OBLIQUE MULTIWAVELETS IN HILBERT SPACES

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(Communicated by David R. Larson)

ABSTRACT. In this paper, we elucidate the relationship between two consecutive levels of a multiresolution in the general setting of a Hilbert space. We first prove a result on an extendability problem and then derive, as a consequence, characterizations of oblique multiwavelets in a Hilbert space.

1. INTRODUCTION

This is a sequel to our earlier papers [4, 7, 5, 10, 8]. Here, we again consider wavelet-type problems in a general Hilbert space setting, rather than in the usual concrete case of $L^2(\mathbf{R}^d)$, the Hilbert space of square integrable complex-valued functions on $\mathbf{R}^d$.

It is well known that one of the major issues in the study of a multiresolution $\{V_n\}$ of $L^2(\mathbf{R}^d)$ (see e.g. [3, 6]) is the relationship between any two consecutive multiresolution spaces V_n and V_{n+1} (between only V_0 and V_1 in the *stationary* case). This paper elucidates this relationship in the setting of a Hilbert space.

In [1], wavelet-type objects called *oblique multiwavelets* in $L^2(\mathbf{R})$ are introduced. These are generalizations of biorthogonal multiwavelets and it is noted in [1] that they have more flexible properties. Characterizations of oblique multiwavelets in $L^2(\mathbf{R})$ are obtained in [1] and [2], using the machinery of the Fourier transform on $L^2(\mathbf{R})$. This paper extends these characterizations to the general setting of oblique multiwavelets in a Hilbert space H , where the Fourier transform is no longer available and the roles of the translation operator and the dilation operator on $L^2(\mathbf{R})$ are replaced by certain unitary operators on H . Indeed, we first prove a result on a related extendability problem and then obtain, as a consequence, the above results on oblique multiwavelets in a Hilbert space.

Throughout this paper, H denotes a complex Hilbert space. A sequence $\{v_n\}$ in H is a *Riesz basis* for its closed linear span $\overline{\text{span}}\{v_n\}$ if there exist positive constants A and B such that

$$(1.1) \quad A \sum |a_n|^2 \leq \left\| \sum a_n v_n \right\|^2 \leq B \sum |a_n|^2, \quad \forall \{a_n\} \in \ell^2.$$

Two sequences $\{v_n\}$ and $\{\tilde{v}_n\}$ in H are *biorthogonal* if

$$(1.2) \quad \langle v_n, \tilde{v}_m \rangle = \delta_{n,m} \quad \forall n, m,$$

where $\langle x, y \rangle$ denotes the inner product of two vectors x and y in H . (See [11].)

Received by the editors August 24, 1998.

2000 *Mathematics Subject Classification.* Primary 46C99, 47B99, 46B15.

Key words and phrases. Riesz basis, biorthogonal system, oblique projection, multiwavelets.

Let $U = (U_1, \dots, U_d)$ be an ordered d -tuple of distinct unitary operators on a Hilbert space H such that $U_k U_j = U_j U_k$, $k, j = 1, \dots, d$. We shall use the multi-index notation $U^m = U_1^{m_1} \cdots U_d^{m_d}$ for $m = (m_1, \dots, m_d) \in \mathbf{Z}^d$, with the convention that U_j^0 is the identity operator on H , $j = 1, \dots, d$. We also assume that U^m is the identity operator only if $m = 0$.

For a subset S of H , let $\langle S \rangle$ denote the closed linear span of S , and

$$U^{\mathbf{Z}^d}(S) := \{U^n s : n \in \mathbf{Z}^d, s \in S\}.$$

If $V = \{v_1, \dots, v_r\}$ and $W = \{w_1, \dots, w_p\}$ are finite subsets of H such that

$$(1.3) \quad \{\langle v_k, U^n w_j \rangle\}_{n \in \mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d), \quad k = 1, \dots, r, \quad j = 1, \dots, p,$$

then the function $\Phi_{V,W}$ defined almost everywhere on $\mathbf{R}^d$ by

$$(1.4) \quad \Phi_{V,W}(u) := \left(\sum_{n \in \mathbf{Z}^d} \langle v_k, U^n w_j \rangle e^{in \cdot u} \right)_{1 \leq k \leq r, 1 \leq j \leq p}$$

is in $L^2(\mathbf{T}^d)^{r \times p}$, the space of all $r \times p$ matrices with entries in $L^2(\mathbf{T}^d)$ (often identified with $L^2(\mathbf{T}^d, \mathbf{C}^{r \times p})$, where $\mathbf{C}^{r \times p}$ is the space of all $r \times p$ complex matrices equipped with the operator norm, or depending on the context, with the Frobenius norm). For simplicity, we write $L^2(\mathbf{T}^d)^p$ for $L^2(\mathbf{T}^d)^{1 \times p}$.

If V and W are closed linear subspaces of H such that $V \cap W = \{0\}$ and the vector sum $V_1 = V + W$ is closed, then we write $V_1 = V \oplus W$ and call this a *direct sum*. In this case, we define the maps $P_{V//W}$ and $P_{W//V}$ from V_1 to V_1 by

$$(1.5) \quad P_{V//W}(v + w) = v, \quad P_{W//V}(v + w) = w, \quad v \in V, w \in W,$$

and call $P_{V//W}$ the (oblique) projection of V_1 on V along W and $P_{W//V}$ the (oblique) projection of V_1 on W along V . For the special case when $W = V^\perp$, the orthogonal complement of V in V_1 , we write $V_1 = V \oplus^\perp W$ for the *orthogonal* direct sum.

We now give a summary of the contents of this paper. In section 2, we obtain necessary and sufficient conditions for the existence of solutions to an extendability problem in a Hilbert space that is related to multiresolution. Section 3 summarizes some useful results on the connection between $L^2(\mathbf{T}^d)$ and dilation matrices. The final section elucidates the relationship between two consecutive levels of a multiresolution-type structure and derives characterizations of oblique multiwavelets in a Hilbert space.

2. AN EXTENDABILITY PROBLEM

Throughout this section, let $Y = \{y_1, \dots, y_s\}$ and $\tilde{Y} = \{\tilde{y}_1, \dots, \tilde{y}_s\}$ be finite subsets of H such that $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\tilde{Y})$ are biorthogonal Riesz bases for $V_1 = \langle U^{\mathbf{Z}^d}(Y) \rangle = \langle U^{\mathbf{Z}^d}(\tilde{Y}) \rangle$. Moreover, let $X = \{x_1, \dots, x_r\}$ be a finite subset of H , where $r < s$, let $U^{\mathbf{Z}^d}(X)$ be a Riesz basis for its closed linear span V_0 and let

$$(2.1) \quad V_0 \subset V_1.$$

In this case, we have the biorthogonal expansions

$$(2.2) \quad f = \sum_{j=1}^s \sum_{n \in \mathbf{Z}^d} \langle f, U^n \tilde{y}_j \rangle U^n y_j = \sum_{j=1}^s \sum_{n \in \mathbf{Z}^d} \langle f, U^n y_j \rangle U^n \tilde{y}_j, \quad f \in V_1,$$

$$(2.3) \quad \Phi_{Y,\tilde{Y}} = \Phi_{\tilde{Y},Y} = I_s \quad a.e.,$$

where I_s is the $s \times s$ identity matrix, and the entries of the matrix functions $\Phi_{V,Y}$ and $\Phi_{V,\tilde{Y}}$ are L^2 -functions for any finite subset V of H .

