INJECTIVE RESOLUTIONS OF NOETHERIAN RINGS AND COGENERATORS

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ABSTRACT. We give new construction of injective resolutions of complexes and bimodules. Applying this construction to an injective resolution of a Noetherian ring, we construct a Σ -embedding cogenerator for the category of modules of projective dimension $\leq n$. Moreover, for a Noetherian projective k -algebra R , we show that R satisfies the Auslander condition if and only if the flat dimension of every R -module M is equal to or larger than the one of the injective hull $E(M)$.

0. INTRODUCTION

Let R be a ring, and let $0 \rightarrow R \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ be a minimal injective resolution of a left free module R of one generator. Bass studied injective resolutions of left Noetherian rings, in particular, of commutative Noetherian rings [B1], [B2]. In the case of commutative Noetherian rings, he showed that for a prime ideal P of R , the injective dimension of R_P is equal to the flat dimension of the injective hull $E(R/P)$ of R/P [B2], [X]. He also showed that R is Gorenstein if and only if E^n is the direct sum of indecomposable injective R -modules $E(R/P)$, where P are prime ideals of height n , for every $n \geq 0$ [B2]. In other words, for every $n \geq 0$, E^n is the direct sum of indecomposable injective R -modules I of which the flat dimensions are equal to n . Xu studied categorical properties of Gorenstein rings [X]. In the case of non-commutative Noetherian rings, Auslander provided the homological condition, which is called the Auslander condition, as one face of non-commutative versions of Gorenstein rings [FGR]. For a non-commutative Noetherian ring R satisfying the Auslander condition, Hoshino showed that every indecomposable injective left R -module I of flat dimension n is a direct summand of E^n [H1]. By using dualities of derived categories, we showed that if the injective dimensions of ${}_R R$ and R_R are finite, then every indecomposable injective left R -module appears in some E^n [Mi]. In this paper, by using a new construction of injective resolutions of complexes, we provide results concerning injective resolutions of Noetherian rings, projective dimension, flat dimension of R -modules. Moreover we study cogenerators for categories of modules of finite flat dimension in order to describe categorical properties of module categories.

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In Section 1, from the point of view of derived categories we give a new construction of injective resolutions of complexes (Lemma 1.1). And we define Σ -embedding cogenerators for additive categories, and study the relation between an injective resolution of a module U and an additive category arising from U (Propositions 1.4, 1.5). In Section 2, we show that $\bigoplus_{i=0}^n E^i$ is a Σ -embedding cogenerator for the category of left R -modules of projective dimension $\leq n$ (Theorem 2.2). As a corollary, we show that if the left injective dimension of R is finite, then $\bigoplus_{i \geq 0} E^i$ is a Σ -embedding cogenerator for the category of left R -modules of finite flat dimension (Corollary 2.3). Moreover, we get another proof of a result [Mi], Corollary 4.7 (Corollary 2.4). In Section 3, we study the case of a projective k -algebra R over a commutative ring k . For a bimodule ${}_A M_B$, we construct an injective resolution of M in $\text{Mod } A \otimes_k B^{op}$ by using an injective resolution of M in $\text{Mod } A$ (Theorem 3.5). As an application of this, we study injective resolutions of flat modules (Proposition 3.8), and we show that $\bigoplus_{i=0}^n E^i$ is an injective cogenerator for the category of left R -modules of flat dimension $\leq n$ (Theorem 3.9). In Section 4, we apply results of Section 3 to projective k -algebras satisfying the Auslander condition. As a consequence, we get the non-commutative version of categorical properties in commutative Gorenstein rings. In particular, in the case of a Noetherian projective k -algebra R over a commutative ring k , we show that R satisfies the Auslander condition if and only if the flat dimension of every R -module M is equal to or larger than the one of the injective hull $E(M)$ (Theorem 4.1).

1. INJECTIVE RESOLUTIONS OF COMPLEXES

Throughout this paper, we assume that all rings have non-zero unity, and that all modules are unital. From the point of view of derived categories, we give the following “The piled resolution lemma”. For a ring R , we denote by $\text{Mod } R$ (resp., $\text{mod } R$, $\text{Inj } R$) the category of left (resp., finitely presented left, injective left) R -modules. Let $\mathcal{A}$ be an additive category, $\mathcal{C}(\mathcal{A})$ the category of complexes of objects in $\mathcal{A}$, $\mathcal{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$, and $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ the full subcategories of $\mathcal{K}(\mathcal{A})$ generated by bounded below complexes, bounded above complexes, bounded complexes, respectively. For an abelian category $\mathcal{A}$, let $\mathcal{D}^*(\mathcal{A})$ be the quotient category of $\mathcal{K}^*(\mathcal{A})$ by the multiplicative system of quasi-isomorphisms, where $*$ = nothing or $+$.

