STRONGLY ASYMPTOTICALLY STABLE FROBENIUS-PERRON OPERATORS

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*Dedicated to Professor Alexandra Bellow in celebration of her achievements
 in all the aspects of being that involve mathematics*

ABSTRACT. Let (X, Σ, μ) be a σ -finite measure space and let $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ be a Frobenius-Perron operator.

In 1997 Bartoszek and Brown proved that if T overlaps supports and if there exists $h \in L^1(X, \Sigma, \mu)$, $h > 0$ on X , such that $Th = h$, then T is (strongly) asymptotically stable.

In the note we prove that instead of assuming that $h > 0$ on X , it is enough to assume that $h \geq 0$ and $h \neq 0$. More precisely, we prove that T is asymptotically stable if and only if T overlaps supports and there exists $h \in L^1(X, \Sigma, \mu)$, $h \geq 0$, $h \neq 0$, such that $Th = h$.

1. INTRODUCTION

Let (X, Σ, μ) be a σ -finite measure space (throughout the paper, (X, Σ, μ) denotes a given σ -finite measure space). A measurable transformation $\phi : X \rightarrow X$ is called nonsingular if $\mu(\phi^{-1}(A)) = 0$ whenever $\mu(A) = 0$. A positive contraction $S : L^\infty(X, \Sigma, \mu) \rightarrow L^\infty(X, \Sigma, \mu)$ is called a Koopman operator if there exists a nonsingular transformation $\phi : X \rightarrow X$ such that $Su = u \circ \phi$ for every $u \in L^\infty(X, \Sigma, \mu)$. Using the Radon-Nikodym theorem it can be shown (see [5]) that every Koopman operator has a predual; that is, given a Koopman operator $S : L^\infty(X, \Sigma, \mu) \rightarrow L^\infty(X, \Sigma, \mu)$, there exists a positive contraction $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ such that $T' = S$, where T' is the dual of T ; the operator T is called a Frobenius-Perron operator (for a detailed discussion of the Frobenius-Perron operators see the book by Lasota and Mackey [5]).

Let $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ be a positive contraction.

We say that T overlaps supports if for every $f, g \in L^1(X, \Sigma, \mu)$, $f \geq 0$, $f \neq 0$, $g \geq 0$, $g \neq 0$, there exists $n \in \mathbb{N}$ (n depends on f and g) such that $T^n f \wedge T^n g \neq 0$.

If $f \in L^1(X, \Sigma, \mu)$, $f \geq 0$, $\|f\| = 1$, then it is often the custom to call f a density. The operator T is called (strongly) asymptotically stable if there exists a unique density f such that the sequence $(T^n g)_{n \in \mathbb{N}}$ converges in the norm topology of $L^1(X, \Sigma, \mu)$ to f for every density g in $L^1(X, \Sigma, \mu)$.

If $g \in L^1(X, \Sigma, \mu)$, we say that g is T -invariant (or, simply, invariant) if $Tg = g$.

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In 1997 Bartoszek and Brown [2] proved that if $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ is a Frobenius-Perron operator which overlaps supports and has an invariant density f such that $f > 0$ on X , then T is asymptotically stable. Our goal in this note is to prove that a Frobenius-Perron operator is asymptotically stable if and only if it overlaps supports and has an invariant density.

It is of interest to point out that there exist Frobenius-Perron operators to which the Bartoszek-Brown result [2] does not apply. More precisely, it may happen that an asymptotically stable Frobenius-Perron operator has a (necessarily unique) invariant density f and the set $\{f = 0\}$ has strictly positive measure, as we can see from the following examples.

Example 1.1. The following example seems to be the simplest one: let $X = \{1, 2\}$, let Σ be the collection of all the subsets of X , let μ be the counting measure defined on (X, Σ) , and let $\phi : X \rightarrow X$ be defined by $\phi(1) = \phi(2) = 1$. Then the Frobenius-Perron operator T generated by ϕ is strongly asymptotically stable and has the invariant density $f \in L^1(X, \Sigma, \mu)$ defined by $f(1) = 1$ and $f(2) = 0$.

Example 1.2. Let $\phi : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $\phi(1) = \phi(2) = 1$ and $\phi(n) = n - 1$ for every $n \geq 3$. If we consider the counting measure on $\mathbb{N}$, then obviously ϕ is nonsingular. The corresponding Frobenius-Perron operator $T : l^1 \rightarrow l^1$ is defined by

$$T(a_1, a_2, a_3, a_4, a_5, \dots) = (a_1 + a_2, a_3, a_4, a_5, \dots)$$

for every $(a_n)_{n \in \mathbb{N}} \in l^1$. Clearly, T is strongly asymptotically stable and $(1, 0, 0, 0, \dots)$ is the invariant density of T .

