ON INTERPOLATION FAMILIES OF WAVELET SETS

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ABSTRACT. The question of which groups are isomorphic to groups of interpolation maps for interpolation families of wavelet sets was raised by Dai and Larson. In this article it is shown that any finite group is isomorphic to a group of interpolation maps for some interpolation family of wavelet sets.

A *dyadic orthonormal (or orthogonal) wavelet* is a function $\psi \in L^2(\mathbb{R})$ (Lebesgue measure), with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - l) : n, l \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

For certain measurable sets $E \subseteq \mathbb{R}$ the normalized characteristic function $\frac{1}{\sqrt{2\pi}}\chi_E$ is the Fourier transform of such a wavelet. There are several characterizations of such sets. In [DL] they are called *wavelet sets*, and the corresponding wavelet is called an *s-elementary wavelet*. In [FW],[HWW1],[HWW2] they are the support sets of *MFS (minimal frequency supported) wavelets*.

In [DL], a method of *operator-theoretic interpolation* between certain special pairs of wavelets, and between single wavelets and special families of them is developed. The interpolation pairs of wavelet sets are extensively discussed and the method yields a new construction of a class of wavelets due to Meyer. An interpolation pair is the simplest case of an interpolation family (see definitions below). Affiliated to an interpolation family is a group (under composition) of measure preserving transformations of $\mathbb{R}$ called the interpolation maps. **Problem E** in [DL] asked for a characterization of all groups which are isomorphic to a group of interpolation maps for some interpolation family of wavelet sets. They showed, by example, that the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ of order 2 and 3 could be realized in this way, but they proved no cases of other groups. The purpose of this paper is to prove that every finite group can be realized in this way.

Before introducing the characterization of wavelet sets, we will need some more terminology. In [DL], two measurable sets E, F are called *translation congruent modulo* 2π if there is a measurable bijection $\phi : E \to F$ such that $\phi(s) - s$ is an integral multiple of 2π for each $s \in E$. Analogously, two measurable sets G, H are called *dilation congruent modulo* 2 if there is a measurable bijection $\tau : G \to H$ such that for each $s \in G$ there is an integer n depending on s, such that $\tau(s) = 2^n s$. We note in passing that if E_i and F_i are measurable subsets of $\mathbb{R}$ such that E_i and

F_i are translation congruent modulo 2π for each $i \in \{1, 2, .., n\}$, and $E_i \cap E_j = \emptyset$, $F_i \cap F_j = \emptyset$ for any distinct $i, j \in \{1, 2, .., n\}$, then $\bigcup_{i=1}^{i=n} E_i$ and $\bigcup_{i=1}^{i=n} F_i$ are translation congruent modulo 2π. Likewise, if E_i and F_i are measurable subsets of $\mathbb{R}$ such that E_i and F_i are dilation congruent modulo 2 for each $i \in \{1, 2, .., n\}$, and $E_i \cap E_j = \emptyset$, $F_i \cap F_j = \emptyset$ for any distinct $i, j \in \{1, 2, .., n\}$, then $\bigcup_{i=1}^{i=n} E_i$ and $\bigcup_{i=1}^{i=n} F_i$ are dilation congruent modulo 2.

A measurable subset $G \subseteq \mathbb{R}$ is called a 2-*dilation generator of a partition* of $\mathbb{R}$ if the sets $2^n G := \{2^n s : s \in G\}, n \in \mathbb{Z}$, are disjoint and $\mathbb{R} \setminus \bigcup_n 2^n G$ is a null set. Likewise a measurable subset $H \subseteq \mathbb{R}$ is called a 2π-*translation generator of a partition of* $\mathbb{R}$ if the sets $H + 2n\pi := \{s + 2n\pi : s \in H\}, n \in \mathbb{Z}$, are disjoint and $\mathbb{R} \setminus \bigcup_n (H + 2n\pi)$ is a null set.

We have the following *characterization of wavelet sets* (see Lemma 4.3 in [DL]): Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set if and only if E is both a 2-dilation generator of a partition (modulo null sets) of $\mathbb{R}$ and a 2π-translation generator of a partition (modulo null sets) of $\mathbb{R}$. Equivalently E is a wavelet set if and only if E is both translation congruent to $[0, 2\pi)$ modulo 2π and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ modulo 2.

