R}^{m \times k}$ is any right inverse of W , then $\tilde{v} = \hat{\psi}(0)M$ and $\tilde{a}(\varepsilon) = Wa(\varepsilon)M$.

Corollary 2.9. *The generator ψ is linearly independent if and only if*

- (i) T satisfies N1 and N2; and
- (ii) $\mathcal{W} = \mathbb{R}^{1 \times m}$.

The interested reader is encouraged to see [HJ] for additional results along these lines.

3. PROOFS

Throughout this section, we write $\phi \subset V$ to mean that the entries of the generator ϕ are elements of V . We recall some results from [BD].

Lemma 3.1. *For any closed shift-invariant space V of finite local dimension, there exists $r > 0$ such that if $f \in V$ vanishes on $[-r, 0]$, then $f|_1 \in V_1^+$.*

Lemma 3.2. *For any closed shift-invariant space V of finite local dimension, there is a compactly supported generator $\phi = [\phi_1, \dots, \phi_k] \subset V$ such that $\phi|_1$ is a basis for V_1^+ and $V^+ = R(\phi)$.*

The topology used in [BD] is that of uniform convergence on compact sets, but the arguments used there also apply to the topology of L_{loc}^1 .

Proof of Proposition 2.1. Suppose $\sigma \in \Sigma(V)$. Then there exists $f \in V^+$ such that $f|_1 = \psi\sigma$. Since V is refinable, $f(\cdot/2) \in V^+$. But, $f(\cdot/2) = \psi(\cdot/2)\sigma = \psi a(0)\sigma$ on $[0, 1]$. So $a(0)\sigma \in \Sigma(V)$. $\square$

Proof of Proposition 2.2. (i)

$$\begin{aligned} e_\lambda \left(\frac{\cdot}{2} \right) &= \sum_j \psi \left(\frac{\cdot - 2j}{2} \right) \ell(j) = \sum_{j, \varepsilon} \psi(\cdot - 2j - \varepsilon) a(\varepsilon) \ell(j) \\ &= \sum_{j, \varepsilon} \psi(\cdot - (2j + \varepsilon)) \ell(2j + \varepsilon) \beta_\lambda = e_\lambda \beta_\lambda. \end{aligned}$$

- (ii) There exists $\gamma \in \mathbb{R}^{n \times n}$ such that $\lambda = \lambda'\gamma$. Set $\beta := \beta_\lambda$ and $\beta' := \beta_{\lambda'}$. Then $\lambda'\gamma\beta = \lambda\beta = a(0)\lambda = a(0)\lambda'\gamma = \lambda'\beta'\gamma$.

Since the columns of λ' form a basis, $\gamma = \beta'\gamma\beta^{-1}$. Define ℓ by Eqs. (2.1) and ℓ' similarly, but with λ' in place of λ . Then, $\ell(0) = \lambda = \lambda'\gamma = \ell'(0)\gamma$. Now, suppose $2j + \varepsilon > 0$ and $\ell(j) = \ell'(j)\gamma$. Then

$$\ell(2j + \varepsilon) = a(\varepsilon)\ell(j)\beta^{-1} = a(\varepsilon)\ell'(j)\gamma\beta^{-1} = \ell'(2j + \varepsilon)\beta'\gamma\beta^{-1} = \ell'(2j + \varepsilon)\gamma.$$

It follows that $e_\lambda = e_{\lambda'}\gamma$.

(iii) Set $V := S(e_\lambda)$. Let ϕ be as in Lemma 3.2. Since $\phi \subset V$, we have $S(\phi) \subset S(e_\lambda)$. Conversely, since $e_\lambda \in V^+ = R(\phi) \subset S(\phi)$, we have $S(e_\lambda) \subset S(\phi)$. $\square$

The proof of Theorem 2.3 will require the following lemma.

Lemma 3.3. *Let V be a local FSI space. Suppose $\phi = [\phi_1, \dots, \phi_n] \subset V^+$ is such that $\text{span}\{\phi_1|, \dots, \phi_n|\} = V_1^+$. Then $V^+ = S(\phi)^+ = R(\phi)$.*

Proof. Let $f \in V^+$. We recursively construct a sequence $c : \mathbb{Z}_+ \rightarrow \mathbb{R}^n$ so that

$$f = f_N := \sum_{j=0}^N \phi(\cdot - j)c(j) \text{ on } [0, N+1].$$

This is the so-called “peeling-off argument” from [BD]. Since $f \in V^+$ and ϕ_1 spans V_1^+ , $f = \phi c(0)$ on $[0, 1]$ for some $c(0) \in \mathbb{R}^n$. Now suppose we have $c(0), \dots, c(N)$ such that $f = f_N$ on $[0, N+1]$. Then $(f - f_N)(\cdot + N+1) \in V^+$. So there exists $c(N+1)$ such that $(f - f_N)(\cdot + N+1) = \phi c(N+1)$ on $[0, 1]$. For this value for $c(N+1)$, $f = f_{N+1}$ on $[0, N+2]$. So $V^+ \subset R(\phi) \subset S(\phi)^+$.

