RELATIVE BOUNDEDNESS AND RELATIVE COMPACTNESS FOR LINEAR OPERATORS IN BANACH SPACES

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(Communicated by David R. Larson)

ABSTRACT. If A and B are linear operators acting between Banach spaces, we show that compactness of B relative to A does not in general imply that B has A -bound zero. We do, however, give conditions under which the above implication is valid.

Let X , Y , and Z be Banach spaces, and $A : \mathcal{D}(A) \subset X \rightarrow Y$, $B : \mathcal{D}(B) \subset X \rightarrow Z$ be densely defined linear operators. B is said to be *A-bounded* if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there are nonnegative constants α and β so that

$$(1) \quad \|Bx\| \leq \alpha\|x\| + \beta\|Ax\| \quad \text{for all } x \in \mathcal{D}(A).$$

This standard definition may be found in [1, Ch. IV] together with the following stronger notion: B is *compact relative to A* if for any sequence $x_n \in \mathcal{D}(A)$, such that x_n and Ax_n are bounded, Bx_n must have a convergent subsequence.

We call the greatest lower bound β_0 of all possible β for which (1) holds the *A-bound* of B . It is a folk theorem of the subject (see, e.g., [2, 3, 4] for special cases) that B is compact relative to A implies that B has A -bound zero. Unfortunately this is false in general, as the following demonstrates (see also [1, Example IV.1.8]).

Example 1. Let $X = Y = L_1[0, 1]$, $Z = \mathbb{C}$, $\mathcal{D}(A) = AC[0, 1]$, $Ax = x'$, $Bx = x(0)$.

We note that if $x \in \mathcal{D}(A)$, then $|x|$ attains its average at some $t \in [0, 1]$, i.e., $\int_0^1 |x| = |x(t)|$. Thus

$$(2) \quad \|Bx\| = |x(0)| \leq |x(t)| + |x(0) - x(t)| \leq \|x\| + \|Ax\|,$$

which proves that the A -bound of B is not greater than 1. Moreover (2) also shows that if x_n and Ax_n are bounded, then so is Bx_n and hence B is compact relative to A . On the other hand, the sequence given by $y_n(t) = 1 - tn$ on $[0, n^{-1}]$, $y_n(t) = 0$ on $[n^{-1}, 1]$ satisfies

$$\|y_n\| = (2n)^{-1}, \quad \|Ay_n\| = \|By_n\| = 1,$$

Received by the editors July 24, 1998.

2000 *Mathematics Subject Classification.* Primary 47A55, 47B07.

Key words and phrases. Relatively bounded operators, relatively compact operators.

The first author's research was supported by NSERC of Canada.

The second author acknowledges appointment as a Post Doctoral Fellow of the Pacific Institute for the Mathematical Sciences at the University of Calgary.

so if

$$\|By_n\| \leq \alpha\|y_n\| + \beta\|Ay_n\| \quad \text{for all } n \in \mathbb{N},$$

then $1 \leq \beta$, and thus 1 is in fact the A -bound of B .

Folk theorems are usually valid if one imposes “reasonable” assumptions, and several possibilities are given in the following. Such results are frequently used, e.g., in studying boundary value problems in L_p spaces and deriving eigenvalue asymptotics for corresponding linear differential operators; cf. [2]–[5].

Recall ([1, § III.5.2]) that if A is closed, then by definition its domain $\mathcal{D}(A)$ equipped with the norm $\|x\|_X + \|Ax\|_Y$ becomes a Banach space which we denote by $\mathcal{D}_A$. In this case relative boundedness (compactness) of B is equivalent to boundedness (compactness) of B considered as an operator from $\mathcal{D}_A$ to Z ([1, § IV.1.1–3]).

Theorem 2. *If B is compact relative to A , then B has A -bound zero if either*

- (a) *B is closable, or*
- (b) *A is closed and $\mathcal{D}_A$ is reflexive.*

Remark. Note that for closed A , $\mathcal{D}_A$ is certainly reflexive when X and Y are so. Indeed, by definition $\mathcal{D}_A$ is isometrically isomorphic to the subspace $\Gamma(A) = \{(x, Ax) \mid x \in \mathcal{D}(A)\}$ of the space $X \times Y$. If both X and Y are reflexive and A is closed, then $X \times Y$ is reflexive and $\Gamma(A)$ is a closed subspace, and the claim follows.

Proof. Assume that the statement is not true. Then for some $\beta_0 > 0$ and any $n \in \mathbb{N}$ there exists $x_n \in \mathcal{D}(A)$ such that $\|x_n\| + \|Ax_n\| = 1$ and

$$(3) \quad \|Bx_n\| \geq n\|x_n\| + \beta_0\|Ax_n\|.$$

Since operators compact relative to A are also A -bounded, for some $C > 0$ we have

$$\|Bx_n\| \leq C(\|x_n\| + \|Ax_n\|) = C,$$

which together with (3) shows that $\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Moreover, by definition Bx_n should contain a convergent subsequence, say By_n .