For a complex $X^\bullet := (X^i, d^i)$, we define the following truncation:

$$\tau_{\leq n} X^\bullet : \dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0 \rightarrow \dots$$

For a sequence $\{X_i^\bullet; f_i : X_{i+1}^\bullet \rightarrow X_i^\bullet\}_{i \geq 1}$ of complexes in $\mathcal{K}(\text{Mod } R)$, we have the following distinguished triangle in $\mathcal{K}(\text{Mod } R)$:

$$X^\bullet \rightarrow \prod_i X_i^\bullet \xrightarrow{1\text{-shift}} \prod_i X_i^\bullet \rightarrow .$$

We denote $X^\bullet$ by $\varprojlim X_i^\bullet$, and call it the homotopy limit of the sequence. According to [BN], for a complex $X^\bullet \in \mathcal{K}(\text{Mod } R)$, $X^\bullet$ is isomorphic to $\varprojlim \tau_{\leq n} X^\bullet$ in $\mathcal{D}(\text{Mod } R)$.

Lemma 1.1. *Let R be a ring, and $L^\bullet$ a complex $\dots \rightarrow 0 \rightarrow L^0 \rightarrow L^1 \rightarrow L^2 \rightarrow \dots$ in $\mathcal{K}^+(\text{Mod } R)$. If $0 \rightarrow L^i \rightarrow I_i^0 \rightarrow I_i^1 \rightarrow \dots$ is an injective resolution of L^i ($i \geq 0$), then there is a quasi-isomorphism from $L^\bullet$ to a complex of the following form in*

$K^+(\text{Mod } R) :$

$$\dots \rightarrow 0 \rightarrow I_0^0 \rightarrow \bigoplus_{i=0}^1 I_i^{1-i} \rightarrow \bigoplus_{i=0}^2 I_i^{2-i} \rightarrow \dots \rightarrow \bigoplus_{i=0}^n I_i^{n-i} \rightarrow \dots$$

Proof. Let $I_i^\bullet$ be a complex $I_i^0 \rightarrow I_i^1 \rightarrow \dots \rightarrow I_i^j \rightarrow \dots$ ($i \geq 0$); then L^i has a quasi-isomorphism to $I_i^\bullet$ in $D^+(\text{Mod } R)$. By the inductive step, we construct a complex $V_k^\bullet \in K^+(\text{Inj } R)$ which has a quasi-isomorphism $\tau_{\leq k} L^\bullet \rightarrow V_k^\bullet$ in $K^+(\text{Mod } R)$ as follows. First, we take $V_0^\bullet := I_0^\bullet$. Assume we have a complex $V_{k-1}^\bullet$ which satisfies the above condition. Since $I_k^\bullet[-k+1]$ belongs to $K^+(\text{Inj } R)$, the quasi-isomorphism $\tau_{\leq k-1} L^\bullet \rightarrow V_{k-1}^\bullet$ gives a bijection $\text{Hom}_{K^+(\text{Mod } R)}(V_{k-1}^\bullet, I_k^\bullet[-k+1]) \rightarrow \text{Hom}_{K^+(\text{Mod } R)}(\tau_{\leq k-1} L^\bullet, I_k^\bullet[-k+1])$. Then, for the quasi-isomorphism $L_k \rightarrow I_k^\bullet$ and a distinguished triangle $\tau_{\leq k} L^\bullet \rightarrow \tau_{\leq k-1} L^\bullet \rightarrow L^k[-k+1] \rightarrow$ in $K^+(\text{Mod } R)$, we have the following commutative diagram in $K^+(\text{Mod } R)$:

$$\begin{array}{ccc} \tau_{\leq k-1} L^\bullet & \longrightarrow & L^k[-k+1] \\ \downarrow & & \downarrow \\ V_{k-1}^\bullet & \longrightarrow & I_k^\bullet[-k+1] \end{array}$$

Therefore we can choose $V_{k-1}^\bullet \rightarrow I_k^\bullet[-k+1]$ as a map between complexes. By taking a mapping cone of $V_{k-1}^\bullet \rightarrow I_k^\bullet[-k+1]$, we have the following morphism between distinguished triangles in $K^+(\text{Mod } R)$:

$$\begin{array}{ccccccc} \tau_{\leq k} L^\bullet & \longrightarrow & \tau_{\leq k-1} L^\bullet & \longrightarrow & L^k[-k+1] & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ V_k^\bullet & \longrightarrow & V_{k-1}^\bullet & \longrightarrow & I_k^\bullet[-k+1] & \longrightarrow & \end{array}$$

