GROWTH OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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(Communicated by Albert Baernstein II)

ABSTRACT. We consider the differential equation $f'' + A(z)f' + B(z)f = 0$, where $A(z)$ and $B(z)$ are entire functions. Provided $\rho(B) < \rho(A)$ and $T(r, A) \sim \log M(r, A)$ as $r \rightarrow \infty$ outside a set of finite logarithmic measure, we prove that all nonconstant solutions f of this equation are of infinite order.

1. INTRODUCTION

We consider the second order linear differential equation

$$(1.1) \quad f'' + A(z)f' + B(z)f = 0,$$

where $A(z), B(z) \not\equiv 0$ are entire functions and neither of them are polynomials. It is well known that each solution f of the equation (1.1) is an entire function, and if f_1 and f_2 are any two linearly independent solutions of (1.1), then at least one of f_1, f_2 must be of infinite order (see [10]).

In 1988, Gundersen [3] proved the following

Theorem A. *Let $A(z)$ and $B(z)$ be entire functions, where*

- (i) $\rho(A) < \rho(B)$, or
- (ii) $A(z)$ is a polynomial and $B(z)$ is transcendental.

Then every nonconstant solution f of (1.1) has infinite order.

In the same paper, Gundersen also proved

Theorem B. *If f is a nonconstant solution of (1.1), where*

- (i) $\rho(B) < \rho(A) < \frac{1}{2}$, or
- (ii) $A(z)$ is transcendental with $\rho(A) = 0$ and $B(z)$ is a polynomial,

then $\rho(f) = \infty$.

More recently, Hellerstein, Miles and Rossi [8] proved the following

Theorem C. *If $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) \leq \frac{1}{2}$, then any nonconstant solution of (1.1) has infinite order.*

Received by the editors August 18, 1998 and, in revised form, October 27, 1998.

1991 *Mathematics Subject Classification.* Primary 34A20; Secondary 30D05, 30D20, 30D35.

Key words and phrases. Differential equation, entire function, finite order, Nevanlinna characteristic.

This work was supported in part by the Finnish Academy research grant 37701.

In the proofs of Theorem B and Theorem C, the main tool is the classical $\cos \rho\pi$ theorem, which guarantees that the minimum modulus of $A(z)$ on a sequence of circles $|z| = R_n$, $n = 1, 2, \dots$, may be suitably large, if $\rho(A) < \frac{1}{2}$. When $\rho(A) = \frac{1}{2}$, the proof is more complicated. We know that if an entire function has a finite deficient value, then its order is $> \frac{1}{2}$ (see [1]). Thus, a natural question is: If $\rho(B) < \rho(A)$ and $A(z)$ has no finite deficient values, does every nonconstant solution of (1.1) have infinite order? This problem remains open here. In fact, we prove the same conclusion under a related condition on $A(z)$. Namely, a slight modification of the argument due to Hellerstein, Miles and Rossi in [8] implies

Theorem. *If $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) < \infty$ and $T(r, A) \sim \log M(r, A)$ as $r \rightarrow \infty$ outside a set G of finite logarithmic measure, then any nonconstant solution of (1.1) has infinite order.*

The notation $T(r, A) \sim \log M(r, A)$ of course means that

$$\lim_{r \rightarrow \infty} \frac{T(r, A)}{\log M(r, A)} = 1$$

holds outside of G . An entire function $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ is said to have Fejér gaps (see [6] or [9]) if

$$\sum_{n=1}^{\infty} \lambda_n^{-1} < \infty.$$

If an entire function has Fejér gaps, then Murai [9] proved that $f(z)$ can have no finite deficient values. In the course of his proof, Murai showed that

$$T(r, f) \sim \log M(r, f)$$

holds as $r \rightarrow \infty$ outside a set of finite logarithmic measure. Therefore, we obtain

Corollary 1. *If $A(z)$ and $B(z)$ are entire functions with $\rho(B) < \rho(A) < \infty$ and $A(z)$ has Fejér gaps, then any nonconstant solution of (1.1) has infinite order.*