Example 1.3. In the previous two examples the L^1 -spaces were generated by counting measures. Let us now consider an example that involves a continuous measure. To this end, let $X = [0, 2]$, let Σ be the σ -algebra of all Lebesgue measurable subsets of $[0, 2]$, let λ be the Lebesgue measure on $[0, 2]$, and let $\phi : [0, 2] \rightarrow [0, 2]$ be defined by $\phi(x) = 2x \bmod 1$ for $x \in [0, 1]$ and $\phi(x) = x - 1$ for $x \in (1, 2]$. Using the example discussed on p. 116 of Horowitz [3], we conclude that if we let T be the Frobenius-Perron operator corresponding to ϕ , then T is strongly asymptotically stable and $1_{[0,1]}$ is the invariant density of T .

Bartoszek and Brown proved (see Proposition 1 of [2]) that if $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ is a positive contraction which overlaps supports, and if there exists a T -invariant density g such that $g > 0$ on X , then the sequence $(T^n)_{n \in \mathbb{N}}$ converges in the weak operator topology to a one-dimensional projection; however, the example discussed on pp. 362-364 of Akcoglu and Boivin [1] shows that, in general, T is not strongly asymptotically stable. Thus, in Theorem 1 of [2] and in the main result of this note, we cannot assume that T is merely a positive contraction rather than a Frobenius-Perron operator.

The paper is organized as follows: in the next section (Section 2) we discuss several lemmas which are needed in the last section (Section 3), where we prove the main result of this note.

2. INVARIANT DENSITIES AND SUPPORT OVERLAPPING

Let $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ be a positive contraction.

Given $A \in \Sigma$, set $\Sigma_A = \{B \in \Sigma \mid B \subseteq A\}$. Clearly, Σ_A is a σ -algebra on A and the function $\mu_A : \Sigma_A \rightarrow \mathbb{R} \cup \{+\infty\}$, $\mu_A(B) = \mu(B)$ for every $B \in \Sigma_A$

is a measure on (A, Σ_A) . Set also $L^1(A) = \{g \in L^1(X, \Sigma, \mu) \mid g1_A = g\}$ and $L^\infty(A) = \{u \in L^\infty(X, \Sigma, \mu) \mid u1_A = u\}$.

A measurable subset A of X is called T -absorbing if $Tg \in L^1(A)$ whenever $g \in L^1(A)$. If A is T -absorbing, we define an operator $T_{(A)} : L^1(A, \Sigma_A, \mu_A) \rightarrow L^1(A, \Sigma_A, \mu_A)$ as follows: if $g \in L^1(A, \Sigma_A, \mu_A)$, then let $\tilde{g} \in L^1(X, \Sigma, \mu)$ be such that $\tilde{g} = g$ on A and $\tilde{g} = 0$ on $X \setminus A$; now set $T_{(A)}g = (T\tilde{g})|_A$, where $(T\tilde{g})|_A$ is the restriction of $T\tilde{g}$ to A .

Lemma 2.1. *Let $\psi : X \rightarrow X$ be a nonsingular transformation, and let*

$$T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$$

be the Frobenius-Perron operator corresponding to ψ and to μ . If $A \in \Sigma$, $\mu(A) > 0$ is a T -absorbing set, then $T_{(A)}$ is also a Frobenius-Perron operator.

The proof of the above lemma follows from Theorem 1.9, p. 119 of Krengel's book [4]. Indeed, since $T'g = g \circ \psi$ whenever $g \in L^\infty(X, \Sigma, \mu)$, it follows that A is absorbing if and only if $\psi(A) \subseteq A$ μ -a.e. Thus, (by possibly redefining ψ on a μ -negligible set) $T_{(A)}$ is a Frobenius-Perron operator corresponding to the restriction of ψ to A .

Given a positive contraction $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ we will consider the Hopf decomposition (see Section 3.1 of [4]) of X into the conservative part C and the dissipative part $D = X \setminus C$ generated by T . We will also use the fact (see the proof of Corollary 1 of [7]) that if T overlaps supports and has an invariant density, then the invariant density is unique (that is, if T overlaps supports, and f, g are densities such that $Tf = f$ and $Tg = g$, then $f = g$).