Since $\phi \subset V$ and $S(\phi)$ is the smallest closed shift-invariant space containing ϕ , $S(\phi)$ is a subspace of V . This, in turn, implies that $S(\phi)^+ \subset V^+$. $\square$

Proof of Theorem 2.3. Let the columns of λ form a basis for $\Sigma(V)$. We first show that $V^+ = S(e_\lambda)^+$. By Lemma 3.3, it is sufficient to show that $e_\lambda \subset V^+$ since $e_{\lambda|} = \psi\lambda$, which spans V_1^+ .

Let $\phi = [\phi_1, \dots, \phi_n] \subset V^+$ be such that ϕ_1 is a basis for V_1^+ . Then $e_{\lambda|} = \phi_1\gamma$ for some $\gamma \in \mathbb{R}^{n \times n}$. Since $e_\lambda = e_{\lambda}(2\cdot)\beta$, $e_\lambda = e_{\lambda}(2^{-k}\cdot)\beta^{-k} = \phi(2^{-k}\cdot)\gamma\beta^{-k}$ on $[0, 2^k]$. Since V is refinable, $\phi(2^{-k}\cdot)\gamma\beta^{-k} \subset V^+$. And since ϕ_1 is a basis for V_1^+ , it follows that $V^+ = R(\phi)$. So, for each $k \in \mathbb{N}$, there exists a sequence c_k such that

$$e_\lambda = \sum_{j=0}^{2^k} \phi(\cdot - j)c_k(j) \quad \text{on } [0, 2^k].$$

Since ϕ_1 is a basis, the set $\{\phi(\cdot - j)|_{[0, 2^k]} \mid j = 0, 1, \dots, 2^k - 1\}$ is linearly independent. It follows that the sequence $c(j) := c_k(j)$ ($j \in \mathbb{Z}_+$, $2^k > j$) is well-defined and satisfies $e_\lambda = \phi *' c$.

Since V is a local FSI space, it follows that $V = S(\nu)$ for some compactly supported generator ν . Without loss of generality, $\text{supp } \nu \subset [0, \infty)$. Then $\nu \in S(e_\lambda)$ and $e_\lambda \in V$, since $V^+ = S(e_\lambda)^+$. Therefore $V = S(e_\lambda)$. $\square$

Proof of Theorem 2.5. Define

$$h(0) := \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \quad \text{and} \quad h(2j + \varepsilon) := A_\varepsilon h(j) \text{ for } \varepsilon \in \{0, 1\}, 2j + \varepsilon > 0.$$

Then $\mathcal{H}_\Lambda$ is the column space of $[h(0), h(1), h(2), \dots]$. Also (with $\ell(-1) := 0$ for consistency), Eqs. (2.1) give

$$h(j) = \begin{bmatrix} \ell(j-1)\beta_\lambda^j \\ \ell(j)\beta_\lambda^j \end{bmatrix} \text{ for all } j \in \mathbb{Z}_+.$$

Since β_λ is invertible, the column space of $h(j)$ is equal to the column space of $\begin{bmatrix} \ell(j-1) \\ \ell(j) \end{bmatrix}$. Also, $e_\lambda(\cdot + j) = \psi(\cdot + 1)\ell(j-1) + \psi\ell(j)$ on $[-1, 1]$. Hence, $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{H}_\Lambda$ if and only if there exists an $f \in S_\Lambda$ which agrees with $\psi(\cdot + 1)u + \psi v$ on $[-1, 1]$, since ψ is independent. It follows that

$$\mathcal{H}_\Lambda^0 = \{\sigma \in \mathbb{R}^m \mid \psi\sigma = f| \text{ for some } f \in S_\Lambda \text{ s.t. } f|_{[-1,0]} = 0\},$$

making it clear that $\Sigma(S_\Lambda) \subset \mathcal{H}_\Lambda^0$.