Since $V_k^\bullet$ is a mapping cone of $V_{k-1}^\bullet \rightarrow I_k^\bullet[-k+1]$, it is clear that $V_k^\bullet$ belongs to $K^+(\text{Inj } R)$. Since $\tau_{\leq k-1} L^\bullet \rightarrow V_{k-1}^\bullet$ and $L^k \rightarrow I_k^\bullet$ are quasi-isomorphisms in $K^+(\text{Mod } R)$, $\tau_{\leq k} L^\bullet \rightarrow V_k^\bullet$ is a quasi-isomorphism in $K^+(\text{Mod } R)$. We have a sequence of quasi-isomorphisms from $\{\tau_{\leq k} L^\bullet; \tau_{\leq k+1} L^\bullet \rightarrow \tau_{\leq k} L^\bullet\}$ to $\{V_k^\bullet; V_{k+1}^\bullet \rightarrow V_k^\bullet\}$, and then there exists a quasi-isomorphism $\alpha : \varprojlim \tau_{\leq k} L^\bullet \rightarrow \varprojlim V_k^\bullet$. On the other hand, by the above construction of $V_k^\bullet$, the morphism $\tau_{\leq k-1} V_k^\bullet \rightarrow \tau_{\leq k-1} V_{k-1}^\bullet$ is the identity in $C(\text{Mod } R)$ for all k , that is, the morphism $V_k^i \rightarrow V_{k-1}^i$ is the identity in $\text{Mod } R$, for all $0 \leq i \leq k-1$. Therefore, there exists the inverse limit $\varprojlim V_k^\bullet$ which belongs to $K^+(\text{Inj } R)$. According to [BN, Remark 2.3], there exists a quasi-isomorphism $\beta : \varprojlim V_k^\bullet \rightarrow \varprojlim V_k^\bullet$. Similarly, there exists a quasi-isomorphism $\gamma : L^\bullet \rightarrow \varprojlim \tau_{\leq k} L^\bullet$. Since $\varprojlim V_k^\bullet$ belongs to $K^+(\text{Inj } R)$, there exists a right inverse $\tilde{\beta}$ of β in $K^+(\text{Mod } R)$. Put $\delta := \tilde{\beta}\alpha\gamma$; then we have the following commutative diagram in $K^+(\text{Mod } R)$:

$$\begin{array}{ccc} L^\bullet & \xrightarrow{\gamma} & \varprojlim \tau_{\leq k} L^\bullet \\ \delta \downarrow & & \downarrow \alpha \\ \varprojlim V_k^\bullet & \xrightarrow{\beta} & \varprojlim V_k^\bullet \end{array}$$

Hence $L^\bullet \rightarrow \varprojlim V_k^\bullet$ is a quasi-isomorphism in $K^+(\text{Mod } R)$. By construction, $\varprojlim V_k^\bullet$ is the following form:

$$\dots \rightarrow 0 \rightarrow I_0^0 \rightarrow \bigoplus_{i=0}^1 I_i^{1-i} \rightarrow \bigoplus_{i=0}^2 I_i^{2-i} \rightarrow \dots \rightarrow \bigoplus_{i=0}^n I_i^{n-i} \rightarrow \dots$$

□

Remark 1.2. In the proof of Lemma 1.1, by direct calculation, it is not hard to see that the above morphism between distinguished triangles can be chosen to become a commutative diagram in $C(\text{Mod } R)$. Then we can directly construct a quasi-isomorphism $L^\bullet \rightarrow \varprojlim V_k^\bullet$.

Corollary 1.3. *Let R be a ring, and $0 \rightarrow Y_n \rightarrow \dots \rightarrow Y_1 \rightarrow Y_0 \rightarrow X \rightarrow 0$ an exact sequence of left R -modules. If $0 \rightarrow Y_i \rightarrow I_i^0 \rightarrow I_i^1 \rightarrow \dots$ is an injective resolution of Y_i ($0 \leq i \leq n$), then X has the following injective resolution:*

$$0 \rightarrow X \rightarrow Q \rightarrow \bigoplus_{i=0}^n I_i^{i+1} \rightarrow \bigoplus_{i=0}^n I_i^{i+2} \rightarrow \dots,$$

where Q is a direct summand of $\bigoplus_{i=0}^n I_i^i$.

Proof. By Lemma 1.1, the complex $\dots \rightarrow 0 \rightarrow Y_n \rightarrow \dots \rightarrow Y_1 \rightarrow Y_0 \rightarrow 0 \rightarrow \dots$ is isomorphic to the following complex in $D^+(\text{Mod } R)$:

$$V^\bullet : \dots \rightarrow 0 \rightarrow V^{-n} \rightarrow V^{-n+1} \rightarrow \dots \rightarrow V^{-1} \xrightarrow{d^{-1}} V^0 \xrightarrow{d^0} V^1 \rightarrow \dots,$$

where V^k is isomorphic to $\bigoplus_{i=0}^{n+k} I_{i-k}^i$ if $0 > k \geq -n$, and is isomorphic to $\bigoplus_{i=0}^n I_i^{i+k}$ if $k \geq 0$. Since V^k is injective ($-n \leq k \leq -1$) and $H^i V^\bullet = 0$ for $i \neq 0$, $\text{Im } d^{-1}$ is injective. By $H^0 V^\bullet \cong X$, $\text{Ker } d^0$ is isomorphic to $X \oplus \text{Im } d^{-1}$. Since the complex $V^0 \rightarrow V^1 \rightarrow \dots$ is an injective resolution of $X \oplus \text{Im } d^{-1}$, we get the statement. □