If $\rho(A) = \rho(B)$, the conclusion of our theorem is false in general. For example, if $P(z)$ is a polynomial, then $f(z) = e^{P(z)}$ solves (1.1) for arbitrary $A(z)$ with $B = -P'' - (P')^2 - A(z)P'$. If $\rho(B) < \rho(A)$, but $T(r, A) \sim \log M(r, A)$ does not hold as $r \rightarrow \infty$ outside a set of finite logarithmic measure, then the conclusion of our theorem also fails in general. For example, if $A(z) = e^z$, then $\log M(r, A) = r$ and $T(r, A) = r/\pi$, so that

$$\frac{T(r, A)}{\log M(r, A)} = \frac{1}{\pi}.$$

Now, the equation $f'' + e^z f' - f = 0$ is of the form (1.1), with a solution $f(z) = e^{-z} - 1$ of finite order.

2. SOME KNOWN RESULTS AND LEMMAS

To prove the theorem, we first establish a series of lemmas. To this end, we recall some notions and results. For $E \subset [1, \infty)$, we denote the complement of E in $[1, \infty)$ by CE , i.e. $CE = [1, \infty) \setminus E$. We define the linear measure of E by

$m(E) = \int_1^\infty \chi_E(t) dt$ where χ_E is the characteristic function of E , and the logarithmic measure of E by

$$m_l(E) = \int_1^\infty \frac{\chi_E(t)}{t} dt.$$

The upper and lower logarithmic density of E are defined by

$$\overline{\log \text{dens}} E = \overline{\lim}_{r \rightarrow \infty} \frac{m_l(E \cap [1, r])}{\log r}$$

and

$$\underline{\log \text{dens}} E = \underline{\lim}_{r \rightarrow \infty} \frac{m_l(E \cap [1, r])}{\log r},$$

respectively. We assume familiarity with the notations and fundamental results of the Nevanlinna theory (see [5]). In particular (see [5], p. 13),

$$|T(r, f) - T_0(r, f)| = O(1), \quad r \rightarrow \infty,$$

where

$$T_0(r, f) = \int_0^r \frac{A(t, f)}{t} dt$$

is the Ahlfors–Shimizu characteristic, and $A(t, f)$ is the average number of solutions of $f(z) = a$ in $|z| \leq t$ as a varies over the Riemann sphere. We will denote the order and lower order of f by $\rho(f)$ and $\mu(f)$.

Lemma 1 ([8, p. 697]). *For a nonconstant entire function f , let*

$$\varphi(r) = \frac{1}{2\pi} \int_0^{2\pi} n(r, e^{i\alpha}, f) d\alpha$$

be the mean covering number of the unit circle under the map f restricted to $\{z : |z| \leq r\}$. Then there exists a set E_1 with $m_l(E_1) < \infty$ such that

$$\lim_{\substack{r \rightarrow \infty \\ r \notin E_1}} \frac{\varphi(r)}{A(r, f)} = 1.$$

Lemma 2 ([2, p. 90]). *Let $f(z)$ be a meromorphic function of finite order ρ , and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_2(\varepsilon)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E_2(\varepsilon) \cup [0, 1]$ and for all integers $k > j \geq 0$ in a given finite set of pairs (k, j) , we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 3. *Suppose that $f(z)$ is an entire function of order $\rho < \infty$. Let $n(r) = n(r, \frac{1}{f})$ and suppose $n(1) \geq 1$. For $K > 1$, $h > 0$, let*

$$E_3(K, h) = \{r \geq e : n(r) > Kn(r/e^h)\}.$$

Then

$$(2.1) \quad \underline{\log \text{dens}} CE_3(K, h) \geq 1 - \frac{h\rho}{\log K},$$

and consequently,

$$(2.2) \quad \overline{\log \text{dens}} E_3(K, h) \leq h\rho / \log K.$$

Proof. Fixing $\varepsilon > 0$, there exists $r_0 > e$ such that

$$(2.3) \quad 1 \leq n(r) < r^{\rho+\varepsilon}$$

for all $r > r_0$. Consider now the inequality

$$(2.4) \quad n(t) \leq Kn(te^h).$$

First of all, if there exists $r_1 > r_0$ such that for all $r > r_1$, the inequality (2.4) holds for all $t \in [r_1, r]$, then (2.1) is trivial. So, we may assume, without loss of generality, that for any given $r > r_0$, there exists a value $t \in [r_0, r]$ such that (2.4) does not hold. Let