Now suppose $\sigma \in \mathcal{H}_\Lambda^0$. Then there exists $f \in S_\Lambda$ such that f vanishes on $[-1, 0]$ and $f| = \psi\sigma$. By Lemma 3.1, there is a $k \in \mathbb{N}$ such that if $g \in S_\Lambda$ vanishes on $[-2^k, 0]$, then $g| \in S_{\Lambda|}^+$. We show $a(0)^k\sigma \in \Sigma(S_\Lambda)$ for this k . First, note that

$$f(2^{-k}\cdot)| = \psi(2^{-k}\cdot)|\sigma = \psi a(0)^k\sigma.$$

Since $f(2^{-k}\cdot)$ vanishes on $[-2^k, 0]$, $a(0)^k\sigma$ is in $\Sigma(S_\Lambda)$. Therefore σ is in $\Sigma(S_\Lambda)$, because $\Sigma(S_\Lambda)$ is preserved by $a(0)$ (it is $a(0)$ -invariant and $a(0)$ is invertible). $\square$

Proof of Corollary 2.6. The equivalence of (ii) and (iii) is an immediate consequence of Theorem 2.5. It is also obvious that (iv) $\implies$ (ii) $\implies$ (i). We prove (i) $\implies$ (ii) and (ii) $\implies$ (iv).

First, let V be a local refinable FSI subspace of $S(\psi)$ such that $\Lambda = \Sigma(V)$. By Theorem 2.3, $V = S_\Lambda$. So $\Lambda = \Sigma(S_\Lambda)$, proving (i) $\implies$ (ii). To prove (ii) $\implies$ (iv), by Lemma 3.3, it is enough to point out that $e_{\lambda|} = \psi\lambda$ is a basis for $S_{\Lambda|}^+ = \psi\Lambda$. $\square$

Proof of Theorem 2.7. Let the columns of λ form a basis for Λ and recall that $e_\lambda = \psi *' \ell$ where ℓ is given by Eqs. (2.1). Then $e_\lambda(\cdot + j)| = \psi\ell(j)$. Let L be the column space of $[\ell(0), \ell(1), \ell(2), \dots]$. Then $S(e_\lambda)| = \text{span } \psi$ if and only if $L = \mathbb{R}^m$. We show that $L = \mathcal{L}_\Lambda$.

Clearly $\Lambda \subset L$, since $\lambda = \ell(0)$. Also, by Eqs. (2.1) and since β is invertible, $a(\varepsilon)L \subset L$ for $\varepsilon = 0, 1$. So $\mathcal{L}_\Lambda \subset L$.

Now, the columns of $\ell(0) = \lambda$ are in $\mathcal{L}_\Lambda$ and, if $\mathcal{L}_\Lambda$ contains the columns of $\ell(m)$, then it must contain the columns of $\ell(2m + \varepsilon)$ for $\varepsilon = 0, 1$. Hence $L \subset \mathcal{L}_\Lambda$. $\square$

Proof of Theorem 2.8. Suppose T satisfies N1 and N2. Define $v := \hat{\psi}(0)$. Let W , $\tilde{v}$, and $(\tilde{a}(0), \tilde{a}(1))$ be as defined before Theorem 2.8. Define $\tilde{T} := \tilde{a}(0) + \tilde{a}(1)$. For any left eigenvector $\tilde{u}$ of $\tilde{T}$, $u := \tilde{u}W$ is a left-eigenvector of T with the same eigenvalue. Moreover, $\tilde{v}\tilde{T} = 2\tilde{v} \neq 0$, since the rows of W are linearly independent and

$$\tilde{v}\tilde{T}W = \tilde{v}WT = vT = 2v = 2\tilde{v}W.$$

That is, $\tilde{T}$ satisfies N1 and N2, so there exists a unique $\tilde{\psi} \in \mathcal{D}'(\mathbb{R})$ satisfying

$$\widehat{\tilde{\psi}}(0) = \tilde{v} \quad \text{and} \quad \tilde{\psi} = \tilde{\psi}(2\cdot)\tilde{a}(0) + \tilde{\psi}(2\cdot - 1)\tilde{a}(1).$$

It follows that $\widehat{\tilde{\psi}W}(0) = v$ and $\tilde{\psi}W$ satisfies Eq. (1.1). So $\tilde{\psi}W = \psi$, proving that $\sigma \in \mathcal{W}^\perp \implies \psi\sigma = 0$. Moreover, since W is of full rank, $\text{span } \psi = \text{span } \tilde{\psi}$.