For a left R -module U , we denote by $\text{Add } U$ (resp., $\text{add } U$) the category of left R -modules which are direct summands of direct sums of copies of U (resp., direct summands of finite direct sums of copies of U), and denote by $\text{Res}^n(U)$ (resp., $\text{res}^n(U)$) the full subcategory of $\text{Mod } R$ (resp., $\text{mod } R$) consisting of left R -modules X which have exact sequences $0 \rightarrow U_n \rightarrow \dots \rightarrow U_1 \rightarrow U_0 \rightarrow X \rightarrow 0$, where $U_i \in \text{Add } U$ (resp., $\text{add } U$) ($0 \leq i \leq n$). We denote the injective dimension of ${}_R U$ (resp., the flat dimension of ${}_R U$) by $\text{idim}_R U$ (resp., $\text{fdim}_R U$). Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full subcategory of $\mathcal{A}$. We call an object $X \in \mathcal{A}$ a Σ -embedding cogenerator (resp., a finitely embedding cogenerator, a cogenerator) for $\mathcal{B}$ provided that every object in $\mathcal{B}$ admits an injection to some direct sum (resp., finite direct sum, direct product) of copies of X in $\mathcal{A}$.

Proposition 1.4. *Let R be a left coherent ring, U a finitely presented left R -module, and $0 \rightarrow U \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots$ an injective resolution of ${}_R U$. Then $\bigoplus_{i=0}^n E^i$ is a finitely embedding cogenerator for $\text{res}^n(U)$.*

Proof. Given $X \in \text{res}^n(U)$, we have the following exact sequence:

$$0 \rightarrow U_n \rightarrow \dots \rightarrow U_1 \rightarrow U_0 \rightarrow X \rightarrow 0,$$

where $U_i \in \text{add } U$ ($n \geq i \geq 0$). It is easy to see that every U_j has the following injective resolution:

$$0 \rightarrow U_j \rightarrow I_j^0 \rightarrow I_j^1 \rightarrow I_j^2 \rightarrow \dots,$$

where $I_j^i \in \text{add}(E^i)$ for all $i \geq 0$. By Corollary 1.3, we get an injection from X to $\bigoplus_{i=0}^n I_i^i$. Since $\bigoplus_{i=0}^n I_i^i$ belongs to $\text{add}(\bigoplus_{i=0}^n E^i)$, we complete the proof. □

Proposition 1.5. *Let R be a left Noetherian ring, U a left R -module, and $0 \rightarrow U \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots$ an injective resolution of ${}_R U$. Then $\bigoplus_{i=0}^n E^i$ is a Σ -embedding cogenerator for $\text{Res}^n(U)$.*

Proof. The same as the proof of Proposition 1.4. $\square$

2. INJECTIVE RESOLUTIONS OF RINGS AND Σ -EMBEDDING COGENERATORS

In and after this section, we fix a minimal injective resolution of a left free module R of one generator:

$$0 \rightarrow {}_R R \rightarrow E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} E^2 \rightarrow \dots$$

Proposition 2.1. *Let R be a left coherent ring. Then $\bigoplus_{i=0}^n E^i$ is a finitely embedding cogenerator for the category of finitely presented left R -modules of projective dimension $\leq n$.*

Proof. By Proposition 1.4. $\square$

Theorem 2.2. *Let R be a left Noetherian ring. Then $\bigoplus_{i=0}^n E^i$ is a Σ -embedding cogenerator for the category of left R -modules of projective dimension $\leq n$. In particular, every injective left R -module of projective dimension $\leq n$ is a direct summand of $\bigoplus_{i=0}^n E^i$.*

Proof. By Proposition 1.5. $\square$

Corollary 2.3. *Let R be a left Noetherian ring. If $\text{idm}({}_R R)$ is finite, then $\bigoplus_{i \geq 0} E^i$ is a Σ -embedding cogenerator for the category of left R -modules of finite flat dimension.*

Proof. According to [H2], Proposition 6, every left R -module of finite flat dimension is of finite projective dimension. Then we complete the proof by Theorem 2.2. $\square$

We get another proof of a result [Mi], Corollary 4.7.

