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ONE-STEP EXTENSION OF THE BERGMAN SHIFT

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Abstract. In this paper we answer a question of Curto and Fialkow: there exists a quadratically hyponormal weighted shift which is not positively quadratically hyponormal.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal, then T is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$.

Recall the Bram-Halmos criterion for subnormality, which states that an operator $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([1], [2, III.1.9]). Using the Choleski algorithm for operator matrices, it is easy to see that this is equivalent to the positivity of the matrices $(T^{*j}T^i - T^iT^{*j})_{i,j=1}^k$ for $k = 1, 2, \cdots$. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k-hyponormal* whenever the $k \times k$ operator matrix $M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$ is positive, then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([7]). Recall ([3], [4]) that $T \in \mathcal{L}(\mathcal{H})$ is *weakly k-hyponormal* if $\sum_{i=0}^k s_i T^i$ is hyponormal for every complex number s_i ($0 \leq i \leq k$). If $k = 2$, then it is said to be *quadratically hyponormal*. It is known that 2-hyponormal $\Rightarrow$ quadratically hyponormal. In [3, Proposition 7], it is shown that there exists a quadratically hyponormal weighted shift which is not 2-hyponormal.

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Let W_α be a hyponormal weighted shift. We write $D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$ for $s \in \mathbb{C}$, and we let

$$D_n(s) := P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n,$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$. Then $D_n(s)$ is of the form

$$D_n(s) = \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix},$$

where

$$\begin{aligned} q_n &:= u_n + |s|^2 v_n, \\ r_n &:= s\sqrt{w_n}, \\ u_n &:= \alpha_n^2 - \alpha_{n-1}^2, \\ v_n &:= \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2, \\ w_n &:= \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{aligned}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2-step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2 d_n; \tag{0.1}$$

if we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write $d_n = \sum_{i=0}^{n+1} c(n,i)t^i$, then the Maclaurin coefficients $c(n,i)$ satisfy a double-indexed recursive formula; namely

$$\begin{gathered} c(n+2,i) = u_{n+2}\,c(n+1,i) + v_{n+2}\,c(n+1,i-1) - w_{n+1}\,c(n,i-1), \\ c(n,0) = u_0\cdots u_n, \quad c(n,n+1) = v_0\cdots v_n, \quad c(1,1) = u_1v_0 + v_1u_0 - w_0 \end{gathered} \tag{0.2}$$

$(n \geq 0,\ i \geq 1)$.

We begin with:

Definition 1 ([4], [5], [6])**.** Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let W_α be the corresponding weighted shift, and let $c(n,i)$ be the Maclaurin coefficients of the polynomial d_n. We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$, and $c(n,n+1) > 0$ for all $n \geq 0$.

Clearly, positively quadratically hyponormal $\Longrightarrow$ quadratically hyponormal. In 1994, Curto and Fialkow ([4, Problem 4.7]) asked if the converse is true: *if W_α is a quadratically hyponormal weighted shift, does it follow that W_α is positively quadratically hyponormal?* In this paper we answer it negatively.

If the weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ is given by

$$\alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \geq 0),$$

then the corresponding weighted shift is called the *Bergman shift*. It is well known that the Bergman shift is subnormal.

The following is an *one-step extension* of the Bergman shift.

Theorem 2. *For $x > 0$, let T_x be the weighted shift whose weight sequence is given by*

$$\alpha_0 = \sqrt{x}, \quad \alpha_n = \sqrt{\frac{n}{n+1}} \ (n \geq 1).$$

Then we have:

(a) *T_x is positively quadratically hyponormal $\Longleftrightarrow 0 < x \leq \frac{22}{47}$.*
(b) *$0 < x \leq \frac{71}{151} \Longrightarrow T_x$ is quadratically hyponormal.*
(c) *T_x is not quadratically hyponormal for $x = \frac{1}{2}$.*
(d) *T_x is 2-hyponormal $\Longleftrightarrow 0 < x \leq \frac{1}{3}$.*
(e) *T_x is never subnormal for any $x > 0$.*