$$t'_1 = \sup\{t \in [r_0, r] : n(t) > Kn(te^{-h})\}.$$

If $n(t'_1) \geq Kn(t'_1 e^{-h})$, we define $t_1 = t'_1$. Otherwise, we choose t_1 such that $0 < t'_1 - t_1 < \varepsilon$ and

$$n(t_1) > Kn(t_1 e^{-h}).$$

Suppose now, inductively, that t_{m-1} has been defined similarly as for t_1 above. Then let

$$t'_m = \sup\{t \in [r_0, t_{m-1} e^{-h}] : n(t) > Kn(te^{-h})\}.$$

If $n(t'_m) \geq Kn(t'_m e^{-h})$, we define $t_m = t'_m$. Otherwise, we choose t_m such that $0 < t'_m - t_m < \frac{\varepsilon}{2^{m-1}}$ and

$$n(t_m) > Kn(t_m e^{-h}).$$

This inductive process now terminates either if (2.4) is satisfied for all $t \in [r_0, t_{m-1} e^{-h}]$ or if $t_m e^{-h} \leq r_0 < t_m$. From the above inequalities, we have

$$\begin{aligned} r^{\rho+\varepsilon} &\geq n(r) \geq n(t'_1) \geq n(t_1) \geq Kn(t_1 e^{-h}) \geq Kn(t_2) \\ &\geq K^2 n(t_2 e^{-h}) \geq \dots \geq K^{m-1} n(t_m) \geq K^{m-1} n(r_0) \geq K^{m-1}. \end{aligned}$$

Hence

$$(m-1)h < \frac{(\rho+\varepsilon)h}{\log K} \log r,$$

i.e.

$$mh < \frac{(\rho+\varepsilon)h}{\log K} \log r + h.$$

Therefore,

$$\begin{aligned} \int_{CE_3(K,h) \cap [r_0, r]} \frac{dt}{t} &\geq \int_{r_0}^r \frac{dt}{t} - mh - \sum_{j=1}^m \int_{t_j}^{t'_j} \frac{dt}{t} \\ &\geq \log \frac{r}{r_0} - \frac{(\rho+\varepsilon)h}{\log K} \log r - h - \sum_{j=1}^m \log \frac{t'_j}{t_j} \\ &\geq \log \frac{r}{r_0} - \frac{(\rho+\varepsilon)h}{\log K} \log r - h - \sum_{j=1}^m \frac{\varepsilon}{2^{j-1} r_0} \\ &\geq \log \frac{r}{r_0} - \frac{(\rho+\varepsilon)h}{\log K} \log r - h - \frac{2\varepsilon}{r_0}. \end{aligned}$$

Hence,

$$\liminf_{r \rightarrow \infty} \frac{1}{\log r} \int_{CE_3(K,h) \cap [r_0, r]} \frac{dt}{t} \geq 1 - \frac{(\rho + \varepsilon)h}{\log K}.$$

Since $\varepsilon > 0$ is arbitrary, (2.1) and so (2.2) follow by $\varepsilon \rightarrow 0$.

Lemma 4. *Let $T(r) > 1$ be a nonconstant increasing function of finite order ρ in $r \in (1, \infty)$, i.e.*

$$\limsup_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = \rho < \infty.$$