Corollary 2.4. ([Mi, Corollary 4.7]) *Let R be a right coherent and left Noetherian ring. If $\text{idm}({}_R R)$ and $\text{idm}(R_R)$ are at most n , then every indecomposable injective left R -module is a direct summand of $\bigoplus_{i=0}^n E^i$.*

Proof. According to [CE], Chap. VI, Proposition 5.6, for every finitely presented right R -module M and every injective left R -module I , we have the following isomorphism:

$$\begin{aligned} \text{Tor}_i^R(M, I) &\cong \text{Hom}_R(\text{Ext}_R^i(M, R), I) \\ &= 0 \quad \text{for all } i > n. \end{aligned}$$

Therefore, the flat dimension of every indecomposable injective left R -module is at most n . By Corollary 2.3, we get the statement. $\square$

3. INJECTIVE RESOLUTIONS OF PROJECTIVE ALGEBRAS AND COGENERATORS

Let k be a commutative ring. We call a k -algebra R a projective k -algebra if R is projective as a k -module. Let A and B be projective k -algebras. According to [CE], a projective (resp., injective) $A \otimes_k B^{\text{op}}$ -module is projective (resp., injective) as both a right B -module and a left A -module.

Lemma 3.1. *Let R be a left Noetherian ring, $\{Q_\lambda; f_{\mu, \lambda} : Q_\lambda \rightarrow Q_\mu\}_{\lambda \in \Lambda, \lambda \leq \mu}$ a direct system of injective left R -modules. Then $\varinjlim Q_\lambda$ belongs to $\text{Add}(\bigoplus_{\lambda \in \Lambda} Q_\lambda)$.*

Proof. If $\varinjlim Q_\lambda = 0$, then there is nothing to prove. Otherwise, by [CE], Chap. I Ex. 8, and [Ma], $\varinjlim Q_\lambda$ is a direct sum $\bigoplus_{\gamma \in \Gamma} I_\gamma$ of indecomposable injective left R -modules I_γ ($\gamma \in \Gamma$). Given $\gamma \in \Gamma$, we choose a non-zero element x in $I_\gamma \subseteq \bigoplus_{\gamma \in \Gamma} I_\gamma$. Let $\Lambda(x)$ be the set of pairs (y, λ) such that $x = f_\lambda(y)$, where $y \in Q_\lambda$ for some $\lambda \in \Lambda$ and $f_\lambda : Q_\lambda \rightarrow \varinjlim Q_\lambda$ is the structure morphism. Since R is left Noetherian and Λ is a directed set, the set of left annihilator ideals $\text{lann}(y)$ where $(y, \lambda) \in \Lambda(x)$ has a unique maximal left ideal, say, $\text{lann}(y_0)$ where $y_0 \in Q_{\lambda_0}$. Then $\text{lann}(x)$ is equal to $\text{lann}(y_0)$. By [Ma], I_γ is a direct summand of Q_{λ_0} , and hence we complete the proof. $\square$

Remark 3.2. Huisgen-Zimmermann and Smalø get a similar result of Lemma 3.1 in the case of Σ -pure injective modules (see [HS]).

Lemma 3.3. *Let k be a commutative ring, R a left Noetherian projective k -algebra, and $0 \rightarrow R \rightarrow V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow \dots$ an injective resolution of R as left $R \otimes_k R^{op}$ -modules. If F is a flat left R -module, then F has the following injective resolution:*

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots,$$

where I^i belongs to $\text{Add}({}_R V^i)$ for all $i \geq 0$.

Proof. Given a flat left R -module F , we have the following acyclic complex:

$$0 \rightarrow F \rightarrow V^0 \otimes_R F \rightarrow V^1 \otimes_R F \rightarrow V^2 \otimes_R F \rightarrow \dots$$

By [L], Theorem 1.2, there is a direct system $\{P_\lambda; f_{\mu, \lambda} : P_\lambda \rightarrow P_\mu\}_{\lambda \in \Lambda, \lambda \leq \mu}$ of finitely generated free left R -modules such that $\varinjlim P_\lambda$ is isomorphic to F . Then every $V^i \otimes_R F$ is isomorphic to $\varinjlim V^i \otimes_R P_\lambda$. By Lemma 3.1, $V^i \otimes_R F$ is an injective left R -module which belongs to $\text{Add}({}_R V^i)$. Hence the above acyclic complex is an injective resolution of F which satisfies the statement. $\square$

Proposition 3.4. *Let k be a commutative ring, R a left Noetherian projective k -algebra, and $0 \rightarrow R \rightarrow V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow \dots$ an injective resolution of R as left $R \otimes_k R^{op}$ -modules. Then $\bigoplus_{i=0}^n {}_R V^i$ is a Σ -embedding cogenerator for the category of left R -modules of flat dimension $\leq n$. In particular, every injective left R -module of flat dimension $\leq n$ is a direct summand of $\bigoplus_{i=0}^n {}_R V^i$.*