So the simplest wavelet set is $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

Again, following [DL], let E and F be wavelet sets and let $\sigma : E \to F$ be the 1-1, onto map implementing the 2π-translation congruence. Then we can extend σ to a 1-1 map (modulo a null set) of $\mathbb{R}$ onto $\mathbb{R}$ by defining $\sigma(0) = 0$, and

$$\sigma(s) = 2^n \sigma(2^{-n} s)$$

for $s \in 2^n E, n \in \mathbb{Z}$. We adopt the notation σ_E^F for this map, and call it the *interpolation map* for the ordered pair of wavelet sets (E, F). It is shown in [DL] that σ_E^F is a measure-preserving transformation from $\mathbb{R}$ to $\mathbb{R}$.

A measurable function $f : \mathbb{R} \to \mathbb{R}$ is called *2-homogeneous* if $f(2s) = 2f(s)$ for a.e. $s \in \mathbb{R}$. Clearly, σ_E^F defined above is such a 2-homogeneous function. As observed in [DL], any 2-homogeneous function is completely determined a.e. by its values on any subset of $\mathbb{R}$ which is a 2-dilation generator of a partition of $\mathbb{R}$ (in particular, a wavelet set). So σ_E^F is the unique 2-homogeneous extension of the 2π-translation congruence $E \to F$.

We say that a pair of wavelet sets (E, F) is an *interpolation pair* if $(\sigma_E^F) \circ (\sigma_E^F)$ is the identity map on $\mathbb{R}$ a.e. In general, an *interpolation family of wavelet sets based at E_0* is a family $\mathcal{E}$ of wavelet sets with $E_0 \in \mathcal{E}$ such that $\{\sigma_{E_0}^E : E \in \mathcal{E}\}$ is a *group* (of measure-preserving transformations of $\mathbb{R}$) under composition.

In the construction of interpolation families of wavelet sets discussed below, one theorem from Speegle's paper [S] concerning path-connectedness of the s-elementary wavelets plays an important role.

In [S] measurable maps $\tilde{\tau} : \mathbb{R} \to [-2\pi, -\pi) \cup [\pi, 2\pi)$ and $\tilde{d} : \mathbb{R}_0 = (\mathbb{R} \setminus \{0\}) \to [-2\pi, -\pi) \cup [\pi, 2\pi)$ are defined by $\tilde{\tau}(x) = x + 2\pi m(x)$, $\tilde{d}(x) = 2^{n(x)} x$, where m and n are the unique integers which map x into $[-2\pi, -\pi) \cup [\pi, 2\pi)$ under translation by $2\pi m$ and dilation by 2^n respectively. Thus W is a *wavelet set* if and only if $\tilde{d}|_W$ and $\tilde{\tau}|_W$ are bijections (modulo null sets). If $\tilde{d}|_W$ and $\tilde{\tau}|_W$ are injections modulo null sets, then W is called a *sub-wavelet set*.

For a subwavelet set A we define two subsets of $[-2\pi, -\pi) \cup [\pi, 2\pi)$ by $A^d = \tilde{d}(A)$ and $A^\tau = \tilde{\tau}(A)$.

Theorem 1 ([S])**.** *Let A be a sub-wavelet set. Suppose*

(1) *there is an $\epsilon > 0$ such that $A^\tau \subset [-2\pi + \epsilon, -\pi) \cup [\pi, 2\pi - \epsilon)$ and*
(2) *$S := ([-2\pi, -\pi) \cup [\pi, 2\pi)) \setminus A^d$ has non-empty interior.*

Then, there is a wavelet set $W \supset A$.

Example A.9 in [DL] is an example of an interpolation map σ between two wavelet sets E and F for which σ^3 is the identity map, and also for which σ^2 is a 2π-congruence on the initial wavelet set E. That is, $(\sigma^2(x) - x)/2\pi \in \mathbb{Z}$ for all $x \in E$. In this example

$$A = [-\pi, -\frac{\pi}{2}) \cup [\frac{7\pi}{2}, \frac{31\pi}{8}),$$

$$B = [-\frac{\pi}{4}, -\frac{\pi}{8}) \cup [7\pi, \frac{31\pi}{4})$$

and $E = A \cup C$ and $F = B \cup C$ for a certain measurable set C. Therefore if we denote $\sigma^2(E)$ as G, then it can be proved that G is a wavelet set. Moreover it can be checked that $\{\sigma_E^F, \sigma_E^G, \sigma_E^E\}$ is a group isomorphic to $\mathbb{Z}_3$ and so $\{E, F, G\}$ is an interpolation family of wavelet sets based at E.