Theorem 2.1. *Let $\Gamma = \{z_1, \dots, z_{s-r}\}$ be a subset of $V_1 \setminus V_0$ and let $S = X \cup \Gamma$. The following conditions are equivalent:*

- (i) $U^{\mathbf{Z}^d}(S)$ is a Riesz basis for V_1 .
- (ii) $U^{\mathbf{Z}^d}(\Gamma)$ is a Riesz basis for $W_0 := \langle U^{\mathbf{Z}^d}(\Gamma) \rangle$ and $V_0 \oplus W_0 = V_1$.
- (iii) There exist positive constants C_1 and C_2 such that

$$C_1 \leq \Phi_{S,\tilde{Y}}(u) \Phi_{S,\tilde{Y}}(u)^* \leq C_2 \quad a.e.$$

- (iv) The matrices $\Phi_{S,\tilde{Y}}(u)$ are invertible for almost all u , and the functions $u \rightarrow \|\Phi_{S,\tilde{Y}}(u)\|$ and $u \rightarrow \|\Phi_{S,\tilde{Y}}(u)^{-1}\|$ are essentially bounded.
- (v) The operator $R_0 : L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^{s-r} \rightarrow L^2(\mathbf{T}^d)^s$ defined by

$$(2.4) \quad \begin{aligned} R_0(A, B)(u) &= (A(u) \ B(u)) \begin{pmatrix} \Phi_{X,\tilde{Y}}(u) \\ \Phi_{\Gamma,\tilde{Y}}(u) \end{pmatrix} \\ &= A(u) \Phi_{X,\tilde{Y}}(u) + B(u) \Phi_{\Gamma,\tilde{Y}}(u) \end{aligned}$$

is bounded and invertible.

Proof. (i) $\implies$ (iii) follows from [10, Proposition 3.4].

(iii) $\implies$ (ii): Again by [10, Proposition 3.4], $U^{\mathbf{Z}^d}(S)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(S) \rangle$, which is a subset of V_1 . In particular, the set S is linearly independent, $U^{\mathbf{Z}^d}(\Gamma)$ is a Riesz basis for $W_0 := \langle U^{\mathbf{Z}^d}(\Gamma) \rangle$ and

$$V_0 \oplus W_0 = \langle U^{\mathbf{Z}^d}(S) \rangle \subset V_1 = \langle U^{\mathbf{Z}^d}(Y) \rangle.$$

Since $\#(S) = s = \#(Y)$, by [5, Theorem 2.4], $\langle U^{\mathbf{Z}^d}(S) \rangle = V_1$.

By [10, Theorem 2.1], we have (ii) $\implies$ (i). Finally, (iii) $\iff$ (iv) follows from standard arguments in operator theory, and (iv) $\iff$ (v) follows from [9, pp. 351-352]. $\square$

It is shown in [5, Theorem 2.5] ([7, Corollary 3.4] for the case $d = 1$) that there exist $z_1, \dots, z_{s-r}$ in $V_1 \cap V_0^\perp$ such that condition (ii) in Theorem 2.1 holds. In the more general setting of a pair of biorthogonal multiresolutions, [10, Theorem 3.6] shows that again there exist $z_1, \dots, z_{s-r}$ satisfying Theorem 2.1 (ii) and some other conditions.

Proposition 2.2. *Suppose that the conditions in Theorem 2.1 hold. Let $\tilde{S}$ be a subset of V_1 such that $U^{\mathbf{Z}^d}(\tilde{S})$ is a Riesz basis for V_1 biorthogonal to $U^{\mathbf{Z}^d}(S)$. Write $\tilde{S} = X' \cup \Gamma'$ such that $\#(X') = r$, $\#(\Gamma') = s - r$, $U^{\mathbf{Z}^d}(X')$ biorthogonal to $U^{\mathbf{Z}^d}(X)$, $U^{\mathbf{Z}^d}(\Gamma')$ biorthogonal to $U^{\mathbf{Z}^d}(\Gamma)$, $U^{\mathbf{Z}^d}(X') \perp U^{\mathbf{Z}^d}(\Gamma)$ and $U^{\mathbf{Z}^d}(\Gamma') \perp U^{\mathbf{Z}^d}(X)$. Then $R_0^{-1} : L^2(\mathbf{T}^d)^s \rightarrow L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^{s-r}$ is given by*

$$(2.5) \quad R_0^{-1}(C)(u) = (C(u)\Phi_{X',Y}(u)^*, \ C(u)\Phi_{\Gamma',Y}(u)^*).$$

Proof. By [5, Theorem 2.4], $\Phi_{S,\tilde{Y}}(u)^{-1} = \Phi_{\tilde{S},Y}(u)^* = (\Phi_{X',Y}(u)^* \ \Phi_{\Gamma',Y}(u)^*)$. By Theorem 2.1, $R_0^{-1}(C)(u) = C(u)\Phi_{S,\tilde{Y}}(u)^{-1} = (C(u)\Phi_{X',Y}(u)^* \ C(u)\Phi_{\Gamma',Y}(u)^*)$. $\square$

Remark 2.3. (i) Under the assumptions of Proposition 2.2, let $V'_0 = \langle U^{\mathbf{Z}^d}(X') \rangle$ and $W'_0 = \langle U^{\mathbf{Z}^d}(\Gamma') \rangle$. Then we arrive at the biorthogonal setting

$$V_0 \oplus W_0 = V_1, \quad V'_0 \oplus W'_0 = V_1, \quad V_0 \perp W'_0, \quad V'_0 \perp W_0.$$

Note that V'_0 may not necessarily equal V_0 and W'_0 may not necessarily equal W_0 .

(ii) If moreover $U^{\mathbf{Z}^d}(X) \perp U^{\mathbf{Z}^d}(\Gamma)$, then $V'_0 = V_0$ and $W'_0 = W_0$, and we arrive at the semiorthogonal setting $V_0 \oplus^\perp W_0 = V_1$.