For any ρ_3 such that $0 \leq \rho_3 < \rho$, if $\rho > 0$, and $\rho_3 = 0$, if $\rho = 0$, define

$$E_4(\rho_3) = \{r \geq 1 : r^{\rho_3} < T(r)\}.$$

Then

$$(2.5) \quad \delta := \overline{\log \text{dens}} E_4(\rho_3) > 0.$$

Proof. If $\rho_3 = 0$, then (2.5) holds trivially. Assume now that $0 < \rho_3 < \rho$. If $\overline{\log \text{dens}} E_4(\rho_3) = 0$, then for any given real numbers $\alpha > 1$, and η such that $0 < \eta < \frac{\alpha-1}{\alpha+1}$, there exists $r_0 > 1$ such that

$$\frac{1}{\log r^\alpha} \int_{E_4(\rho_3) \cap [1, r^\alpha]} \frac{dt}{t} < \eta$$

for all $r > r_0$ by the definition of upper logarithmic density. Therefore,

$$(2.6) \quad \int_{E_4(\rho_3) \cap [r, r^\alpha]} \frac{dt}{t} < \alpha \eta \log r$$

for all $r > r_0$. By (2.6),

$$\begin{aligned} (\alpha - 1) \log r &= \int_r^{r^\alpha} \frac{dt}{t} = \int_{E_4(\rho_3) \cap [r, r^\alpha]} \frac{dt}{t} + \int_{CE_4(\rho_3) \cap [r, r^\alpha]} \frac{dt}{t} \\ &< \alpha \eta \log r + \int_{CE_4(\rho_3) \cap [r, r^\alpha]} \frac{dt}{t}. \end{aligned}$$

Since $\eta < \frac{\alpha-1}{\alpha+1}$, a simple computation results in

$$(2.7) \quad \eta \log r < \int_{CE_4(\rho_3) \cap [r, r^\alpha]} \frac{dt}{t}.$$

By (2.7), there exists $t \in CE_4(\rho_3) \cap [r, r^\alpha]$ such that

$$T(r) \leq T(t) \leq t^{\rho_3} \leq r^{\alpha \rho_3}$$

for all $r > r_0$. This implies that $\rho \leq \alpha \rho_3$. Since $\alpha > 1$ is arbitrary, we obtain $\rho \leq \rho_3$ by $\alpha \rightarrow 1$, a contradiction.

Lemma 5. *Suppose that $\varphi_j(t)$, $j = 1, 2$, are nondecreasing, nonnegative functions in $[1, \infty)$ such that for a real constant C ,*

$$\int_1^r \frac{\varphi_1(t)}{t^2} dt \leq \int_1^r \frac{\varphi_2(t)}{t^2} dt + C$$

for all $r > 1$. For any given constant $K > 1$, define

$$E_5(K) = \{t > 1 : \varphi_1(t) \geq K e \varphi_2(t)\}.$$

Then

$$\overline{\log \text{dens}} E_5(K) \leq \frac{1}{K}.$$

Proof. This is an immediate consequence of [7], Theorem 4, and [6], Theorem 3.

Lemma 6. *Let f be a nonconstant entire function, and $K > 1$ a given constant, and define*

$$E_6(K) = \{t \geq 1 : T_0(t, f) > eKA(t, f)\}.$$

Then

$$\overline{\log \text{dens}} E_6(K) \leq \frac{1}{K}.$$

Proof. By partial integration,

$$\int_1^r \frac{T_0(t, f)}{t^2} dt = \int_1^r \frac{A(t, f)}{t^2} dt + T_0(1, f) - \frac{T_0(r, f)}{r} \leq \int_1^r \frac{A(t, f)}{t^2} dt + T_0(1, f).$$

By Lemma 5,

$$\overline{\log \text{dens}} E_6(K) \leq \frac{1}{K}.$$

Making use of the notations of the preceding lemmas, we easily obtain

Lemma 7. *Suppose that $f(z)$ is a nonconstant entire function of finite order ρ , $0 < \rho < \infty$, satisfying*

$$(2.8) \quad \lim_{\substack{r \rightarrow \infty \\ r \notin G}} \frac{T(r, f)}{\log M(r, f)} = 1,$$

where G is a set of finite logarithmic measure. Then, for any given number $K > \max(\frac{16}{7\delta}, e^{16\rho/\delta})$, there exists a sequence $\{R_n\}$, $R_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$(2.9) \quad \lim_{n \rightarrow \infty} \frac{T(R_n, f)}{\log M(R_n, f)} = 1,$$

$$(2.10) \quad n(R_n) \leq Kn(R_n/e),$$

$$(2.11) \quad \lim_{n \rightarrow \infty} \frac{\varphi(R_n)}{A(R_n, f)} = 1,$$

$$(2.12) \quad T_0(R_n, f) \leq eKA(R_n, f),$$

and

$$(2.13) \quad R_n^{\rho_3} < T(R_n, f) \leq \log M(R_n, f).$$

Proof. Let

$$F := E_4(\rho_3) \setminus (E_1 \cup E_2(1) \cup E_3(K, 1) \cup E_6(K) \cup G).$$