The construction in this paper is a generalization of this example.

Theorem 2. *For any $k \in \mathbb{N}$ with $k \geq 2$ there exists an interpolation family $\mathcal{E}$ of wavelet sets based at $E_0 \in \mathcal{E}$ such that $\{\sigma_{E_0}^E : E \in \mathcal{E}\}$ is isomorphic to the symmetric group of degree k.*

We need the following lemmas to prove the theorem. We use $m(\cdot)$ to denote Lebesgue measure.

Lemma 3. *Let $k \geq 2$ be an integer. Let $A = [-\pi, -\frac{\pi}{2}) \cup [\frac{2^k-1}{2}\pi, \frac{2^{2k-1}-1}{2^k}\pi)$, and denote $A_1 = [-\pi, -\frac{\pi}{2})$, and $A_i = [\frac{2^{k+i-2}-1}{2^{i-1}}\pi, \frac{2^{k+i-1}-1}{2^i}\pi)$ for $2 \leq i \leq k$. Then the following hold:*

(i) *$A = \bigcup_{i=1}^k A_i$ and $m(A_i) = \frac{1}{2^i}\pi$ for all $i \in \{1, ..., k\}$. Moreover for all $i \neq j$, with $i, j \in \{1, ..., k\}$, $A_i \cap A_j = \emptyset$.*
(ii) *For all $i \neq j$, with $i, j \in \{1, ..., k\}$, the number $l = 2^{j-i}$ is the unique positive real number such that $m(A_i) = m(lA_j)$. Moreover the sets A_i and lA_j are 2π-translation congruent.*

Proof of Lemma 3. To prove (i), note that we can write $[\frac{2^k-1}{2}\pi, \frac{2^{2k-1}-1}{2^k}\pi)$ as $2^{k-1}\pi + [-\frac{1}{2}\pi, -\frac{1}{2^k}\pi)$. Also we can write A_i as $A_i = 2^{k-1}\pi + [-\frac{1}{2^{i-1}}\pi, -\frac{1}{2^i}\pi)$ for $2 \leq i \leq k$. Now

$$[-\frac{1}{2}\pi, -\frac{1}{2^k}\pi) = \bigcup_{i=2}^k [-\frac{1}{2^{i-1}}\pi, -\frac{1}{2^i}\pi).$$

This implies $A = \bigcup_{i=1}^k A_i$. It is clear that $m(A_i) = \frac{1}{2^i}\pi$ for all $i \in \{1, ..., k\}$.

To see that $A_i \cap A_j = \emptyset$ for any pair of distinct $i, j \in \{1, ..., k\}$, first we note that since A_1 is contained in $(-\infty, 0)$, we have $A_1 \cap A_i = \emptyset$ for any $i \in \{2, ..., k\}$. Next we observe that for any distinct $i, j \in \{2, ..., k\}$,

$$[-\frac{1}{2^{i-1}}\pi, -\frac{1}{2^i}\pi) \cap [-\frac{1}{2^{j-1}}\pi, -\frac{1}{2^j}\pi) = \emptyset.$$

So by the way the A_i's are defined, we have $A_i \cap A_j = \emptyset$ for any distinct pair of $i, j \in \{2, ..., k\}$. Therefore $A_i \cap A_j = \emptyset$ for any distinct pair of $i, j \in \{1, ..., k\}$.

To prove (ii), we first note that from (i) we have $m(A_i) = \frac{1}{2^i}\pi$ for all $i \in \{1, ..., k\}$. Since the A_i's are intervals, for any $l > 0$, $m(lA_i)$ must be $l \cdot \frac{1}{2^i}\pi$. So for fixed $i \neq j$ the positive number l which makes $m(A_i) = m(lA_j)$ must be unique. Now let $l = 2^{j-i}$ and observe that $m(A_i) = m(lA_j)$.

Last of all, in order to see the 2π-translation congruence between A_i and $l \cdot A_j$ we recall that $A_1 = [-\pi, -\frac{\pi}{2})$ and that A_i can be written as

$$A_i = 2^{k-1}\pi + [-\frac{1}{2^{i-1}}\pi, -\frac{1}{2^i}\pi)$$

for $2 \leq i \leq k$ as in the proof of (i). Thus the following are easily obtained:
1) For $j \neq 1$, $A_1 = 2^{j-1}A_j - 2^{k+j-2}\pi$;
2) For $i \neq 1$, $A_i = 2^{k-1}\pi + 2^{1-i}A_1$;
3) For $i \neq 1, j \neq 1, i \neq j$, $A_i = 2^{j-i}A_j - 2^{k+j-i-1}\pi + 2^{k-1}\pi$.