Theorem 2.4. *Suppose that the conditions in Theorem 2.1 hold. Let*

$$\begin{aligned} A_j(u) &= \sum_{n \in \mathbf{Z}^d} a_j(n) e^{in \cdot u}, \quad a_j \in \ell^2(\mathbf{Z}^d), \quad j = 1, \dots, r, \\ B_k(u) &= \sum_{n \in \mathbf{Z}^d} b_k(n) e^{in \cdot u}, \quad b_k \in \ell^2(\mathbf{Z}^d), \quad k = 1, \dots, s-r, \\ C_\ell(u) &= \sum_{n \in \mathbf{Z}^d} c_\ell(n) e^{in \cdot u}, \quad c_\ell \in \ell^2(\mathbf{Z}^d), \quad \ell = 1, \dots, s, \end{aligned}$$

and

$$A = (A_1 \dots A_r), \quad B = (B_1 \dots B_{s-r}), \quad C = (C_1 \dots C_s).$$

The following conditions are equivalent:

$$\begin{aligned} \text{(i)} \quad & \sum_{\ell=1}^s \sum_{n \in \mathbf{Z}^d} c_\ell(n) U^n y_\ell = \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} a_j(n) U^n x_j + \sum_{k=1}^{s-r} \sum_{n \in \mathbf{Z}^d} b_k(n) U^n z_k. \\ \text{(ii)} \quad & \text{(Reconstruction algorithm)} \end{aligned}$$

$$\begin{aligned} \text{(2.6)} \quad C(u) &= R_0(A, B)(u) \\ &= A(u) \Phi_{X, \tilde{Y}}(u) + B(u) \Phi_{\Gamma, \tilde{Y}}(u). \end{aligned}$$

(iii) (Decomposition algorithm)

$$\begin{aligned} \text{(2.7)} \quad (A(u), B(u)) &= R_0^{-1}(C)(u), \text{ i.e.,} \\ A(u) &= C(u) \Phi_{X', Y}(u)^*, \\ B(u) &= C(u) \Phi_{\Gamma', Y}(u)^*. \end{aligned}$$

Proof. (i) $\implies$ (ii): Suppose that (i) holds. Write (i) as $y = x + z$ such that $y \in V_1$, $x \in V_0$ and $z \in W_0$. Then by [10, Proposition 3.3],

$$\begin{aligned} C(u) &= \Phi_{\{y\}, \tilde{Y}}(u) = \Phi_{\{y\}, \tilde{S}}(u) \Phi_{\tilde{Y}, S}(u)^* \\ &= (\Phi_{\{y\}, X'}(u) \Phi_{\{y\}, \Gamma'}(u)) \Phi_{S, \tilde{Y}}(u) \\ &= (\Phi_{\{x\}, X'}(u) \Phi_{\{z\}, \Gamma'}(u)) \Phi_{S, \tilde{Y}}(u) \\ &= (A(u) B(u)) \Phi_{S, \tilde{Y}}(u), \end{aligned}$$

since $z \perp U^{\mathbf{Z}^d}(X')$ and $x \perp U^{\mathbf{Z}^d}(\Gamma')$. Hence (ii) holds.

The proof of (ii) $\iff$ (iii) is obvious. Suppose that (ii) holds. Let

$$x = \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} a_j(n) U^n x_j \quad \text{and} \quad z = \sum_{k=1}^{s-r} \sum_{n \in \mathbf{Z}^d} b_k(n) U^n z_k.$$

Let $y = x + z$ be expressed as $\sum_{\ell=1}^s \sum_{n \in \mathbf{Z}^d} c'_\ell(n) U^n y_\ell$, where $c'_\ell \in \ell^2(\mathbf{Z}^d)$, $\ell = 1, \dots, s$. If $C'_\ell(u) = \sum_{n \in \mathbf{Z}^d} c'_\ell(n) e^{in \cdot u}$, $\ell = 1, \dots, s$, then by the implication (i) $\implies$ (ii)

established above and (ii), $(C'_1(u) \dots C'_s(u)) = (A(u) \ B(u)) \ \Phi_{S, \tilde{Y}}(u) = C(u)$. Hence $c'_\ell = c_\ell$, $\ell = 1, \dots, s$, and so (i) holds. $\square$

Corollary 2.5. *Suppose that the conditions in Theorem 2.1 hold. If*

$$y = \sum_{\ell=1}^s \sum_{n \in \mathbf{Z}^d} c_\ell(n) U^n y_\ell$$

and

$$C(u) = \left(\sum_{n \in \mathbf{Z}^d} c_1(n) e^{in \cdot u} \dots \sum_{n \in \mathbf{Z}^d} c_s(n) e^{in \cdot u} \right),$$

then

$$P_{V_0//W_0}(y) = \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} a_j(n) U^n x_j \quad \text{and} \quad P_{W_0//V_0}(y) = \sum_{k=1}^{s-r} \sum_{n \in \mathbf{Z}^d} b_k(n) U^n z_k,$$

where

$$\left(\sum_{n \in \mathbf{Z}^d} a_1(n) e^{in \cdot u} \dots \sum_{n \in \mathbf{Z}^d} a_r(n) e^{in \cdot u} \right) = C(u) \Phi_{X', Y}(u)^*$$

and

$$\left(\sum_{n \in \mathbf{Z}^d} b_1(n) e^{in \cdot u} \dots \sum_{n \in \mathbf{Z}^d} b_{s-r}(n) e^{in \cdot u} \right) = C(u) \Phi_{\Gamma', Y}(u)^*.$$

3. SOME CONNECTIONS BETWEEN $L^2(\mathbf{T}^d)$ AND DILATION MATRICES

For the rest of this paper, let M be a $d \times d$ matrix with integer entries such that $m = |\det(M)| \geq 2$. Such a matrix is often called a *dilation* matrix. Let M^T be the transpose of M . Let $\mathcal{C}_M = \{\gamma_0, \gamma_1, \dots, \gamma_{m-1}\}$ (respectively $\mathcal{C}_{M^T} = \{\eta_0, \eta_1, \dots, \eta_{m-1}\}$) be a full set of coset representatives of $\mathbf{Z}^d/M\mathbf{Z}^d$ (respectively $\mathbf{Z}^d/M^T\mathbf{Z}^d$), where $\gamma_0 = \eta_0 = 0$. Then $\mathbf{Z}^d$ is the disjoint union of the cosets $M\mathbf{Z}^d + \gamma_j, j = 0, 1, \dots, m-1$ (respectively $M^T\mathbf{Z}^d + \eta_j, j = 0, 1, \dots, m-1$).

It is well known that the following orthogonality relations hold:

$$(3.1) \quad \sum_{k=0}^{m-1} e^{i2\pi(\eta_j - \eta_\ell) \cdot M^{-1}\gamma_k} = m \delta_{j,\ell}, \quad j, \ell = 0, 1, \dots, m-1,$$

and

$$(3.2) \quad \sum_{j=0}^{m-1} e^{i2\pi\eta_j \cdot M^{-1}(\gamma_k - \gamma_\ell)} = m \delta_{k,\ell}, \quad k, \ell = 0, 1, \dots, m-1.$$

Let

$$(3.3) \quad Q(u) = \frac{1}{\sqrt{m}} \left(e^{i\gamma_k \cdot u} e^{i2\pi\eta_\ell \cdot M^{-1}\gamma_k} I_r \right)_{k,\ell=0}^{m-1}, \quad u \in \mathbf{R}^d,$$

where I_r is the $r \times r$ identity matrix. By (3.1) and (3.2), $Q(u)$ is a unitary $mr \times mr$ matrix (indeed the product of two unitary matrices).