Proof. Let X be a left R -module of flat dimension $\leq n$; then X has the following flat resolution:

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow X \rightarrow 0.$$

By Lemma 3.3, every F_j has the following injective resolution:

$$0 \rightarrow F_j \rightarrow I_j^0 \rightarrow I_j^1 \rightarrow I_j^2 \rightarrow \dots,$$

where $I_j^i \in \text{Add}({}_R V^i)$ for all $i \geq 0$. By Corollary 1.3, we get an injection from X to $\bigoplus_{i=0}^n I_i^i$. Since $\bigoplus_{i=0}^n I_i^i$ belongs to $\text{Add}(\bigoplus_{i=0}^n {}_R V^i)$, we complete the proof. $\square$

Let k be a commutative ring, A a projective k -algebra. We denote by $S_\bullet(A)$ the bar resolution of A , i.e. the complex $(S_n(A), d_{n+1} : S_{n+1}(A) \rightarrow S_n(A))_{n \geq 0}$ such that $S_n(A)$ is the $(n+2)$ -fold tensor product over k of A with itself, and that $d_n(a_0 \otimes \dots \otimes a_{n+1}) = \sum_{0 \leq i \leq n+1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1}$. In this case, $S_\bullet(A)$ is a projective resolution of A in $\text{Mod } A \otimes_k A^{op}$.

Theorem 3.5. *Let k be a commutative ring, A a k -algebra and B a projective k -algebra. For an $(A - B)$ -bimodule M , let $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$ be an injective resolution of M as left A -modules. Then M has the following injective resolution in $\text{Mod } A \otimes_k B^{op}$:*

$$0 \rightarrow M \rightarrow V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow \dots$$

such that $V^n = \bigoplus_{i=0}^n \text{Hom}_k({}_B S_{i-1}(B)_k, {}_A I_k^{n-i})$ for all $n \geq 0$, where $S_{-1}(B) = B$.

Proof. It is easy to see that we have the following exact sequence in $\text{Mod } A \otimes_k B^{op}$:

$$0 \rightarrow \text{Hom}_B({}_B B_B, {}_A M_B) \rightarrow \text{Hom}_B({}_B S_0(B)_B, {}_A M_B) \rightarrow \text{Hom}_B({}_B S_1(B)_B, {}_A M_B) \rightarrow \dots$$

Then ${}_A M_B$ is isomorphic to $\text{Hom}_B({}_B S_{\bullet}(B)_B, {}_A M_B)$ in $D^+(\text{Mod } A \otimes_k B^{op})$. By adjointness, for every $n \geq 0$, we have the following isomorphism in $\text{Mod } A \otimes_k B^{op}$:

$$\text{Hom}_B({}_B S_n(B)_B, {}_A M_B) \cong \text{Hom}_k({}_B S_{n-1}(B)_k, {}_A M_k).$$

Since B is k -projective, by [CE], Chap. IX, Proposition 2.3a, $\text{Hom}_k({}_B S_{n-1}(B)_k, {}_A I_k^i)$ is injective as a left $A \otimes_k B^{op}$ -module for every $n, i \geq 0$. Therefore, for every $n \geq 0$, $\text{Hom}_k({}_B S_{n-1}(B)_k, {}_A M_k)$ has the following injective resolution in $\text{Mod } A \otimes_k B^{op}$:

$$0 \rightarrow \text{Hom}_k({}_B S_{n-1}(B)_k, {}_A M_k) \rightarrow \text{Hom}_k({}_B S_{n-1}(B)_k, {}_A I_k^0) \rightarrow \text{Hom}_k({}_B S_{n-1}(B)_k, {}_A I_k^1) \rightarrow \dots$$

By Lemma 1.1, we complete the proof. $\square$

Remark 3.6. In the proof of Theorem 3.5, we cannot induce an $(A - B)$ -bimodule morphism $\text{Hom}_k({}_B S_{n-1}(B)_k, {}_A M_k) \rightarrow \text{Hom}_k({}_B S_n(B)_k, {}_A M_k)$ from any morphism of $\text{Hom}_B({}_B S_n(B)_k, {}_B S_{n-1}(B)_k)$ in the case of non-commutative algebras. Then the above complex $V^\bullet$ cannot be described by a double complex in general.

For an injective left module I over a ring R , we denote by $\text{Ps}(I)$ the category of left R -modules which are direct summands of direct products of copies of I .