Thus all the cases are covered and in each case the 2π-translation congruence between A_i and $2^{j-i}A_j$ is evident. Note that the Lebesgue measures of A_i and $2^{j-i}A_j$ are clearly the same. $\qquad\square$

As usual, we denote the symmetric group of degree k as S_k, where the elements are bijections $p : \{1, ..., k\} \rightarrow \{1, ..., k\}$ (permutations of $\{1, ..., k\}$) and the binary operation is functional composition. We also need the following lemma.

Lemma 4. *Let k, A, A_1 and A_i be as in Lemma 3. For each $p \in S_k$, let $A_p := \bigcup_{i=1}^k 2^{p(i)-i}A_{p(i)}$. Then the following hold:*
(i) *$\{A_p : p \in S_k\}$ consists of mutually 2π-translation and 2-dilation congruent elements.*
(ii) *A_p is a sub-wavelet set and satisfies the criteria in Theorem 1 for all $p \in S_k$.*

Proof of Lemma 4. To prove (i), note first that if $p = id \in S_k$, then $A_p = A$. Note also that from Lemma 3, for all $i \neq j$, with $i, j \in \{1, ..., k\}$, $A_i \cap A_j = \emptyset$. We claim that for fixed $p \in S_k$, and for all $i \neq j$, with $i, j \in \{1, ..., k\}$,

$$(2^{p(i)-i}A_{p(i)}) \cap (2^{p(j)-j}A_{p(j)}) = \emptyset.$$

Indeed, let $l \in \{1, ..., k\}$ be the unique integer such that $p(l) = 1$. Then $2^{p(l)-l}A_{p(l)}$ is the only interval among the sets $2^{p(i)-i}A_{p(i)}$ such that $2^{p(l)-l}A_{p(l)} \subseteq (-\infty, 0)$. Therefore $(2^{p(l)-l}A_{p(l)}) \cap (2^{p(j)-j}A_{p(j)}) = \emptyset$ for any $j \neq l$. Now for any $i \neq l$ such that $i \in \{1, ..., k\}$, we have $p(i) \neq 1$ and

$$2^{p(i)-i}A_{p(i)} = 2^{p(i)-i+k-1}\pi + [-\frac{1}{2^{i-1}}\pi, -\frac{1}{2^i}\pi).$$

Thus $(2^{p(i)-i}A_{p(i)}) \cap (2^{p(j)-j}A_{p(j)}) = \emptyset$ for any distinct $i, j \neq l$ with $i, j \in \{1, ..., k\}$. Hence for all $i \neq j$ such that $i, j \in \{1, ..., k\}$, we must have $(2^{p(i)-i}A_{p(i)}) \cap (2^{p(j)-j}A_{p(j)}) = \emptyset$.

Now for any fixed $p \in S_k$ and for each $i \in \{1, ..., k\}$ we have

$$m(2^{p(i)-i}A_{p(i)}) = 2^{p(i)-i} \cdot \frac{1}{2^{p(i)}}\pi = \frac{1}{2^i}\pi = m(A_i).$$

So by Lemma 3, the sets A_i and $2^{p(i)-i}A_{p(i)}$ are 2π-translation congruent for any $i \in \{1, ..., k\}$. Since the sets A_i are pairwise disjoint, and the sets $2^{p(i)-i}A_{p(i)}$ are also pairwise disjoint, it follows that A is 2π-translation congruent to A_p for any $p \in S_k$.

Likewise, observe that $A_{p(i)}$ and $2^{p(i)-i}A_{p(i)}$ are 2-dilation congruent for any $i \in \{1, ..., k\}$. For any fixed $p \in S_k$, $p : \{1, ..., k\} \rightarrow \{1, ..., k\}$ is a bijection, so $\{A_i : i \in \{1, ..., k\}\} = \{A_{p(i)} : i \in \{1, ..., k\}\}$. Once again since the sets A_i are pairwise disjoint, the sets $2^{p(i)-i}A_{p(i)}$ are also pairwise disjoint, it follows that A is 2-dilation congruent to A_p for any $p \in S_k$.