Let q be a fixed positive integer and let the map

$$J : \underbrace{L^2(\mathbf{T}^d)^{q \times r} \oplus \dots \oplus L^2(\mathbf{T}^d)^{q \times r}}_{m \text{ times}} \longrightarrow L^2(\mathbf{T}^d)^{q \times r}$$

be defined by

$$\begin{aligned} (3.4) \quad J(G_0, \dots, G_{m-1})(u) &= (G_0(M^T u) \dots G_{m-1}(M^T u)) \Omega(u) \\ &= G_0(M^T u) + G_1(M^T u) e^{i\gamma_1 \cdot u} + \dots + G_{m-1}(M^T u) e^{i\gamma_{m-1} \cdot u} \\ &= \sum_{j=0}^{m-1} \sum_{n \in \mathbf{Z}^d} g_j(n) e^{i(Mn + \gamma_j) \cdot u}, \end{aligned}$$

where $G_j(u) = \sum_{n \in \mathbf{Z}^d} g_j(n) e^{in \cdot u}$, $g_j \in \ell^2(\mathbf{Z}^d)^{q \times r}$, $j = 0, 1, \dots, m-1$, and

$$(3.5) \quad \Omega(u) = \begin{pmatrix} I_r \\ e^{i\gamma_1 \cdot u} I_r \\ \vdots \\ e^{i\gamma_{m-1} \cdot u} I_r \end{pmatrix}, \quad u \in \mathbf{R}^d.$$

Theorem 3.1. *The map J is a unitary operator, and its inverse*

$$J^* : L^2(\mathbf{T}^d)^{q \times r} \longrightarrow \underbrace{L^2(\mathbf{T}^d)^{q \times r} \oplus \dots \oplus L^2(\mathbf{T}^d)^{q \times r}}_{m \text{ times}}$$

is given by $J^(G)(u) = (F_0(u), \dots, F_{m-1}(u))$, where*

$$(3.6) \quad G(u) = \sum_{n \in \mathbf{Z}^d} g(n) e^{in \cdot u}, \quad F_j(u) = \sum_{n \in \mathbf{Z}^d} g(Mn + \gamma_j) e^{in \cdot u}, \quad j = 0, 1, \dots, m-1.$$

Proof. Since

$$\|J(G_0, \dots, G_{m-1})\|^2 = \sum_{j=0}^{m-1} \sum_{n \in \mathbf{Z}^d} \|g_j(n)\|^2 = \sum_{j=0}^{m-1} \|G_j\|^2,$$

the map J is isometric. For any G in $L^2(\mathbf{T}^d)^{q \times r}$,

$$\begin{aligned} G(u) &= \sum_{n \in \mathbf{Z}^d} g(n) e^{in \cdot u} = \sum_{j=0}^{m-1} \sum_{n \in \mathbf{Z}^d} g(Mn + \gamma_j) e^{i(Mn + \gamma_j) \cdot u} \\ &= \sum_{j=0}^{m-1} F_j(M^T u) e^{i\gamma_j \cdot u}, \end{aligned}$$

where F_j , $j = 0, 1, \dots, m-1$, are given by (3.6). By (3.4), $G = J(F_0, \dots, F_{m-1})$. Therefore J is surjective, and the inverse of J has the desired form. $\square$

For any G in $L^2(\mathbf{T}^d)^{q \times r}$, define a function G_{mod} in $L^2(\mathbf{T}^d)^{q \times mr}$ by

$$(3.7) \quad G_{mod}(u) = \frac{1}{\sqrt{m}} (G(u) \ G(u + 2\pi(M^T)^{-1}\eta_1) \ \dots \ G(u + 2\pi(M^T)^{-1}\eta_{m-1})).$$

Remark 3.2. $Q(u) = \Omega_{mod}(u)$, where $\Omega(u)$ is given by (3.5).

We summarize below some properties of G_{mod} which will be needed in the next section. We omit their proofs.

Proposition 3.3. For any G in $L^2(\mathbf{T}^d)^{q \times r}$,

- (i) $G_{mod}(u) = (J^*G)(M^T u) Q(u)$,
- (ii) $J^*(G)(u) = G_{mod}((M^T)^{-1}u) Q((M^T)^{-1}u)^*$,
- (iii) $\frac{1}{m} \sum_{j=0}^{m-1} G(u + 2\pi(M^T)^{-1}\eta_j) = \sum_{n \in \mathbf{Z}^d} g(Mn) e^{iMn \cdot u}$, where

$$G(u) = \sum_{n \in \mathbf{Z}^d} g(n) e^{in \cdot u}.$$

Consider next the special case $q = mr$.

Proposition 3.4. Let $G \in L^2(\mathbf{T}^d)^{mr \times r}$. Suppose that the $mr \times mr$ matrices $G_{mod}(u)$ are invertible for almost all u in $\mathbf{R}^d$ and G_{mod}^{-1} is in $L^2(\mathbf{T}^d)^{mr \times mr}$. Let

$$(3.8) \quad \tilde{G}(u) = \sqrt{m} (G_{mod}(u)^{-1})^* \begin{pmatrix} I_r \\ 0_r \\ \vdots \\ 0_r \end{pmatrix}_{mr \times r}, \quad u \in \mathbf{R}^d.$$

Then $\tilde{G}$ is in $L^2(\mathbf{T}^d)^{mr \times r}$ and $G_{mod}(u)^{-1} = \tilde{G}_{mod}(u)^*$.

4. CHARACTERIZATIONS OF OBLIQUE MULTIWAVELETS

We shall follow the same notations as in sections 1 and 3. As before, let $U = (U_1, \dots, U_d)$ be an ordered d -tuple of distinct commuting unitary operators on a Hilbert space H . Let D be another unitary operator on H such that

$$(4.1) \quad U^n D = D U^{Mn}, \quad n \in \mathbf{Z}^d,$$

where M is a $d \times d$ matrix as in section 3. Then for every n in $\mathbf{Z}^d$, there exist a unique ℓ in $\{0, 1, \dots, m-1\}$ and a unique p in $\mathbf{Z}^d$ such that $n = Mp + \gamma_\ell$. Hence

$$(4.2) \quad D U^n = U^p D U^{\gamma_\ell}.$$

Let $X = \{x_1, \dots, x_r\}$ be a finite subset of H such that

$$(4.3) \quad \{\langle x_j, U^n x_k \rangle\}_{n \in \mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d), \quad j, k = 1, \dots, r.$$

Let $L = \{(\ell, j) : \ell = 0, 1, \dots, m-1, j = 1, \dots, r\}$ with *lexicographical* ordering in (ℓ, j) ,

$$(4.4) \quad y_{\ell, j} = D U^{\gamma_\ell} x_j, \quad (\ell, j) \in L,$$

and let $Y = \{y_{\ell, j} : (\ell, j) \in L\}$. Let

$$(4.5) \quad V_0 = \langle U^{\mathbf{Z}^d}(X) \rangle \quad \text{and} \quad V_1 = \langle U^{\mathbf{Z}^d}(Y) \rangle.$$

For the time being, we do *not* assume that $V_0 \subset V_1$.