Now, let the entries of $\phi = [\phi_1, \dots, \phi_j]$ form a basis for $\text{span } \psi$. Then there exists $V \in \mathbb{R}^{j \times m}$ such that $\psi = \phi V$. Let $\mathcal{V}$ be the row space of V , that is, $\mathcal{V} := \{uV \mid u \in \mathbb{R}^{1 \times j}\}$. Evidently, $v \in \mathcal{V}$. We will show that $\mathcal{V}a(\varepsilon) \subset \mathcal{V}$ for $\varepsilon = 0, 1$. Consequently, $\mathcal{W} \subset \mathcal{V}$. So $\psi\sigma = 0 \implies \sigma \in \mathcal{V}^\perp \subset \mathcal{W}^\perp$.

For any $\sigma \in \mathbb{R}^m$,

$$\phi V \sigma = \psi \sigma = \psi(2 \cdot) a(0) \sigma + \psi(2 \cdot - 1) a(1) \sigma = \phi(2 \cdot) V a(0) \sigma + \phi(2 \cdot - 1) V a(1) \sigma.$$

And, since the entries of ϕ are linearly independent, $V \sigma = 0 \implies V a(\varepsilon) \sigma = 0$ for $\varepsilon = 0, 1$. Since $\sigma \in \mathbb{R}^m$ was arbitrary, it follows that $\mathcal{V}a(\varepsilon) \subset \mathcal{V}$ for $\varepsilon = 0, 1$.

Lastly, let $\tilde{\mathcal{W}}$ be the smallest subset of $\mathbb{R}^{1 \times k}$ satisfying $\tilde{v} \in \tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}\tilde{a}(\varepsilon) \subset \tilde{\mathcal{W}}$ ($\varepsilon = 0, 1$). Then $v = \tilde{v}W \in \tilde{\mathcal{W}}W$ and $\tilde{\mathcal{W}}W a(\varepsilon) = \tilde{\mathcal{W}}\tilde{a}(\varepsilon)W \subset \tilde{\mathcal{W}}W$. So $\mathcal{W} \subset \tilde{\mathcal{W}}W$. This, together with the fact that $\dim \mathcal{W} = \text{rank } W = k$, implies that $\tilde{\mathcal{W}} = \mathbb{R}^{1 \times k}$. Since $\tilde{\psi}\sigma = 0 \iff \sigma \in \tilde{\mathcal{W}}^\perp$, ψ is independent. $\square$

4. EXAMPLES

Example 4.1. We present all local refinable FSI spaces of piecewise polynomials with integer breakpoints and prove the list is complete (cf. [GL],[LLS] for related results).

For any $r, m \in \mathbb{Z}$ satisfying $-1 \leq r < m$, the space $\mathcal{S}_r^m$ of all r times continuously differentiable piecewise polynomials of degree at most m with integer breakpoints is defined by

$$\mathcal{S}_r^m := \{f \in C^r(\mathbb{R}) \mid f|_{(j, j+1)} \in \Pi_m \text{ for } j \in \mathbb{Z}\}.$$

Note that $\mathcal{S}_r^m = \sum_{j=r+1}^m \mathcal{S}_{j-1}^j$. In fact, we will show that every local refinable shift-invariant subspace of $\mathcal{S}_{-1}^m$ is of the form

$$\sum_{j \in J} \mathcal{S}_{j-1}^j \text{ for some } J \subset \{0, \dots, m\}.$$

In particular, every local refinable shift-invariant subspace of $\mathcal{S}_{-1}^m$ is a sum of refinable *principal shift-invariant (PSI)* spaces (a subspace V of L_{loc}^1 is a principal shift-invariant space if $V = S(\phi)$ for some single scalar-valued generator ϕ). This is not true of shift-invariant spaces in general. For example, the only refinable PSI subspace of the space generated by $\chi_{[0,1]}$ and $\chi_{[0,1/2]}$ is (the proper subspace) $\mathcal{S}_{-1}^0$.

Define $\psi := [\pi_0, \dots, \pi_m]$, where $\pi_j(x) := x^j$. Then the entries of ψ are linearly independent, $\mathcal{S}_{-1}^m = S(\psi)$, and ψ is refinable with mask $a(0) = d$, $a(1) = cd$, where

$$c := \left[\binom{j}{i} \right]_{i,j=0}^m \quad \text{and} \quad d := \text{diag}(2^{-j})_{j=0}^m.$$

Since $a(0)$ is diagonal with distinct eigenvalues, the eigenvectors are

$$\lambda_0 := [1, 0, \dots, 0], \lambda_1 := [0, 1, 0, \dots, 0], \dots, \lambda_m := [0, \dots, 0, 1],$$

and the $a(0)$ -invariant spaces are $\Lambda_J := \text{span}\{\lambda_j \mid j \in J\}$, $J \subset \{0, \dots, m\}$. It is easy to verify that, for each j , e_{λ_j} is the well-known truncated power function

$$e_{\lambda_j} : x \mapsto x_+^j := (\max\{0, x\})^j$$

and $S(e_{\lambda_j}) = \mathcal{S}_{j-1}^j$. It follows that for any $J \subset \{0, \dots, m\}$,

$$S_{\Lambda_J} = \sum_{j \in J} S(e_{\lambda_j}) = \sum_{j \in J} \mathcal{S}_{j-1}^j.$$

Example 4.2. We consider the case of local dimension $m = 2$ with $a(0)$ invertible in order to illustrate the main results of this paper.