Lemma 2.2. *Assume that $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ is a Frobenius-Perron operator which overlaps supports and has a (necessarily) unique invariant density f . Then $C = \{f > 0\}$ and $\inf_{n \in \mathbb{N}} T'^n 1_D = 0$, where T' is the dual of T .*

Proof. It is well known that $\{f > 0\} \subseteq C$. If we assume that $\{f > 0\} \neq C$, then using the observations made on p. 126 of Krengel's book [4], and taking into consideration that $\{f > 0\}$ is a T -absorbing set, we obtain that $C \setminus \{f > 0\}$ is also T -absorbing. We obtain a contradiction since on one hand $\{f > 0\}$ and $C \setminus \{f > 0\}$ are T -absorbing sets, while on the other hand T overlaps supports.

Clearly, in order to complete the proof of the lemma, it is enough to prove that $\sup_n T'^n 1_C = 1_X$. To this end, note that since T' is a Koopman operator, it follows that $\sup_n T'^n 1_C = 1_B$ for some $B \in \Sigma$. Assume that $B \neq X$; that is, assume that $\mu(X \setminus B) > 0$. Then there exists $g \in L^1(X, \Sigma, \mu)$, $g \geq 0$, $g \neq 0$, $g1_B = 0$. Since

$$\begin{aligned} \int (T^n f) \wedge (T^n g) d\mu &= \int (f \wedge (T^n g)) 1_C d\mu \leq \int (T^n g) 1_C d\mu \\ &\leq \int g (T'^n 1_C) d\mu \leq \int g 1_B d\mu = 0 \end{aligned}$$

for every $n \in \mathbb{N}$, we obtain a contradiction since T overlaps supports. Q.E.D.

3. ASYMPTOTIC STABILITY

As mentioned in the Introduction, our goal in this section is to prove the main result of this note (Corollary 3.2 below).

Theorem 3.1. *Let T be a positive contraction of $L^1(X, \Sigma, \mu)$ with dual T' and with conservative and dissipative parts C and D , respectively. The following assertions are equivalent:*

- (a) *T is strongly asymptotically stable.*
- (b) *$\inf_{n \in \mathbb{N}} T'^n 1_D = 0$ and $T_{(C)}$ is strongly asymptotically stable.*

Proof. (a) $\Rightarrow$ (b). Using a result of Helmsberg (see Theorem 3.3, p. 175 of [4]), we obtain that $\inf_{n \in \mathbb{N}} T'^n 1_D = 0$.

Clearly, the fact that T is strongly asymptotically stable implies that $T_{(C)}$ is also strongly asymptotically stable.

(b) $\Rightarrow$ (a). Since $T_{(C)}$ is strongly asymptotically stable, there exists $h \in L^1(C, \Sigma_C, \mu_C)$, $h \geq 0$, $\|h\| = 1$ such that $T_{(C)}h = h$. Let $f \in L^1(X, \Sigma, \mu)$ be defined as follows: $f = h$ on C and $f = 0$ on D .

In order to complete the proof of the theorem, it is enough to prove that $(T^n g)_{n \in \mathbb{N}}$ converges to f in the norm topology of $L^1(X, \Sigma, \mu)$ whenever g is a density in $L^1(X, \Sigma, \mu)$. Since every $u \in L^1(X, \Sigma, \mu)$ can be written in the form $u = u1_C + u1_D$, and since the strong asymptotic stability of $T_{(C)}$ implies that $(T^n(u1_C))_{n \in \mathbb{N}}$ converges in norm to $f \cdot \int u1_C d\mu$, it follows that it is actually enough to prove that $(T^n g)_n$ converges to f whenever g is a density in $L^1(X, \Sigma, \mu)$ such that $g1_D = g$.

To this end, set $T_D = 1_D T 1_D$ and note that D is a strong zero set in the terminology of [8] (that is, the sequence of operators $(T_D^n)_{n \in \mathbb{N}}$ converges strongly to zero). Indeed,

$$\int T_D^n u d\mu = \int u((1_D T' 1_D)^n 1_X) d\mu \leq \int u(T'^n 1_D) d\mu$$

for every $n \in \mathbb{N}$ and $u \in L^1(X, \Sigma, \mu)$, $u \geq 0$. Since $(T'^n 1_D)_n$ is a decreasing sequence (of elements of $L^\infty(X, \Sigma, \mu)$), and since $\inf_n T'^n 1_D = 0$, it follows that

$$\lim_{n \rightarrow \infty} \|T_D^n u\| = 0 \text{ for every } u \in L^1(X, \Sigma, \mu).$$

Now, let g be a density in $L^1(X, \Sigma, \mu)$ such that $g1_D = g$, and let $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Since D is a strong zero set, it follows that there exists $n_1 \in \mathbb{N}$ such that $\|T_D^n g\| < \frac{\epsilon}{3}$ for every $n \geq n_1$.