We have shown that A is both 2π-translation congruent and 2-dilation congruent to A_p for any $p \in S_k$. Therefore $\{A_p : p \in S_k\}$ consists of mutually 2π-translation and 2-dilation congruent elements.

To prove (ii), namely to check that A_p is a sub-wavelet set and satisfies the criteria in Theorem 1 for all $p \in S_k$, it is enough to check any one of them since (i) has been established. Without loss of generality, let us look at

$$A_{id} = A = [-\pi, -\frac{\pi}{2}) \cup [\frac{2^k - 1}{2}\pi, \frac{2^{2k-1} - 1}{2^k}\pi).$$

First of all, since $2[-\pi, -\frac{1}{2}\pi) = [-2\pi, -\pi)$ we have $\tilde{d}(x) = 2x$ for $x \in [-\pi, -\frac{\pi}{2})$. Furthermore, since $2^{-k+2}[\frac{2^k-1}{2}\pi, \frac{2^{2k-1}-1}{2^k}\pi) = [\frac{2^k-1}{2^{k-1}}\pi, \frac{2^{2k-1}-1}{2^{2k-2}}\pi)$ which is contained in $[\pi, 2\pi)$, it follows that $\tilde{d}(x) = 2^{-k+2}x$ for $x \in [\frac{2^k-1}{2}\pi, \frac{2^{2k-1}-1}{2^k}\pi)$. Therefore $\tilde{d}|_A$ is injective. As for the injectiveness of $\tilde{\tau}|_A$, recall again that $A = \bigcup_{i=1}^{k} A_i$ with $A_1 = [-\pi, -\frac{\pi}{2})$ and $A_i = 2^{k-1}\pi + [-\frac{1}{2^{i-1}}\pi, -\frac{1}{2^i}\pi)$ for $2 \leq i \leq k$. Thus it is immediate that $\tilde{\tau}(x) = x + 2\pi$ for $x \in A_1$ and $\tilde{\tau}(x) = x + 2\pi + 2^{k-1}\pi$ for $x \in A_i$ and $2 \leq i \leq k$. Thus $\tilde{\tau}|_A$ is injective. So A is a sub-wavelet set.

Since the above formulae for $\tilde{d}|_A$ and $\tilde{\tau}|_A$ show that

$$A^\tau = [\pi, 2\pi - \frac{1}{2^k}\pi),$$

$$A^d = [-2\pi, -\pi) \cup [\frac{2^k - 1}{2^{k-1}}\pi, \frac{2^{2k-1} - 1}{2^{2k-2}}\pi),$$

A satisfies the criteria in Theorem 1. $\qquad\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let $k \geq 2$ be an integer. Let A, A_i and A_p be the same sets as defined in Lemma 3 for each $1 \leq i \leq k$ and each $p \in S_k$. By Lemma 3, $A_{id} = A$ is a sub-wavelet set and satisfies the criteria in Theorem 1. Thus there exists a measurable set C, such that $C \cap A = \emptyset$ and $W_{id} := C \cup A$ is a wavelet set. Also by Lemma 3, $\{A_p : p \in S_k\}$ consists of mutually 2π-translation and 2-dilation congruent elements. Therefore, such a set C satisfies $C \cap A_p = \emptyset$ and $W_p := C \cup A_p$ is a wavelet set for every $p \in S_k$.

Let $\mathcal{E} := \{W_p : p \in S_k\}$. We will prove that $\{\sigma_{W_{id}}^{W_p} : W_p \in \mathcal{E}\}$ is isomorphic to the group S_k. Thus it will follow that $\mathcal{E}$ is the desired interpolation family of wavelet sets based at $E_0 = W_{id}$.

Denote $\sigma_p := \sigma_{W_{id}}^{W_p}$. Obviously $\sigma_p(C) = C$ and for any $x \in C$, $\sigma_p(x) = x$. Also since σ_p implements the 2π-translation congruence between W_{id} and W_p, the restriction of σ_p on the interval A_i is a linear 1-1 map onto the interval $2^{p(i)-i} \cdot A_{p(i)}$ and it maps the left end point of A_i to the left end point of $2^{p(i)-i} \cdot A_{p(i)}$ for each $i \in \{1, ..., k\}$ and each $p \in S_k$.