Then

$$E_4(\rho_3) \subset F \cup E_1 \cup E_2(1) \cup E_3(K, 1) \cup E_6(K) \cup G,$$

and so

$$\delta < \overline{\log \text{dens}} F + \delta/16 + 7\delta/16.$$

Thus

$$\frac{\delta}{2} < \overline{\log \text{dens}} F.$$

Therefore, there exists a sequence $\{R_n\}$, $R_n \in F$, tending to infinity such that (2.9)–(2.13) hold.

Lemma 8. *Suppose that $f(z)$ is a meromorphic function of finite order ρ and a sequence $\{R_n\}$ has been defined as in Lemma 7. Then, for any given number h , $0 < h < \infty$, we have*

$$T(R_n e^h, f) < e^{h\rho} T(R_n, f)(1 + o(1))$$

as $n \rightarrow \infty$.

Proof. See [11], pp. 40–43.

Lemma 9. *Suppose that $f(z)$ is a nonconstant entire function satisfying the conditions of Lemma 7. Let $\{R_n\}$ be defined as in Lemma 7, let $0 < \rho_2 < \rho_3 < \rho$, and define*

$$E(R_n) := \{ \theta \in [0, 2\pi) : \log |f(R_n e^{i\theta})| \leq R_n^{\rho_2} \}.$$

Then, for any given constant $\varepsilon > 0$, there exists n_0 such that

$$(2.14) \quad m(E(R_n)) \leq \varepsilon$$

for all $n \geq n_0$.

Proof. Since $\delta(\infty, f) = 1$, it follows from (2.9) that

$$\begin{aligned} \left(1 - \frac{\varepsilon}{4\pi}\right) \log M(R_n, f) &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(R_n e^{i\theta})| d\theta \\ &\leq R_n^{\rho_2} + \frac{m(CE(R_n))}{2\pi} \log M(R_n, f), \end{aligned}$$

where $CE(R_n) = [0, 2\pi) \setminus E(R_n)$. It now follows that

$$(2.15) \quad 2\pi \left(1 - \frac{\varepsilon}{4\pi}\right) - \frac{2\pi R_n^{\rho_2}}{\log M(R_n, f)} < m(CE(R_n)).$$

Since $\rho_2 < \rho_3$, $R_n \in E_4(\rho_3)$ and $T(R_n, f) \leq \log M(R_n, f)$, from (2.15) we obtain

$$(2.16) \quad 2\pi \left(1 - \frac{\varepsilon}{2\pi}\right) < m(CE(R_n)),$$

provided that $n \geq n_0$ and n_0 is sufficiently large. Therefore,

$$m(E(R_n)) \leq \varepsilon.$$

3. PROOF OF THE THEOREM

Assume that f is a nonconstant entire solution of finite order $\rho(f)$ of (1.1). If $f(z)$ has finitely many zeros only, then $f(z) = P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials. Substituting this into (1.1) gives

$$(3.1) \quad P'' + 2P'Q' + PQ'' + (Q')^2 P + A(z)(P' + Q'P) + B(z)P = 0.$$

Since $\rho(B) < \rho(A)$, it follows from (3.1) that $P' + Q'P \equiv 0$. Thus $Q' \equiv 0$ and $P' \equiv 0$. Then f is a constant, a contradiction. Therefore, we may assume that

$f(z)$ has infinitely many zeros $\{a_\nu\}$. Now let $\{R_n\}$ be the sequence determined by Lemma 7 with respect to $A(z)$, and proceed to estimate

$$\frac{1}{2\pi} \int_{E(R_n)} \left(Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta.$$