(a) Since B is closable, $y_n \rightarrow 0$, and By_n is convergent, it follows that $By_n \rightarrow 0$, yielding a contradiction with (3) for all n large enough.

(b) Since $\mathcal{D}_A$ is reflexive, y_n has a subsequence z_n weakly convergent in $\mathcal{D}_A$, and it follows that Az_n is weakly convergent in Y . Being closed, the operator A is weakly closed as well so $z_n \rightarrow 0$ in X (weakly) shows that z_n converges weakly to zero in $\mathcal{D}_A$. Then we deduce that $Bz_n \rightarrow 0$ weakly, and since Bz_n is strongly convergent by construction, in fact $Bz_n \rightarrow 0$ strongly, which again contradicts (3). The theorem is proved. $\square$

We mentioned earlier that if A is closed and B is compact relative to A , then B is compact as an operator from $\mathcal{D}_A$ to Z . If, in addition, Z has the *approximation property* (see [6, §VII.3]), then B is a limit in the uniform operator topology of finite-rank operators, i.e., for any given $\varepsilon > 0$ there exists a finite-rank operator¹ $F_\varepsilon \in \mathfrak{B}(\mathcal{D}_A, Z)$ such that

$$\|Bx - F_\varepsilon x\| \leq \varepsilon(\|x\| + \|Ax\|) \quad \text{for all } x \in \mathcal{D}(A).$$

¹ $\mathfrak{B}(X, Y)$ denotes the Banach space of bounded linear transformations from a Banach space X to a Banach space Y .

From this we derive the inequality

$$(4_{\varepsilon}) \quad \|Bx\| \leq \varepsilon(\|x\| + \|Ax\|) + \|F_{\varepsilon}x\|,$$

which is actually of limited applicability since F_{ε} might not even be bounded as a mapping from X to Z . This inconvenience can be overcome in the following situation.

Lemma 3. *Suppose that A is closed and that $\mathcal{D}(A^*)$ is dense in Y^* . If B is compact relative to A and Z has the approximation property, then for any $\varepsilon > 0$ there exists a finite-rank operator $F'_{\varepsilon} \in \mathfrak{B}(X, Z)$ such that, for all $x \in \mathcal{D}(A)$,*

$$(5) \quad \|Bx\| \leq \varepsilon(\|x\| + \|Ax\|) + \|F'_{\varepsilon}x\|.$$

Remark. $\mathcal{D}(A^*)$ is dense in Y^* if, say, A is closable and X and Y are reflexive (see [1, Ch. III.5]).

Proof. The operator $F_{\varepsilon/2}$ from $(4_{\varepsilon/2})$ has the form

$$F_{\varepsilon/2}x = \sum_{k=1}^N (\langle x, f_k \rangle + \langle Ax, g_k \rangle) z_k,$$

where $f_k \in X^*$, $g_k \in Y^*$, and $z_k \in Z$, $k = 1, \dots, N$, are vectors depending on ε but not x and the brackets denote duality between the corresponding spaces. Since $\mathcal{D}(A^*)$ is dense in Y^* , for any $\delta_k > 0$ there exists $g'_k \in \mathcal{D}(A^*)$ such that $\|g_k - g'_k\| \leq \delta_k$. Take $\delta_k := \varepsilon(2^{k+1}\|z_k\|)^{-1}$ and with g'_k chosen as explained above set

$$F'_{\varepsilon}x = \sum_{k=1}^N (\langle x, f_k \rangle + \langle Ax, g'_k \rangle) z_k = \sum_{k=1}^N \langle x, f_k + A^*g'_k \rangle z_k.$$

Then $F'_{\varepsilon} \in \mathfrak{B}(X, Z)$ and

$$\|F_{\varepsilon/2}x - F'_{\varepsilon}x\| \leq \|Ax\| \sum_{k=1}^N \|g_k - g'_k\| \|z_k\| \leq \frac{\varepsilon}{2} \|Ax\|.$$

Combining this with $(4_{\varepsilon/2})$ finishes the proof. $\square$

Corollary 4. *Suppose that A is bounded away from zero, i.e., that $\|Ax\| \geq \delta\|x\|$ for some $\delta > 0$ and all $x \in \mathcal{D}(A)$. Then, under the conditions of Lemma 3, for any $\varepsilon > 0$ there exists a finite-rank operator $F''_{\varepsilon} \in \mathfrak{B}(X, Z)$ such that, for all $x \in \mathcal{D}(A)$,*

$$(6) \quad \|Bx\| \leq \varepsilon\|Ax\| + \|F''_{\varepsilon}x\|.$$

Inequalities (5), (6) are frequently used, e.g., while studying negative spectra of Schrödinger-type operators (see [3, 7]).

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