By (4.3), the function

$$(4.6) \quad \Phi_{X, X}(u) = \left(\sum_{n \in \mathbf{Z}^d} \langle x_j, U^n x_k \rangle e^{in \cdot u} \right)_{1 \leq j \leq r, 1 \leq k \leq r}$$

is in $L^2(\mathbf{T}^d)^{r \times r}$ and the function

$$(4.7) \quad \Phi_{Y,Y}(u) = \left(\sum_{n \in \mathbf{Z}^d} \langle y_{\ell,j}, U^n y_{p,k} \rangle e^{in \cdot u} \right)_{(\ell,j) \in L, (p,k) \in L}$$

is in $L^2(\mathbf{T}^d)^{mr \times mr}$. Define a function $\Phi_{X,X}^{[m]}$ in $L^2(\mathbf{T}^d)^{mr \times mr}$ by

$$(4.8) \quad \Phi_{X,X}^{[m]}(u) = \begin{pmatrix} \Phi_{X,X}(u) & 0_r & \cdots & 0_r \\ 0_r & \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_r \\ 0_r & \cdots & 0_r & \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_{m-1}) \end{pmatrix}.$$

Theorem 4.1. *We have the following relations between X and Y .*

- (i) $V_1 = D(V_0)$.
- (ii) $\Phi_{Y,Y}(M^T u) = Q(u) \Phi_{X,X}^{[m]}(u) Q(u)^*$.
- (iii) $U^{\mathbf{Z}^d}(X)$ is a Riesz basis for V_0 if and only if $U^{\mathbf{Z}^d}(Y)$ is a Riesz basis for V_1 .

Proof. (i) and (iii) follow easily from (4.2) and the assumption that D is unitary.

By (4.1) and (4.4), for $t, p = 0, 1, \dots, m-1$, $j, k = 1, \dots, r$,

$$\langle y_{t,j}, U^n y_{p,k} \rangle = \langle DU^{\gamma_t} x_j, DU^{Mn+\gamma_p} x_k \rangle = \langle x_j, U^{Mn+\gamma_p-\gamma_t} x_k \rangle.$$

Hence the (t, p) -block of $\Phi_{Y,Y}(M^T u)$ is

$$\left(\sum_{n \in \mathbf{Z}^d} \langle y_{t,j}, U^n y_{p,k} \rangle e^{in \cdot M^T u} \right)_{j,k=1}^r = \sum_{n \in \mathbf{Z}^d} (\langle x_j, U^{Mn+\gamma_p-\gamma_t} x_k \rangle)_{j,k=1}^r e^{iMn \cdot u}.$$

Using (3.3), the (t, p) -block of $Q(u) \Phi_{X,X}^{[m]}(u) Q(u)^*$ is

$$\begin{aligned} (*) &= \frac{1}{m} \sum_{\ell=0}^{m-1} e^{i(\gamma_t \cdot u + 2\pi\eta_\ell \cdot M^{-1}\gamma_t)} \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_\ell) e^{-i(\gamma_p \cdot u + 2\pi\eta_\ell \cdot M^{-1}\gamma_p)} \\ &= \frac{1}{m} \sum_{\ell=0}^{m-1} G(u + 2\pi(M^T)^{-1}\eta_\ell), \end{aligned}$$

where

$$\begin{aligned} G(u) &= \Phi_{X,X}(u) e^{i(\gamma_t - \gamma_p) \cdot u} = \left(\sum_{n \in \mathbf{Z}^d} \langle x_j, U^n x_k \rangle e^{i(n+\gamma_t-\gamma_p) \cdot u} \right)_{j,k=1}^r \\ &= \sum_{n \in \mathbf{Z}^d} (\langle x_j, U^{n+\gamma_p-\gamma_t} x_k \rangle)_{j,k=1}^r e^{in \cdot u}. \end{aligned}$$

Hence by Proposition 3.3,

$$(*) = \sum_{n \in \mathbf{Z}^d} (\langle x_j, U^{Mn+\gamma_p-\gamma_t} x_k \rangle)_{j,k=1}^r e^{iMn \cdot u},$$

which is the (t, p) -block of $\Phi_{Y,Y}(M^T u)$. Hence (ii) holds. $\square$

Henceforth, assume that $U^{\mathbf{Z}^d}(X)$ is a Riesz basis for V_0 so that $U^{\mathbf{Z}^d}(Y)$ is a Riesz basis for $V_1 = D(V_0)$. Let $\tilde{X} = \{\tilde{x}_1, \dots, \tilde{x}_r\}$ be a subset of V_0 such that $U^{\mathbf{Z}^d}(\tilde{X})$ is a Riesz basis for V_0 biorthogonal to $U^{\mathbf{Z}^d}(X)$. Since D is unitary, $\{DU^n x_j : n \in \mathbf{Z}^d, j = 1, \dots, r\}$ and $\{DU^n \tilde{x}_j : n \in \mathbf{Z}^d, j = 1, \dots, r\}$ are biorthogonal Riesz bases for V_1 . Let

$$(4.9) \quad \tilde{y}_{\ell,j} = DU^{\gamma_\ell} \tilde{x}_j, \quad (\ell, j) \in L,$$

and let $\tilde{Y} = \{\tilde{y}_{\ell,j} : (\ell, j) \in L\}$. It is not difficult to show that $U^{\mathbf{Z}^d}(Y)$ and $U^{\mathbf{Z}^d}(\tilde{Y})$ are biorthogonal Riesz bases for V_1 .

Let $W = \{w_1, \dots, w_q\}$ be a subset of V_1 , for some positive integer q . Then

$$(4.10) \quad w_k = \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} g_{k,j}(n) DU^n x_j, \quad k = 1, \dots, q,$$

where $g_{k,j} \in \ell^2(\mathbf{Z}^d)$, $k = 1, \dots, q$, $j = 1, \dots, r$. Let

$$(4.11) \quad g(n) = (g_{k,j}(n))_{1 \leq k \leq q, 1 \leq j \leq r}, \quad n \in \mathbf{Z}^d.$$

Then g is in $\ell^2(\mathbf{Z}^d)^{q \times r}$ and (4.10) can be written in matrix form as

$$(4.12) \quad \begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} g(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix}.$$

Define a function G in $L^2(\mathbf{T}^d)^{q \times r}$ by

$$(4.13) \quad G(u) = \sum_{n \in \mathbf{Z}^d} g(n) e^{in \cdot u}, \quad u \in \mathbf{R}^d.$$