Proof. (a) We use an idea of Curto ([3, Proposition 7]). Suppose T_x is hyponormal and hence $0 < x \leq \frac{1}{2}$. Write $d_n(t) = \sum_{i=0}^{n+1} c(n,i)t^i$. From (0.2) we can check directly that

$$\begin{cases} c(0,0) = x, \\ c(0,1) = \frac{1}{2}x, \end{cases} \qquad \begin{cases} c(1,0) = x(\frac{1}{2} - x), \\ c(1,1) = x(\frac{1}{3} - \frac{1}{2}x), \\ c(1,2) = \frac{1}{6}x, \end{cases}$$

$$\begin{cases} c(2,0) = \frac{x}{6}(\frac{1}{2} - x), \\ c(2,1) = \frac{x}{6}(\frac{1}{2} - x), \\ c(2,2) = \frac{x}{12}(1 - x), \\ c(2,3) = \frac{x}{12}(1 - x), \end{cases} \qquad \begin{cases} c(3,0) = \frac{x}{72}(\frac{1}{2} - x), \\ c(3,1) = \frac{x}{60}(\frac{1}{2} - x), \\ c(3,2) = \frac{11}{360}x(\frac{1}{2} - x), \\ c(3,3) = \frac{x}{720}(16 - 21x), \\ c(3,4) = \frac{x}{45}(1 - x), \end{cases}$$

and

$$\begin{cases} c(4,0) = \frac{x}{1440}(\frac{1}{2} - x), \\ c(4,1) = \frac{x}{1080}(\frac{1}{2} - x), \\ c(4,2) = \frac{x}{480}(\frac{1}{2} - x), \\ c(4,3) = \frac{x}{8640}(22 - 47x), \\ c(4,4) = x(\frac{1}{270} - \frac{7}{1440}x), \\ c(4,5) = \frac{x}{270}(1 - x). \end{cases}$$

Observe

$$c(n,i) \geq 0 \quad \text{for all } 0 \leq n, i \leq 3 \text{ with } 0 \leq i \leq n+1. \tag{2.1}$$

Note that

$$c(n,0) = u_0 \cdots u_n \geq 0 \quad \text{for all } n \geq 0 \tag{2.2}$$

and

$$c(n,1) = u_0 \cdots u_{n-1} \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2) \geq 0 \quad (n \geq 2). \tag{2.3}$$

Claim I. $c(n,2) \geq 0$ for all $n \geq 1$.

Proof of Claim I. Since for $n \geq 2$,

$$\begin{aligned}
&v_{n+2}c(n+1,1) - w_{n+1}c(n,1) \\
&= v_{n+2}u_0 \cdots u_n \alpha_{n+1}^2 \left(\alpha_{n+2}^2 - \alpha_n^2\right) - w_{n+1}u_0 \cdots u_{n-1}\alpha_n^2 \left(\alpha_{n+1}^2 - \alpha_{n-1}^2\right) \\
&= u_0 \cdots u_{n-1}\Big(v_{n+2}u_n\alpha_{n+1}^2(\alpha_{n+2}^2 - \alpha_n^2) - w_{n+1}\alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)\Big) \\
&= u_0 \cdots u_{n-1} \cdot \frac{24}{n(n+1)^2(n+2)^2(n+3)^2(n+4)} \geq 0,
\end{aligned} \tag{2.4}$$

it follows that if $c(n+1,2) \geq 0$, then for $n \geq 2$,

$$c(n+2,2) = u_{n+2}c(n+1,2) + v_{n+2}c(n+1,1) - w_{n+1}c(n,1) \geq 0.$$

But since $c(n,2) \geq 0$ for $n = 1,2,3$, the above argument gives that $c(n,2) \geq 0$ for all $n \geq 1$. This proves Claim I.