By Lemma 2.2 of Lin [6] (see also Lemma 2.1 of [8]), it follows that there exist $g_1, g_2, g_3, \dots, g_{n_1} \in L^1(X, \Sigma, \mu)$, $g_i \geq 0$, $g_i 1_C = g_i$ for every $i = 1, 2, 3, \dots, n_1$ such that $T^{n_1} g = T_D^{n_1} g + \sum_{i=1}^{n_1} T^{n_1-i} g_i$.

Since $g_i 1_C = g_i$, it follows that $\lim_{n \rightarrow \infty} \left\| T^n g_i - \left(\int g_i d\mu \right) f \right\| = 0$ for every $i = 1, 2, \dots, n_1$. Accordingly, there exists $n_2 \in \mathbb{N}$ such that $\left\| T^n g_i - \left(\int g_i d\mu \right) f \right\| \leq \frac{\epsilon}{3} \|g_i\|$ for every $n \geq n_2$ and $i = 1, 2, \dots, n_1$.

Let $n_\epsilon = n_1 + n_2$.

Since

$$\sum_{i=1}^{n_1} \|g_i\| = \left\| \sum_{i=1}^{n_1} g_i \right\| = \|T^{n_1} g - T_D^{n_1} g\| = \|T^{n_1} g\| - \|T_D^{n_1} g\| > 1 - \frac{\epsilon}{3},$$

it follows that

$$\begin{aligned}
 \|T^n g - f\| &= \|T^{n-n_1}(T_D^{n_1} g + \sum_{k=1}^{n_1} T^{n_1-k} g_k) - f\| \\
 &\leq \left\| T^{n-n_1} T_D^{n_1} g - \left(1 - \sum_{i=1}^{n_1} \|g_i\|\right) f \right\| \\
 &\quad + \sum_{k=1}^{n_1} \left\| T^{n-k} g_k - \left(\int g_k d\mu\right) f \right\| \\
 &\leq \|T_D^{n_1} g\| + \left(1 - \sum_{i=1}^{n_1} \|g_i\|\right) + \frac{\epsilon}{3} \sum_{i=1}^{n_1} \|g_i\| \\
 &< \frac{\epsilon}{3} + \left(1 - \left(1 - \frac{\epsilon}{3}\right)\right) + \frac{\epsilon}{3} = \epsilon
 \end{aligned}$$

for every $n \geq n_\epsilon$.

Q.E.D.

Corollary 3.2. *Let $T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$ be a Frobenius-Perron operator. The following assertions are equivalent:*

- (a) *T is strongly asymptotically stable.*
- (b) *T has an invariant density and overlaps supports.*

Proof. (a) $\Rightarrow$ (b). If g is a density, then the sequence $(T^n g)_{n \in \mathbb{N}}$ converges in the norm topology of $L^1(X, \Sigma, \mu)$ to a density f which is T -invariant.

Let g and h be densities in $L^1(X, \Sigma, \mu)$. Since $\lim_{n \rightarrow \infty} \|T^n g - T^n h\| = 0$, and since

$$\|T^m g - T^m h\| = \|T^m g + T^m h - 2((T^m g) \wedge (T^m h))\|$$

for every $m \in \mathbb{N}$, it follows that there exists $n \in \mathbb{N}$ such that $(T^n g) \wedge (T^n h) \neq 0$. Thus, T overlaps supports.

(b) $\Rightarrow$ (a). Let the conservative and the dissipative parts of X generated by T be C and D , respectively.

Since T has an invariant density, it follows that $C \neq \emptyset$.

Since C is a T -absorbing set, Lemma 2.1 tells us that $T_{(C)}$ is a Frobenius-Perron operator, while Lemma 2.2 implies that $T_{(C)}$ satisfies the conditions of Theorem 1 of [2]; hence $T_{(C)}$ is strongly asymptotically stable.

Finally, using Lemma 2.2 and Theorem 3.1 we conclude that T is strongly asymptotically stable.

Q.E.D.

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