Theorem 4.2. *The function G associated with W has the following properties.*

- (i) $\Phi_{W, \tilde{Y}}(u) = J^*(G)(u)$.
- (ii) $G_{\text{mod}}(u) = \Phi_{W, \tilde{Y}}(M^T u) Q(u)$.
- (iii) $U^{\mathbf{Z}^d}(W)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(W) \rangle$ if and only if there exist positive constants C_1 and C_2 such that

$$C_1 \leq G_{\text{mod}}(u) G_{\text{mod}}(u)^* \leq C_2 \quad \text{a.e.}$$

Proof. (i): By (4.10), (4.2), (4.1) and (4.4), for $k = 1, \dots, q$,

$$\begin{aligned} w_k &= \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} \sum_{\ell=0}^{m-1} g_{k,j}(Mn + \gamma_\ell) DU^{Mn + \gamma_\ell} x_j \\ &= \sum_{\ell=0}^{m-1} \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} g_{k,j}(Mn + \gamma_\ell) U^n y_{\ell,j}. \end{aligned}$$

Therefore for $\ell = 0, 1, \dots, m-1$ and $j = 1, \dots, r$,

$$\sum_{n \in \mathbf{Z}^d} \langle w_k, U^n \tilde{y}_{\ell,j} \rangle e^{in \cdot u} = \sum_{n \in \mathbf{Z}^d} g_{k,j}(Mn + \gamma_\ell) e^{in \cdot u}.$$

Hence by Theorem 3.1,

$$\begin{aligned}\Phi_{W,\tilde{Y}}(u) &= \left(\sum_{n \in \mathbf{Z}^d} g(Mn) e^{in \cdot u} \sum_{n \in \mathbf{Z}^d} g(Mn + \gamma_1) e^{in \cdot u} \dots \sum_{n \in \mathbf{Z}^d} g(Mn + \gamma_{m-1}) e^{in \cdot u} \right) \\ &= J^*(G)(u).\end{aligned}$$

(ii) follows from (i) and Proposition 3.3.

By [10, Proposition 3.4], $U^{\mathbf{Z}^d}(W)$ is a Riesz basis for $\langle U^{\mathbf{Z}^d}(W) \rangle$ if and only if there exist positive constants C_1 and C_2 such that

$$C_1 \leq \Phi_{W,\tilde{Y}}(u) \Phi_{W,\tilde{Y}}(u)^* \leq C_2 \quad a.e.$$

Then (iii) follows from the relation

$$(4.14) \quad G_{mod}(u) G_{mod}(u)^* = \Phi_{W,\tilde{Y}}(u) \Phi_{W,\tilde{Y}}(u)^*,$$

which is obvious from (ii). $\square$

We shall now see the effect of the change of bases for V_1 .

Corollary 4.3. *Let a vector y in V_1 be expressed as*

$$y = \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} a_j(n) DU^n x_j = \sum_{\ell=1}^{m-1} \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} c_{\ell,j}(n) U^n y_{\ell,j},$$

where a_j and $c_{\ell,j}$ are in $\ell^2(\mathbf{Z}^d)$, $j = 1, \dots, r$, $\ell = 0, 1, \dots, m-1$. Let

$$\begin{aligned}A_j(u) &= \sum_{n \in \mathbf{Z}^d} a_j(n) e^{in \cdot u}, \quad j = 1, \dots, r, \\ C_{\ell,j}(u) &= \sum_{n \in \mathbf{Z}^d} c_{\ell,j}(n) e^{in \cdot u}, \quad \ell = 0, 1, \dots, m-1, \quad j = 1, \dots, r,\end{aligned}$$

and

$$A = (A_1 \dots A_r), \quad C = ((C_{0,j})_{j=1,\dots,r} \dots (C_{m-1,j})_{j=1,\dots,r}).$$

Then $C = J^*(A)$.

Proof. By Theorem 4.2(i), $C(u) = \Phi_{\{y\},\tilde{Y}}(u) = J^*(A)(u)$. $\square$

Corollary 4.4. *Let $W = \{w_1, \dots, w_q\}$ and $\Gamma = \{z_1, \dots, z_p\}$ be subsets of V_1 such that $\{\langle w_k, U^n z_j \rangle\}_{n \in \mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d)$, $k = 1, \dots, q$, $j = 1, \dots, p$, and*

$$\begin{pmatrix} w_1 \\ \vdots \\ w_q \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} g(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} f(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix},$$

where g is in $\ell^2(\mathbf{Z}^d)^{q \times r}$ and f is in $\ell^2(\mathbf{Z}^d)^{p \times r}$. Define

$$G(u) = \sum_{n \in \mathbf{Z}^d} g(n) e^{in \cdot u}, \quad F(u) = \sum_{n \in \mathbf{Z}^d} f(n) e^{in \cdot u}, \quad u \in \mathbf{R}^d.$$

Then

$$\begin{aligned}(4.15) \quad \Phi_{W,\Gamma}(M^T u) &= G_{mod}(u) \Phi_{X,X}^{[m]}(u) F_{mod}(u)^* \\ &= \frac{1}{m} \sum_{j=0}^{m-1} G(u + 2\pi(M^T)^{-1}\eta_j) \Phi_{X,X}(u + 2\pi(M^T)^{-1}\eta_j) F(u + 2\pi(M^T)^{-1}\eta_j)^*.\end{aligned}$$

If moreover $U^{\mathbf{Z}^d}(X)$ is orthonormal, then

- (i) $\Phi_{W,\Gamma}(M^T u) = G_{\text{mod}}(u) F_{\text{mod}}(u)^*$;
- (ii) assuming $\{\langle w_k, U^n w_j \rangle\}_{n \in \mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d)$, $k, j = 1, \dots, q$, then $U^{\mathbf{Z}^d}(W)$ is orthonormal if and only if $G_{\text{mod}}(u) G_{\text{mod}}(u)^* = I_q$ a.e.

Proof. By [10, Proposition 3.3], Theorem 4.1 and Theorem 4.2,

$$\begin{aligned} \Phi_{W,\Gamma}(M^T u) &= \Phi_{W,\tilde{Y}}(M^T u) \Phi_{Y,Y}(M^T u) \Phi_{\Gamma,\tilde{Y}}(M^T u)^* \\ &= \Phi_{W,\tilde{Y}}(M^T u) Q(u) \Phi_{X,X}^{[m]}(u) Q(u)^* \Phi_{\Gamma,\tilde{Y}}(M^T u)^* \\ &= G_{\text{mod}}(u) \Phi_{X,X}^{[m]}(u) F_{\text{mod}}(u)^*. \end{aligned}$$

The rest of the assertions are straightforward. $\square$

We now apply our previous results to the special case when $V_0 \subset V_1$. The next theorem, which is a consequence of Theorem 2.1 and Theorem 4.2, gives characterizations of oblique multiwavelets in the general setting of a Hilbert space.