Therefore since σ_p is 2-homogeneous, for any integer l, we must have $\sigma_p(2^l C) = 2^l C$ and for any $x \in 2^l C$, $\sigma_p(x) = x$. Also the restriction of σ_p on interval $2^l A_i$ is a linear 1-1 map onto interval $2^l \cdot 2^{p(i)-i} \cdot A_{p(i)}$ and it maps the left-hand end point of

$2^l A_i$ to the left-hand end point of $2^l \cdot 2^{p(i)-i} \cdot A_{p(i)}$ for each $i \in \{1, ..., k\}$ and each $p \in S_k$ and each integer l. In particular,

$$
\begin{aligned}
(\sigma_{p_1} \circ \sigma_{p_2})(A_i) &= \sigma_{p_1}[\sigma_{p_2}(A_i)] = \sigma_{p_1}[2^{p_2(i)-i} \cdot A_{p_2(i)}] \\
&= 2^{p_2(i)-i} \cdot \sigma_{p_1}(A_{p_2(i)}) = 2^{p_2(i)-i} \cdot 2^{p_1(p_2(i))-p_2(i)} \cdot A_{p_1(p_2(i))} \\
&= 2^{p_1 \circ p_2(i)-i} \cdot A_{p_1 \circ p_2(i)} = \sigma_{p_1 \circ p_2}(A_i).
\end{aligned}
$$

Since the restriction of both $\sigma_{p_1} \circ \sigma_{p_2}$ and $\sigma_{p_1 \circ p_2}$ on the interval A_i are linear, 1-1 maps onto the interval $2^{p_1 \circ p_2(i)-i} \cdot A_{p_1 \circ p_2(i)}$ which map the left hand end-point of A_i to that of $2^{p_1 \circ p_2(i)-i} \cdot A_{p_1 \circ p_2(i)}$, it follows that $\sigma_{p_1} \circ \sigma_{p_2}|_{A_i}$ and $\sigma_{p_1 \circ p_2}|_{A_i}$ are identical.

Now since the maps σ_p and their compositions $\sigma_{p_1} \circ \sigma_{p_2}$ for any p, p_1 and p_2 in S_k are 2-homogeneous functions, each map is totally determined by its action on $\{A_i : i = 1, ..., k\} \cup \{C\}$. Therefore $\sigma_{p_1} \circ \sigma_{p_2} = \sigma_{p_1 \circ p_2}$.

Let $\mathcal{H} : S_k \to \{\sigma_p : p \in S_k\}$ be defined as $\mathcal{H}(p) = \sigma_p$. Then clearly $\mathcal{H}$ is a bijection. We only need to check that $\mathcal{H}$ is a homomorphism. Indeed we have $\mathcal{H}(p_1 \circ p_2) = \sigma_{p_1 \circ p_2} = \sigma_{p_1} \circ \sigma_{p_2} = \mathcal{H}(p_1) \circ \mathcal{H}(p_2)$. This ends the proof. $\square$

Corollary 5. *For any finite group G, there exists an interpolation family $\mathcal{E}_G$ of wavelet sets based at some $E_0 \in \mathcal{E}_G$, such that $\{\sigma^E_{E_0} : E \in \mathcal{E}_G\}$ is isomorphic to the group G.*

Proof. Recall that **Cayley's Theorem** states that any group G is isomorphic to a subgroup of $Sym G$. In particular any finite group G of order k is isomorphic to a subgroup of S_k. Thus we only need to consider arbitrary subgroups G of the group S_k for some integer k. With the same sets A and A_i as in the proof of Theorem 2, let

$$
A_g := \bigcup_{i=1}^k 2^{g(i)-i} A_{g(i)}
$$

for each $g \in G \subset S_k$ as before and $W_g := C \cup A_g$ for each $g \in G \subset S_k$. Denote $\mathcal{E}_G := \{W_g : g \in G \subset S_k\}$ and $\sigma_g := \sigma^{W_g}_{W_{id}}$. We will prove that $\{\sigma_g : g \in G\}$ is isomorphic to the group $G \subset S_k$. Then it will follow that $\mathcal{E}_G$ is the desired interpolation family of wavelet sets based at $E_0 = W_{id}$.

Let $\mathcal{K} : G \subset S_k \to \{\sigma_g : g \in G \subset S_k\}$ be defined as $\mathcal{K}(g) = \sigma_g$. Clearly $\mathcal{K}$ is a bijection. $\mathcal{K}$ is also a homomorphism since it is the restriction of the map $\mathcal{H}$ in the proof of Theorem 2 to the subgroup G. Hence $\mathcal{K}$ is an isomorphism. $\square$

Acknowledgments

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