Claim II. For $n \geq 4$, $i \geq 4$,

$$c(n,i) = v_n c(n-1,i-1).$$

Proof of Claim II. First of all we prove that for $n \geq 4$, $i \geq 1$,

$$c(n,i) = v_n c(n-1,i-1) + u_n \cdots u_4 h_i \quad \text{with } h_i := u_3 c(2,i) - w_2 c(1,i-1). \tag{2.5}$$

A simple calculation shows that $u_{n+1}v_n = w_n$ for all $n \geq 3$. Thus if $n = 4$,

$$\begin{aligned}
c(4,i) &= u_4 c(3,i) + v_4 c(3,i-1) - w_3 c(2,i-1) \\
&= v_4 c(3,i-1) + u_4\Big(u_3 c(2,i) + v_3 c(2,i-1) - w_2 c(1,i-1)\Big) - w_3 c(2,i-1) \\
&= v_4 c(3,i-1) + u_4\Big(u_3 c(2,i) - w_2 c(1,i-1)\Big) + (u_4 v_3 - w_3)c(2,i-1) \\
&= v_4 c(3,i-1) + u_4 h_i
\end{aligned}$$

and a similar calculation works for the inductive step. Now

$$h_1 = \frac{1}{36}x(x-\frac{1}{2}), \quad h_2 = \frac{1}{72}x(x-\frac{1}{2}), \quad h_3 = -\frac{1}{144}x^2, \quad \text{and} \quad h_i = 0 \text{ for } i \geq 4,$$

which together with (2.5) proves Claim II.

Claim III. If $c(n,3) \geq 0$, then $c(n+1,3) \geq 0$ for $n \geq 4$.

Proof of Claim III. Since for $n \geq 4$,

$$\begin{aligned}
&v_{n+1}c(n,2) - w_n c(n-1,2) \\
&= v_{n+1}\Big(u_n c(n-1,2) + v_n c(n-1,1) - w_{n-1}c(n-2,1)\Big) - w_n c(n-1,2) \\
&= (v_{n+1}u_n - w_n)c(n-1,2) + v_{n+1}\Big(v_n c(n-1,1) - w_{n-1}c(n-2,1)\Big) \\
&= c(n-1,2) \cdot \frac{4}{n(n+1)^2(n+2)^2(n+3)} + v_{n+1}g_n \geq 0,
\end{aligned}$$

where $g_n := v_n c(n-1,1) - w_{n-1}c(n-2,1) \geq 0$ by (2.4). Therefore if $c(n,3) \geq 0$, then

$$c(n+1,3) = u_{n+1}c(n,3) + v_{n+1}c(n,2) - w_n c(n-1,2) \geq 0,$$

which proves Claim III.

It now follows from (2.1), (2.2), (2.3), Claim I and Claim II that $c(n,i) \geq 0$ for all $n, i \geq 0$ with $0 \leq i \leq n+1$ if and only if $c(n,3) \geq 0$ for all $n \geq 4$. Therefore by Claim III,

$$c(n,i) \geq 0 \text{ for all } n, i \geq 0 \iff c(4,3) \geq 0 \iff x \leq \frac{22}{47}.$$

This proves statement (a).

(b) In view of (a), it suffices to show that if $\frac{22}{47} < x \leq \frac{71}{151}$, then T_x is quadratically hyponormal. Thus suppose $\frac{22}{47} < x \leq \frac{71}{151}$. Then we have:

(i) $c(n,i) \geq 0$ for all $0 \leq n, i \leq 4$ with $0 \leq i \leq n+1$ except for $c(4,3)$;

(ii) (2.2), (2.3), Claim I, Claim II, and Claim III all in the proof of (a) hold.

Observe that

$$\begin{aligned} c(5,3) &= u_5 c(4,3) + v_5 c(4,2) - w_4 c(3,2) \\ &= \frac{1}{30}x\left(\frac{11}{4320} - \frac{47}{8640}x\right) + \frac{1}{4200}x(\frac{1}{2} - x) - \frac{11}{64800}x(\frac{1}{2} - x) \\ &= \frac{1}{604800}x(72 - 151x). \end{aligned}$$

Thus $c(5,3) \geq 0$ since $x \leq \frac{71}{151}$. Therefore by Claim III,

$$c(n,3) \geq 0 \qquad \text{for } n \geq 5 \text{ and } 0 < x \leq \frac{71}{151}.$$

Thus by Claim II, we have

$$c(n,i) \geq 0 \quad \text{for all } n, i \geq 0 \text{ except for } c(n, n-1)\ (n \geq 4). \tag{2.6}$$