Theorem 4.5. *Assume that*

$$(4.16) \quad V_0 \subset V_1.$$

Let $p = mr - r$, $\Gamma = \{z_1, \dots, z_p\} \subset V_1 \setminus V_0$,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} h(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix}, \quad \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} g(n) \begin{pmatrix} DU^n x_1 \\ \vdots \\ DU^n x_r \end{pmatrix},$$

where h is in $\ell^2(\mathbf{Z}^d)^{r \times r}$ and g is in $\ell^2(\mathbf{Z}^d)^{p \times r}$. Define

$$H(u) = \sum_{n \in \mathbf{Z}^d} h(n) e^{in \cdot u}, \quad G(u) = \sum_{n \in \mathbf{Z}^d} g(n) e^{in \cdot u}, \quad u \in \mathbf{R}^d.$$

Let $S = X \cup \Gamma$ and

$$F(u) = \begin{pmatrix} H(u) \\ G(u) \end{pmatrix}, \quad u \in \mathbf{R}^d.$$

The following conditions are equivalent:

- (i) $U^{\mathbf{Z}^d}(S)$ is a Riesz basis for V_1 .
- (ii) $U^{\mathbf{Z}^d}(\Gamma)$ is a Riesz basis for $W_0 := \langle U^{\mathbf{Z}^d}(\Gamma) \rangle$ and $V_0 \oplus W_0 = V_1$.
- (iii) There exist positive constants C_1 and C_2 such that

$$C_1 \leq F_{\text{mod}}(u) F_{\text{mod}}(u)^* \leq C_2 \quad \text{a.e.}$$

- (iv) The matrices $F_{\text{mod}}(u)$ are invertible for almost all u , and the functions $u \rightarrow \|F_{\text{mod}}(u)\|$ and $u \rightarrow \|F_{\text{mod}}(u)^{-1}\|$ are essentially bounded.
- (v) The operator $R : L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^p \rightarrow L^2(\mathbf{T}^d)^r$ defined by

$$\begin{aligned} (4.17) \quad R(A, B)(u) &= (A(M^T u) \ B(M^T u)) \begin{pmatrix} H(u) \\ G(u) \end{pmatrix} \\ &= A(M^T u) H(u) + B(M^T u) G(u) \end{aligned}$$

is bounded and invertible.

Proof. By Theorem 4.2,

$$(4.18) \quad \Phi_{S, \tilde{Y}}(u) = J^*(F)(u),$$

$$(4.19) \quad F_{mod}(u) = \Phi_{S, \tilde{Y}}(M^T u) Q(u),$$

and

$$(4.20) \quad F_{mod}(u) F_{mod}(u)^* = \Phi_{S, \tilde{Y}}(M^T u) \Phi_{S, \tilde{Y}}(M^T u)^*.$$

Next, by composing the operator $R_0 : L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^p \longrightarrow L^2(\mathbf{T}^d)^{mr}$ defined in (2.4) with the unitary operator $J : L^2(\mathbf{T}^d)^{mr} \longrightarrow L^2(\mathbf{T}^d)^r$ defined in (3.4) with $q = 1$, and using (4.18),

$$\begin{aligned} J R_0(A, B)(u) &= (A(M^T u) \ B(M^T u)) \begin{pmatrix} \Phi_{X, \tilde{Y}}(M^T u) \\ \Phi_{\Gamma, \tilde{Y}}(M^T u) \end{pmatrix} \Omega(u) \\ &= (A(M^T u) \ B(M^T u)) F(u) = R(A, B)(u). \end{aligned}$$

Hence

$$(4.21) \quad R = J R_0.$$

By (4.20), (4.19) and (4.21), the equivalence of (i) to (v) in this theorem then follows easily from that in Theorem 2.1. $\square$

Remark 4.6. Suppose that $U^{\mathbf{Z}^d}(X)$ is orthonormal. Then $U^{\mathbf{Z}^d}(S)$ is orthonormal if and only if $F_{mod}(u)$ are unitary for almost all $u \in \mathbf{R}^d$.

The next two results are analogues of Proposition 2.2 and Theorem 2.4 respectively.

Proposition 4.7. *Suppose that the conditions in Theorem 4.5 hold and $p = mr - r$. Let $\tilde{F} \in L^2(\mathbf{T}^d)^{mr \times r}$ be given as in Proposition 3.4 such that*

$$(4.22) \quad \tilde{F}_{mod}(u)^* = F_{mod}(u)^{-1}.$$

(i) *Then $R^{-1} : L^2(\mathbf{T}^d)^r \longrightarrow L^2(\mathbf{T}^d)^r \oplus L^2(\mathbf{T}^d)^p = L^2(\mathbf{T}^d)^{mr}$ is given by*

$$\begin{aligned} (4.23) \quad R^{-1}(C)(M^T u) &= C_{mod}(u) \tilde{F}_{mod}(u)^* \\ &= \frac{1}{m} \sum_{j=0}^{m-1} C(u + 2\pi(M^T)^{-1}\eta_j) \tilde{F}(u + 2\pi(M^T)^{-1}\eta_j)^*. \end{aligned}$$

(ii) *Write*

$$\tilde{F}(u) = \begin{pmatrix} \tilde{H}(u) \\ \tilde{G}(u) \end{pmatrix}, \text{ where } \tilde{H}(u) = \sum_{n \in \mathbf{Z}^d} \tilde{h}(n) e^{in \cdot u}, \quad \tilde{G}(u) = \sum_{n \in \mathbf{Z}^d} \tilde{g}(n) e^{in \cdot u}, \quad u \in \mathbf{R}^d,$$

$\tilde{h}$ is in $\ell^2(\mathbf{Z}^d)^{r \times r}$ and $\tilde{g}$ is in $\ell^2(\mathbf{Z}^d)^{p \times r}$. Let $X' = \{x'_1, \dots, x'_r\}$ and $\Gamma' = \{z'_1, \dots, z'_p\}$ be subsets of V_1 defined by

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_r \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} \tilde{h}(n) \begin{pmatrix} DU^n \tilde{x}_1 \\ \vdots \\ DU^n \tilde{x}_r \end{pmatrix}, \quad \begin{pmatrix} z'_1 \\ \vdots \\ z'_p \end{pmatrix} = \sum_{n \in \mathbf{Z}^d} \tilde{g}(n) \begin{pmatrix} DU^n \tilde{x}_1 \\ \vdots \\ DU^n \tilde{x}_r \end{pmatrix},$$

and let $\tilde{S} = X' \cup \Gamma'$. Then $U^{\mathbf{Z}^d}(\tilde{S})$ is a Riesz basis for V_1 biorthogonal to $U^{\mathbf{Z}^d}(S)$, $U^{\mathbf{Z}^d}(X')$ biorthogonal to $U^{\mathbf{Z}^d}(X)$, $U^{\mathbf{Z}^d}(\Gamma')$ biorthogonal to $U^{\mathbf{Z}^d}(\Gamma)$, $U^{\mathbf{Z}^d}(X') \perp U^{\mathbf{Z}^d}(\Gamma)$ and $U^{\mathbf{Z}^d}(\Gamma') \perp U^{\mathbf{Z}^d}(X)$.