To this end, we apply the differential Poisson–Jensen formula (see [5], p. 22), to obtain

$$\begin{aligned} (3.2) \quad \frac{zf'(z)}{f(z)} &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})| \frac{2zRe^{i\varphi}}{(Re^{i\varphi} - z)^2} d\varphi + \sum_{|a_\nu| \leq r/e} \left(\frac{z}{z - a_\nu} + \frac{\bar{a}_\nu z}{R^2 - \bar{a}_\nu z} \right) \\ &\quad + \sum_{r/e < |a_\nu| \leq r} \left(\frac{z}{z - a_\nu} + \frac{\bar{a}_\nu z}{R^2 - \bar{a}_\nu z} \right) + \sum_{r < |a_\nu| < R} \left(\frac{z}{z - a_\nu} + \frac{\bar{a}_\nu z}{R^2 - \bar{a}_\nu z} \right) \\ &:= f_1 + f_2 + f_3 + f_4, \end{aligned}$$

where $|z| = r < R$; see also [8], pp. 696–697.

To apply (3.2), we choose $R = \frac{R_n e^h}{2}$, $h = 1 + \log 4$. By the first main theorem, Lemma 8 and (2.12), we first obtain

$$|f_1(R_n e^{i\theta})| \leq 18e^{1+h\rho} T_0(R_n, f) \leq 18e^{2+h\rho} K A(R_n, f),$$

so that

$$(3.3) \quad \frac{1}{2\pi} \int_{E(R_n)} |f_1(R_n e^{i\theta})| d\theta \leq \frac{9e^{2+h\rho}}{\pi} K A(R_n, f) m(E(R_n)).$$

By the formula (3.5) in [8], we have

$$(3.4) \quad \frac{1}{2\pi} \int_{E(R_n)} |Re f_2(R_n e^{i\theta})| d\theta \leq \frac{2}{\pi} n(R_n/e) m(E(R_n))$$

and by [8], (3.3) and (3.4),

$$(3.5) \quad \frac{1}{2\pi} \int_{E(R_n)} |Re f_3(R_n e^{i\theta})| d\theta \leq n(R_n) - n(R_n/e).$$

From (3.7) in [8], Lemma 8 and (2.12), we deduce

$$\begin{aligned} (3.6) \quad \frac{1}{2\pi} \int_{E(R_n)} (Re f_4(R_n e^{i\theta}))^+ d\theta &\leq \frac{1}{2\pi} \left(n \left(\frac{R_n e^h}{2} \right) - n(R_n) \right) m(E(R_n)) \\ &\leq \frac{1}{2\pi} n \left(\frac{R_n e^h}{2} \right) m(E(R_n)) \leq \frac{1}{2\pi \log 2} N(R_n e^h) m(E(R_n)) \\ &\leq \frac{1}{2\pi \log 2} T(R_n e^h, f) m(E(R_n)) \leq \frac{e^{h\rho}}{2\pi \log 2} T(R_n, f) m(E(R_n)) \\ &\leq \frac{e^{h\rho}}{\pi \log 2} T_0(R_n, f) m(E(R_n)) \leq \frac{e^{1+h\rho}}{\pi \log 2} K A(R_n, f) m(E(R_n)). \end{aligned}$$

Combining now (3.2)–(3.6), we obtain

$$\begin{aligned} (3.7) \quad \frac{1}{2\pi} \int_{E(R_n)} \left(Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta &\leq \frac{10e^{2+h\rho} K}{\pi} A(R_n, f) m(E(R_n)) \\ &\quad + \frac{2}{\pi} n(R_n) m(E(R_n)) + n(R_n) - n(R_n/e). \end{aligned}$$