Note that $c(4,3) < 0$. Again by Claim II,

$$c(n, n-1) < 0 \quad \text{for all } n \geq 4. \tag{2.7}$$

Note that $d_n(t) \geq 0$ for $n = 0,1,2,3$. Observe by Claim II that if $n \geq 6$, then

$$\begin{aligned} &c(n,n-2)t^{n-2} + c(n,n-1)t^{n-1} + c(n,n)t^n \\ &\quad = v_n \cdots v_6 t^{n-5}\left(c(5,3)t^3 + c(5,4)t^4 + c(5,5)t^5\right). \end{aligned}$$

Thus if $c(5,3)t^3 + c(5,4)t^4 + c(5,5)t^5 \geq 0$ for every $t \geq 0$, then $d_n(t) \geq 0$ for every $n \geq 6$ and every $t \geq 0$ because other Maclaurin coefficients are nonnegative. Thus it will suffice to show that if $\frac{22}{47} < x \leq \frac{71}{151}$, then

$$c(n,n-2)t^{n-2} + c(n,n-1)t^{n-1} + c(n,n)t^n \geq 0 \quad \text{for } n = 4, 5 \tag{2.8}$$

(this also implies that $d_n(t) \geq 0$ for $n = 4, 5$).

Claim IV. If $\frac{22}{47} < x \le \frac{71}{151}$, then

$$c(4,2)t^2 + c(4,3)t^3 + c(4,4)t^4 \ge 0 \quad \text{for every } t \ge 0.$$

Proof of Claim IV. There are two cases to consider.

Case 1 ($0 < t \le \frac{7}{24}$). From (0.2) we have

$$u_5 c(4,3) + v_5 c(4,2) = c(5,3) + w_4 c(3,2).$$

Since by (2.6), $c(5,3) + w_4 c(3,2) \ge 0$ we have

$$u_5 c(4,3) + v_5 c(4,2) \ge 0,$$

so that

$$c(4,2) + \frac{7}{24} c(4,3) \ge 0.$$

Since $c(4,2) \ge 0$ and $c(4,3) < 0$, it follows that if $0 < t \le \frac{7}{24}$, then $c(4,2)+c(4,3)t \ge 0$. Since $c(4,4) \ge 0$ we have that $c(4,2)t^2 + c(4,3)t^3 + c(4,4)t^4 \ge 0$.

Case 2 ($t \ge \frac{7}{24}$). From (0.2) we have

$$u_5 c(4,4) + v_5 c(4,3) = c(5,4) + w_4 c(3,3).$$

A straightforward calculation shows that if $x \le \frac{71}{151}$, then

$$\begin{aligned} c(5,4) + w_4 c(3,3) &= \frac{11}{37800} x \left(1 - \frac{47}{22} x\right) + \frac{1}{180} x \left(\frac{1}{45} - \frac{7}{240} x\right) \\ &= \frac{47}{113400} x \left(1 - \frac{711}{376} x\right) \ge 0. \end{aligned}$$

Thus $u_5 c(4,4) + v_5 c(4,3) \ge 0$, so that $c(4,3) + \frac{7}{24} c(4,4) \ge 0$, which implies that if $t \ge \frac{7}{24}$, then $c(4,3) + c(4,4)t \ge 0$. Since $c(4,2) \ge 0$ we have that $c(4,2)t^2 + c(4,3)t^3 + c(4,4)t^4 \ge 0$.