Proof. (i): Since $R = JR_0$, by Proposition 3.3, (4.19) and (4.22),

$$\begin{aligned} R^{-1}(C)(u) &= R_0^{-1}(J^*C)(u) \\ &= C_{mod}((M^T)^{-1}u) Q((M^T)^{-1}u)^* \Phi_{S,\tilde{Y}}(u)^{-1} \\ &= C_{mod}((M^T)^{-1}u) F_{mod}((M^T)^{-1}u)^{-1} \\ &= C_{mod}((M^T)^{-1}u) \tilde{F}_{mod}((M^T)^{-1}u)^*. \end{aligned}$$

(ii): By Theorem 4.2 and Theorem 4.5 (with X and Y there replaced by $\tilde{X}$ and $\tilde{Y}$, respectively), $U^{\mathbf{Z}^d}(\tilde{S})$ is a Riesz basis for V_1 and

$$(4.24) \quad \tilde{F}_{mod}(u) = \Phi_{\tilde{S},Y}(M^T u) Q(u).$$

By [10, Proposition 3.3], (4.19), (4.24) and (4.22),

$$\Phi_{S,\tilde{S}}(u) = \Phi_{S,\tilde{Y}}(u) \Phi_{\tilde{S},Y}(u)^* = F_{mod}((M^T)^{-1}u) \tilde{F}_{mod}((M^T)^{-1}u)^* = I_{mr} \text{ a.e.}$$

Hence $U^{\mathbf{Z}^d}(\tilde{S})$ is biorthogonal to $U^{\mathbf{Z}^d}(S)$. The rest of the assertions are obvious. $\square$

Theorem 4.8. *Suppose that the conditions in Theorem 4.5 hold. Let $p = mr - r$,*

$$\begin{aligned} A_j(u) &= \sum_{n \in \mathbf{Z}^d} a_j(n) e^{in \cdot u}, \quad a_j \in \ell^2(\mathbf{Z}^d), \quad j = 1, \dots, r, \\ B_k(u) &= \sum_{n \in \mathbf{Z}^d} b_k(n) e^{in \cdot u}, \quad b_k \in \ell^2(\mathbf{Z}^d), \quad k = 1, \dots, p, \\ P_j(u) &= \sum_{n \in \mathbf{Z}^d} p_j(n) e^{in \cdot u}, \quad p_j \in \ell^2(\mathbf{Z}^d), \quad j = 1, \dots, r, \end{aligned}$$

and

$$A = (A_1 \dots A_r), \quad B = (B_1 \dots B_p), \quad P = (P_1 \dots P_r).$$

The following conditions are equivalent:

- (i) $\sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} p_j(n) D U^n x_j = \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} a_j(n) U^n x_j + \sum_{k=1}^p \sum_{n \in \mathbf{Z}^d} b_k(n) U^n z_k.$
- (ii) (Reconstruction algorithm)

$$(4.25) \quad \begin{aligned} P(u) &= R(A, B)(u) \\ &= A(M^T u) H(u) + B(M^T u) G(u). \end{aligned}$$

- (iii) (Decomposition algorithm)

$$(4.26) \quad (A(u), B(u)) = R^{-1}(P)(u)$$

$$= \frac{1}{m} \sum_{j=0}^{m-1} P((M^T)^{-1}(u + 2\pi\eta_j)) \tilde{F}((M^T)^{-1}(u + 2\pi\eta_j))^*,$$

where $\tilde{F}$ is given in (4.22).

Proof. Write

$$\sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} p_j(n) D U^n x_j = \sum_{\ell=1}^{m-1} \sum_{j=1}^r \sum_{n \in \mathbf{Z}^d} c_{\ell,j}(n) U^n y_{\ell,j},$$

where $c_{\ell,j}$ are in $\ell^2(\mathbf{Z}^d)$, $j = 1, \dots, r$, $\ell = 0, 1, \dots, m-1$. Let

$$C_{\ell,j}(u) = \sum_{n \in \mathbf{Z}^d} c_{\ell,j}(n) e^{in \cdot u}, \quad \ell = 0, 1, \dots, m-1, \quad j = 1, \dots, r,$$

and $C = ((C_{0,j})_{j=1,\dots,r} \dots (C_{m-1,j})_{j=1,\dots,r})$. By Corollary 4.3, $C = J^*(P)$. Then by Theorem 2.4,

$$\begin{aligned} \text{(i)} \quad & \iff C = R_0(A, B) \\ & \iff J^*(P) = R_0(A, B) \\ & \iff P = JR_0(A, B) = R(A, B) \\ & \iff (A, B) = R^{-1}(P). \end{aligned}$$

□

Note that we can also obtain formulae for the projections $P_{V_0//W_0}$ and $P_{W_0//V_0}$, as in Corollary 2.5. We omit the details here.

Remark 4.9. (i) The results in this paper can be applied to the special case when $H = L^2(\mathbf{R}^d)$, $(U_k f)(x) = f(x - e_k)$, $k = 1, \dots, d$, and $(Df)(x) = m^{\frac{1}{2}} f(Mx)$, for x in $\mathbf{R}^d$ and f in $L^2(\mathbf{R}^d)$, where $e_k = (\delta_{k,j})_{j=1,\dots,d}$, $k = 1, \dots, d$, M is a $d \times d$ dilation matrix considered in section 3 and $m = |\det(M)| \geq 2$. In this case,

$$(DU^n f)(x) = m^{\frac{1}{2}} f(Mx - n), \quad n \in \mathbf{Z}^d,$$

and if $V = \{f_1, \dots, f_r\}$ and $W = \{g_1, \dots, g_p\}$ are subsets of $L^2(\mathbf{R}^d)$, then

$$\Phi_{V,W}(u) = \left(\sum_{n \in \mathbf{Z}^d} \hat{f}_k(u + 2\pi n) \overline{\hat{g}_j(u + 2\pi n)} \right)_{1 \leq k \leq r, 1 \leq j \leq p},$$

where $\hat{f}$ is the Fourier transform of a function f in $L^2(\mathbf{R}^d)$.

(ii) If $d = 1$, and U and D are unitary operators on a Hilbert space H such that $UD = DU^2$, using the notations in Theorem 4.5, then

$$F_{mod}(u) = \frac{1}{\sqrt{2}} \begin{pmatrix} H(u) & H(u + \pi) \\ G(u) & G(u + \pi) \end{pmatrix}.$$

In particular, Theorem 4.5 generalizes the characterizations of oblique multiwavelets given in [1, Theorem 3.1] and [2, Proposition 3.1, Theorem 3.2, Corollary 3.4], where the setting $H = L^2(\mathbf{R})$, $(Uf)(x) = f(x-1)$ and $(Df)(x) = \sqrt{2}f(2x)$ is considered.

ACKNOWLEDGEMENT

This research was supported by the Wavelets Strategic Research Programme, National University of Singapore, under a grant from the National Science and Technology Board and the Ministry of Education, Republic of Singapore. We would like to thank Professor S. L. Lee for his helpful comments on this paper.

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