Lemma 3.7. *Let R be a ring, I an injective left R -module.*

(a) *If R is a left Noetherian ring, then $\text{Ps}(I)$ is closed under direct sums.*

(b) *If R is a right coherent ring and $\text{fdim}_R I \leq n$, then every R -module of $\text{Ps}(I)$ is of flat dimension $\leq n$.*

Proof. See [Ma] and [CE], Chap. II, Ex. 2. $\square$

Proposition 3.8. *Let k be a commutative ring, and R a left Noetherian projective k -algebra. Then every flat left R -module F has the following injective resolution:*

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots,$$

where $I^n \in \text{Ps}(\bigoplus_{i=0}^n E^i)$ for all $n \geq 0$.

Proof. By Theorem 3.5, R has the following injective resolution in $R \otimes_k R^{op} - \text{Mod}$:

$$0 \rightarrow R \rightarrow V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow \dots,$$

where $V^n = \bigoplus_{i=0}^n \text{Hom}_k(S_{i-1}(R), E^{n-i})$ for all $n \geq 0$. Since all $S_{i-1}(R)$ are projective k -modules, it is easy to see that every V^n belongs to $\text{Ps}(\bigoplus_{i=0}^n E^i)$. According to Lemmas 3.3 and 3.7, every flat R -module F has the following injective resolution:

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots,$$

where $I^n \in \text{Ps}(\bigoplus_{i=0}^n E^i)$ for all $n \geq 0$. Hence we get the statement. $\square$

Theorem 3.9. *Let k be a commutative ring, and R a left Noetherian projective k -algebra. Then $\bigoplus_{i=0}^n E^i$ is an injective cogenerator for the category of left R -modules of flat dimension $\leq n$.*

Proof. Let X be a left R -module of flat dimension $\leq n$; then X has the following flat resolution :

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow X \rightarrow 0.$$

By Proposition 3.8, every F_j has the following injective resolution:

$$0 \rightarrow F_j \rightarrow I_j^0 \rightarrow I_j^1 \rightarrow I_j^2 \rightarrow \dots,$$

where $I_j^m \in \text{Ps}(\bigoplus_{i=0}^m E^i)$ for all $m \geq 0$. By Corollary 1.3, we get an injection from X to $\bigoplus_{i=0}^n I_i^i$. Since $\bigoplus_{i=0}^n I_i^i$ belongs to $\text{Ps}(\bigoplus_{i=0}^n E^i)$, we complete the proof. $\square$

Corollary 3.10. *Let k be a commutative ring, and R a Noetherian projective k -algebra which is a finitely generated k -module. Then $\bigoplus_{i=0}^n E^i$ is a Σ -embedding cogenerator for the category of left R -modules of flat dimension $\leq n$. In particular, every indecomposable injective left R -module of flat dimension $\leq n$ is a direct summand of $\bigoplus_{i=0}^n E^i$.*

Proof. Since R is a finitely generated k -module, we can replace $\text{Ps}(\bigoplus_{i=0}^m E^i)$ by $\text{Add}(\bigoplus_{i=0}^m E^i)$ in Proposition 3.8. Therefore we can also replace $\text{Ps}(\bigoplus_{i=0}^m E^i)$ by $\text{Add}(\bigoplus_{i=0}^m E^i)$ in the proof of Theorem 3.9. $\square$

4. APPLICATIONS TO THE AUSLANDER CONDITION

Let R be a coherent ring. Auslander defined the Auslander condition for R : for every finitely presented left R -module M and every $i \geq 0$, every submodule N of $\text{Ext}_R^i(M, R)$ satisfies that $\text{Ext}_R^j(N, R) = 0$ for all $0 \leq j < i$. This property implies the right side version of it. Moreover, the Auslander condition is equivalent to the following: $\text{fdim}_R E^i \leq i$ for all $i \geq 0$ (see [FGR] for details). A Noetherian ring R is called Auslander-Gorenstein provided that R satisfies the Auslander condition and that $\text{idm}(R_R)$ and $\text{idm}({}_R R)$ are finite, and is called Auslander regular if R is an Auslander-Gorenstein ring of which global dimension is finite. For an R -module M , we denote by $E(M)$ the injective hull of M .

Theorem 4.1. *Let k be a commutative ring, and R a right coherent and left Noetherian projective k -algebra. Then the following are equivalent.*

- (a) R satisfies the Auslander condition.
- (b) $\text{fdim}_R E(M) \leq \text{fdim}_R M$ for every left R -module M .

Proof. (a) $\Rightarrow$ (b): We may assume $\text{fdim}_R M = n < \infty$. According to Theorem 3.9, M admits an injection to some injective left R -module I in $\text{Ps}(\bigoplus_{i=0}^n E^i)$. By Lemma 3.7, we have $\text{fdim}_R I \leq n$. Since the injective hull $E(M)$ of M is a direct summand of I , we have $\text{fdim}_R E(M) \leq n$.