Claim V. If $\frac{22}{47} < x \le \frac{71}{151}$, then

$$c(5,3)t^3 + c(5,4)t^4 + c(5,5)t^5 \ge 0 \quad \text{for every } t \ge 0.$$

Proof of Claim V. From (0.2) we have

$$\begin{cases} u_6 c(5,4) + v_6 c(5,3) = c(6,4) + w_5 c(4,3), \\ u_6 c(5,5) + v_6 c(5,4) = c(6,5) + w_5 c(4,4). \end{cases} \tag{2.9}$$

By the same argument as in Claim IV, it suffices to show that if $x \le \frac{71}{151}$, then

$$u_6 c(5,4) + v_6 c(5,3) \ge 0 \quad \text{and} \quad u_6 c(5,5) + v_6 c(5,4) \ge 0. \tag{2.10}$$

Indeed, a straightforward calculation shows that if $x \le \frac{71}{151}$, then

$$\begin{aligned} c(6,4) + w_5 c(4,3) &= x \left(\frac{1}{100800} - \frac{151}{7257600} x\right) + \frac{2}{735} x \left(\frac{11}{4320} - \frac{47}{8640} x\right) \\ &= \frac{107}{6350400} x \left(1 - \frac{1809}{856} x\right) \ge 0 \end{aligned}$$

and

$$c(6,5)+w_5c(4,4)=x\left(\frac{11}{453600}-\frac{47}{907200}x\right)+\frac{2}{735}x\left(\frac{1}{270}-\frac{7}{1440}x\right)$$
$$=\frac{109}{3175200}x\left(1-\frac{413}{218}x\right)\geq 0,$$

which together with (2.9) proves (2.10) and hence Claim V. Now Claim IV and Claim V prove (2.8); therefore if $0<x\leq\frac{71}{151}$, then T_x is quadratically hyponormal.

(c) Observe that if $x=\frac{1}{2}$, then

$$d_4(t)=\frac{1}{1080}t^5+\frac{11}{17280}t^4-\frac{1}{11520}t^3,$$

so that

$$\lim_{t\to 0+}\frac{d_4(t)}{t^3}=-\frac{1}{11520}<0,$$

which implies that T_x is not quadratically hyponormal.

(d) Remember ([3, Corollary 5]) that if W_α is the weighted shift with weights $\alpha=\{\alpha_n\}_{n=0}^\infty$, then W_α is 2-hyponormal if and only if

$$\alpha_{n+1}^2(\alpha_{n+2}^2-\alpha_n^2)^2\ \leq\ (\alpha_{n+1}^2-\alpha_n^2)(\alpha_{n+2}^2\alpha_{n+3}^2-\alpha_n^2\alpha_{n+1}^2)\quad(n\geq 0).$$

Thus T_x is 2-hyponormal if and only if

$$\alpha_1^2(\alpha_2^2-x)^2\leq(\alpha_1^2-x)(\alpha_2^2\alpha_3^2-x\,\alpha_1^2);$$

that is,

$$\frac{1}{2}(\frac{2}{3}-x)^2\leq(\frac{1}{2}-x)(\frac{1}{2}-\frac{1}{2}x),$$

or equivalently, $0<x\leq\frac{1}{3}$.

(e) Let W_α be the weighted shift with weights $\alpha=\{\alpha_n\}_{n=0}^\infty$ and let $\beta_n:=\alpha_0^2\cdots\alpha_{n-1}^2$ for $n\geq 1$. Then C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [2, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ supported on $[0,||W_\alpha||^2]$, with $||W_\alpha||^2\in\operatorname{supp}\mu$ such that

$$\beta_n=\int_0^{||W_\alpha||^2}t^n\,d\mu(t)\quad\text{for all } n\geq 1.$$

By an argument of Curto [3, Proposition 8], if W_α is a weighted shift whose restriction to $\bigvee\{e_1,e_2,\cdots\}$ is subnormal, with associated measure μ, then W_α is subnormal if and only if

(i) $\frac{1}{t}\in L^1(\mu)$;

(ii) $\alpha_0^2\leq\left(||\frac{1}{t}||_{L^1(\mu)}\right)^{-1}$.

A straightforward calculation shows that the Bergman shift has measure $d\mu=dt$. Indeed, $\beta_n=\frac{1}{n+1}=\int_0^1 t^n d\mu(t)$ has a solution $d\mu=dt$. Thus $\frac{1}{t}$ is not integrable with respect to μ. This implies that T_x is never subnormal for any $x>0$. □

Corollary 3. *Let T_x be as in Theorem 2. If $\frac{22}{47}<x\leq\frac{71}{151}$, then T_x is quadratically hyponormal and not positively quadratically hyponormal.*

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