As in [8], p. 698, we apply the argument principle to deduce

$$\frac{1}{2\pi} \int_0^{2\pi} n(R_n, e^{i\alpha}, f) d\alpha = \frac{1}{2\pi} \int_{B_{R_n}} \operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} d\theta,$$

where

$$B_{R_n} = \{ \theta \in [0, 2\pi) : |f(R_n e^{i\theta})| > 1 \}.$$

Moreover, by (2.11) and Lemma 1 we conclude

$$\begin{aligned} A(R_n, f) &\leq \frac{1+o(1)}{2\pi} \int_0^{2\pi} \left(\operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ (3.8) \quad &= \frac{1+o(1)}{2\pi} \int_{E(R_n)} \left(\operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ &\quad + \frac{1+o(1)}{2\pi} \int_{CE(R_n)} \left(\operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta. \end{aligned}$$

We now choose ρ_1, ρ_2, ρ_3 such that $\rho(B) < \rho_1 < \rho_2 < \rho_3 < \rho(A)$. If $\theta \in CE(R_n)$, then from (1.1) and Lemma 2 applied to $f(z)$ and Lemma 9 to $A(z)$, we have

$$(3.9) \quad \left| R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right| \leq \frac{e^{R_n^{\rho_1}} + R_n^q}{e^{R_n^{\rho_2}}} \rightarrow 0,$$

as $n \rightarrow \infty$, where q is a constant. It follows from (3.8) and (3.9) that

$$(3.10) \quad A(R_n, f) \leq \frac{2}{2\pi} \int_{E(R_n)} \left(\operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta$$

as n is sufficiently large. By (3.7), (3.10) and (2.10) we deduce that

$$\begin{aligned} (3.11) \quad &\frac{1}{2\pi} \int_{E(R_n)} \left(\operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ &\leq \frac{10e^{2+h\rho}K}{\pi} A(R_n, f) m(E(R_n)) + \frac{2}{\pi} n(R_n) m(E(R_n)) + n(R_n) - \frac{1}{K} n(R_n) \\ &\leq \frac{10e^{2+h\rho}K}{\pi} m(E(R_n)) \frac{1}{\pi} \int_{E(R_n)} \left(\operatorname{Re} R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ &\quad + \frac{2}{\pi} n(R_n) m(E(R_n)) + n(R_n) - \frac{1}{K} n(R_n). \end{aligned}$$

Now fix $0 < \varepsilon < \frac{\pi}{40e^{2+h\rho}K^2}$. By Lemma 9, there exists n_0 such that

$$(3.12) \quad m(E(R_n)) < \varepsilon$$

whenever $n > n_0$. It follows from (3.11) and (3.12) that

$$\begin{aligned} & \frac{1}{2\pi} \int_{E(R_n)} \left(Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ & \leq \frac{10e^{2+h\rho} K \varepsilon}{\pi} \frac{1}{\pi} \int_{E(R_n)} \left(Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ & \quad + \frac{2\varepsilon}{\pi} n(R_n) + n(R_n) - \frac{1}{K} n(R_n) \\ & < \frac{1}{2K} \frac{1}{2\pi} \int_{E(R_n)} \left(Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ & \quad + \frac{1}{20e^{2+h\rho} K^2} n(R_n) + n(R_n) - \frac{1}{K} n(R_n). \end{aligned}$$

Therefore,

$$\begin{aligned} (3.13) \quad & \left(1 - \frac{1}{2K}\right) \frac{1}{2\pi} \int_{E(R_n)} \left(Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} \right)^+ d\theta \\ & < \frac{1}{20e^{2+h\rho} K^2} n(R_n) + n(R_n) - \frac{1}{K} n(R_n). \end{aligned}$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} Re R_n e^{i\theta} \frac{f'(R_n e^{i\theta})}{f(R_n e^{i\theta})} d\theta = n(R_n),$$

it follows from (3.9) and (3.13) that

$$\left(1 - \frac{1+o(1)}{2K}\right) n(R_n) < \frac{1}{20e^{2+h\rho} K^2} n(R_n) + n(R_n) - \frac{1}{K} n(R_n).$$

Hence

$$\frac{1+o(1)}{2} < \frac{1}{20e^{2+h\rho} K}.$$

Taking K as in Lemma 7 large enough, we get a contradiction. The proof of the Theorem is completed.

ACKNOWLEDGEMENT

The authors would like to thank the referee for observations which resulted in a more precise proof of Lemma 3.

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