(b) $\Rightarrow$ (a): If $\text{fdim}_R \text{Im } \delta^{n-1} \leq n$, then $\text{fdim}_R E^{n-1} \leq n$ because of E^{n-1} being the injective hull of $\text{Im } \delta^{n-1}$. Moreover, by dimension shift, we have $\text{fdim}_R \text{Im } \delta^n \leq n + 1$. Hence we complete the proof by induction on n . $\square$

Proposition 4.2. *Let k be a commutative ring, and R a Noetherian projective k -algebra which satisfies the Auslander condition. Then $\bigoplus_{i=0}^n E^i$ is a Σ -embedding cogenerator for the category of left R -modules of flat dimension $\leq n$.*

Proof. By Theorem 3.9, it is sufficient to show that $\text{Ps}(\bigoplus_{i=0}^n E^i) = \text{Add}(\bigoplus_{i=0}^n E^i)$. Clearly, $\text{Ps}(\bigoplus_{i=0}^n E^i)$ contains $\text{Add}(\bigoplus_{i=0}^n E^i)$. Conversely, for every indecomposable injective left R -module I of $\text{Ps}(\bigoplus_{i=0}^n E^i)$, the Auslander condition implies that

$\text{fdim}_R I$ is at most n by Lemma 3.7. According to [H1], Theorem 6.3, I is a direct summand of $\bigoplus_{i=0}^n E^i$, and hence I belongs to $\text{Add}(\bigoplus_{i=0}^n E^i)$. $\square$

Proposition 4.3. *Let k be a commutative ring, and R an Auslander-Gorenstein projective k -algebra of injective dimension n . For a left R -module M , the following are equivalent.*

- (a) $\text{fdim}_R M \leq m$.
- (b) $\text{idim}_R M \leq n$ and M has the following minimal injective resolution:

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots,$$

where $\text{fdim}_R I^i \leq \min(i + m, n)$ for all $i \geq 0$.

Proof. (a) $\Rightarrow$ (b): Let $0 \rightarrow F_m \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ be a flat resolution of M . According to [H2], Proposition 6, every flat left R -module F is of finite projective dimension. Then, by Corollary 1.3, it is easy to see $\text{idim}_R F \leq n$. By Proposition 3.8, it is easy to see that F_i has the following injective resolution:

$$0 \rightarrow F_i \rightarrow I_i^0 \rightarrow I_i^1 \rightarrow \dots \rightarrow I_i^n \rightarrow 0,$$

where $I^j \in \text{Ps}(\bigoplus_{i=0}^j E^i)$ for all $j \geq 0$. Then by Corollary 1.3, M has the following injective resolution:

$$0 \rightarrow M \rightarrow Q^0 \rightarrow Q^1 \rightarrow \dots \rightarrow Q^n \rightarrow 0,$$

where Q^0 is a direct summand of $\bigoplus_{i=0}^{\min(m,n)} I_i^i$, and $Q^j = \bigoplus_{i=0}^{\min(m,n-j)} I_i^{i+j}$ ($j \geq 1$). Then, for every $0 \leq j \leq n$, we have $Q^j \in \text{Ps}(\bigoplus_{i=0}^{\min(j+m,n)} E^i)$. By the definition of Auslander-Gorenstein rings, we have $\text{fdim}_R Q^j \leq \min(j + m, n)$ for all $j \geq 0$. Since every j -th term of a minimal injective resolution of M is a direct summand of Q^j , the condition (b) is satisfied.

(b) $\Rightarrow$ (a): Let $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots \rightarrow I^s \rightarrow 0$ be a minimal injective resolution of M , where $s \leq n$. By dimension shifting, we have $\text{fdim}_R M \leq m$. $\square$

Proposition 4.4. *Let k be a commutative ring, and R an Auslander-Gorenstein projective k -algebra of injective dimension $\leq n + 1$. Then we have $0 \leq \text{fdim}_R M - \text{fdim}_R E(M) \leq n$ for every left R -module M of finite flat dimension.*

Proof. Let M be a left R -module of finite flat dimension. By [H2], Proposition 6, we have $\text{fdim}_R M \leq n + 1$. According to Theorem 4.1, it suffices to consider the case of $\text{fdim}_R M = n + 1$. Let $M \rightarrow E(M)$ be the injective hull of M ; then we have the following short exact sequence:

$$0 \rightarrow M \rightarrow E(M) \rightarrow C \rightarrow 0.$$

If $\text{fdim}_R E(M) = 0$, then $\text{fdim}_R C = n + 2$. By [H2], Proposition 6, this is contradiction. Therefore $\text{fdim}_R E(M) > 0$, and hence we complete the proof. $\square$

Corollary 4.5. *Let k be a commutative ring, and R an Auslander regular projective k -algebra of injective dimension $\leq n + 1$. Then $0 \leq \text{fdim}_R M - \text{fdim}_R E(M) \leq n$ for every left R -module M .*

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