Let $\psi = [\psi_1, \psi_2]$ be a linearly independent generator supported in $[0, 1]$ which is refinable with mask $(a(0), a(1))$. Then $S(\psi)$ must contain all constant functions and we can assume, without loss of generality, that $\psi_1 = \chi_{[0,1]}$ (cf. [H], [J1], [J2]).

It is also assumed that $a(0)$ is invertible.

First, suppose $a(0)$ is diagonalizable, in which case we may assume (by a change of basis for ψ) that $a(0)$ and $a(1)$ are of the form

$$a(0) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & u \\ 0 & t \end{bmatrix},$$

where $s \neq 0$, since $a(0)$ is invertible. Then $T = a(0) + a(1) = \begin{bmatrix} 2 & u \\ 0 & s+t \end{bmatrix}$.

Since ψ is linearly independent, Corollary 2.9 implies $|s+t| < 2$. Then the left 2-eigenspace of T is spanned by $[2-s-t, u]$. If $u = 0$, then the invariant space $\mathcal{W}$ is spanned by $[1, 0]$; and if $s = 1$, then $\mathcal{W}$ is spanned by $[1-t, u]$. In either case, Corollary 2.9 implies that ψ is linearly dependent, hence that $S(\psi) = S(\psi_1)$ which has no proper local refinable FSI subspaces. So we assume $s \neq 0$, $s \neq 1$, and $u \neq 0$ (under these assumptions, $\mathcal{W} = \mathbb{R}^{1 \times 2}$ and ψ is necessarily independent). By rescaling ψ_2 , we may assume that $u = 1$.

There are three possible choices for an $a(0)$ -invariant space Λ :

1. $\Lambda := \text{span}\{\lambda := [1, 0]^T\}$. Then $e_\lambda = \chi_{[0, \infty)}$ and $S_\Lambda = S(\psi_1)$ is the space of piecewise constant polynomials with integer breakpoints.
2. $\Lambda := \mathbb{R}^2$. Then $S_\Lambda = S(\psi)$.
3. (The interesting case) $\Lambda := \text{span}\{\lambda := [0, 1]^T\}$. Begin with $h(0) = [0, 0, 0, 1]^T$, and calculate $h(1) = A_1 h(0)$, $h(2) = A_0 h(1)$, and $h(3) = A_1 h(1)$. The span of $h(0), \dots, h(3)$ is $\{A_0, A_1\}$ -invariant and so equals $\mathcal{H}_\Lambda$. By a simple reduction, we find that $\mathcal{H}_\Lambda$ is also spanned by the four vectors

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ s+t-1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ s \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence $\mathcal{H}_\Lambda^0$ is $\text{span}\{[0, 1]^T, [s+t-1, 0]^T\}$. By Theorem 2.5, Λ is a proper subset of $\Sigma(S_\Lambda)$ whenever $s+t \neq 1$. It follows that $S(\psi)$ and $S(\psi_1)$ are the only local refinable subspaces of $S(\psi)$ when $s+t \neq 1$; but, when $s+t = 1$, there is a third local refinable subspace, $S(e_\lambda)$. Lastly, since $a(1)\lambda = [1, t]^T$, we see that $\mathcal{L}_\Lambda = \mathbb{R}^2$ and so, by Theorem 2.7, $S_{\Lambda|} = \text{span}\{\psi_1, \psi_2\}$ for any values of s and t .

When $a(0)$ is not diagonalizable, we may assume (by a change of basis) that

$$a(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only choices for Λ , in this case, are $\Lambda = \text{span}\{[1, 0]^T\}$, and $\Lambda = \mathbb{R}^2$. So the only

local refinable FSI spaces are $S(\psi_1)$ (which is the space of all piecewise constant polynomials with integer breakpoints) and $S(\psi)$.

It is assumed in this example, as throughout this paper, that ψ_1 and ψ_2 are in $L^1(\mathbb{R})$. A sufficient condition for this is that $|s| + |t| < 1$. In fact, as shown in [JRZ], the subdivision scheme associated with a converges in L^1 in this case.

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