COMPOSITION OPERATORS ON DIRICHLET-TYPE SPACES

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ABSTRACT. The Dirichlet-type space D^p ($1 \leq p \leq 2$) is the Banach space of functions analytic in the unit disc with derivatives belonging to the Bergman space A^p . Let Φ be an analytic self-map of the disc and define $C_\Phi(f) = f \circ \Phi$ for $f \in D^p$. The operator $C_\Phi : D^p \rightarrow D^p$ is bounded (respectively, compact) if and only if a related measure μ_p is Carleson (respectively, compact Carleson). If C_Φ is bounded (or compact) on D^p , then the same behavior holds on D^q ($1 \leq q < p$) and on the weighted Dirichlet space D_{2-p} . Compactness on D^p implies that C_Φ is compact on the Hardy spaces and the angular derivative exists nowhere on the unit circle. Conditions are given which, together with the angular derivative condition, imply compactness on the space D^p . Inner functions which induce bounded composition operators on D^p are discussed briefly.

INTRODUCTION

Let $D = \{z : |z| < 1\}$ and let A denote normalized area measure on D . The Bergman space A^p ($1 \leq p < \infty$) is the Banach space of functions f analytic in the disc D such that $\|f\|_{A^p}^p = \int_D |f(z)|^p dA(z) < \infty$. The Dirichlet-type space D^p is the set of functions analytic in D with derivatives belonging to A^p . The set D^p is a Banach space, with norm given by

$$(1) \quad \|f\|_{D^p} = |f(0)| + \|f'\|_{A^p}.$$

Note that point evaluation is continuous on D^p and $D^p \subset D^q$ if $1 \leq q < p$.

Throughout this paper, Φ denotes an analytic self-map of D . The composition operator C_Φ is defined by $C_\Phi(f) = f \circ \Phi$ for $f \in D^p$. If $C_\Phi(f) \in D^p$ for every $f \in D^p$, then C_Φ is bounded, by the Closed Graph Theorem.

Interest in the spaces D^p is motivated by the work of R. Roan [19] and B. D. MacCluer [14], who studied composition operators on S^p , the space of functions with derivatives in the Hardy space H^p for $p \geq 1$. Other related work appears in [15], where MacCluer and J. H. Shapiro studied composition operators on the weighted Dirichlet spaces D_α .

SECTION 1

Let μ be a finite positive Borel measure on D . For $|\zeta| = 1$ and $0 < \delta \leq 2$, $S(\zeta, \delta)$ is the Carleson set $\{z \in D : |z - \zeta| < \delta\}$. The measure μ is said to be

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Carleson if there is a constant C such that $\mu(S(\zeta, \delta)) \leq C\delta^2$ for all ζ and δ . The measure is said to be compact Carleson if

$$\lim_{\delta \rightarrow 0} \sup_{|\zeta|=1} \frac{\mu(S(\zeta, \delta))}{\delta^2} = 0.$$

Carleson measures have been useful in the study of composition operators in several settings [11, 13, 14, 15, 27, 28].

For $w \in D$, let $N_2(\Phi, w)$ denote the number of zeroes (counting multiplicities) of the equation $\Phi(z) = w$. For $1 \leq p < 2$ and $w \in D$, $N_p(\Phi, w)$ is defined to be the modified counting function

$$N_p(\Phi, w) = \sum \frac{1}{|\Phi'(z)|^{2-p}}$$

where the sum extends over the zeroes of $\Phi - w$, repeated by multiplicity. In particular, $N_p(\Phi, w) = 0$ for $w \notin \Phi(D)$. Let μ_p be the measure defined on D by $d\mu_p(w) = N_p(\Phi, w) dA(w)$, $1 \leq p \leq 2$. Note that μ_p is a finite measure if and only if $\Phi \in D^p$.

The proofs of the first two theorems are standard. A sketch of the proofs is provided in Section 3.

Theorem 1. *The operator C_Φ is bounded on D^p if and only if μ_p is a Carleson measure.*

Theorem 2. *The operator C_Φ is compact on D^p if and only if the measure μ_p is compact Carleson.*

The case $p = 2$ in Theorems 1 and 2 was first proven by MacCluer and Shapiro in their study of the weighted Dirichlet spaces D_α [15]. An analytic function f belongs to D_α ($\alpha > -1$) if $\int_D |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty$. Note that $D_0 = D^2$, the classical Dirichlet space. For $p = 2$ the norm (1) is equivalent to the norm used in [15] for D_0 .

Throughout the remainder of this work C will denote a positive constant, the exact value of which will vary from one appearance to the next.

Theorem 3. *If C_Φ is bounded (respectively, compact) on D^p and $1 \leq q < p$, then C_Φ is bounded (respectively, compact) on D^q .*

Proof. The hypotheses imply that $\Phi \in D^q$ for $1 \leq q \leq p$. Thus μ_q is a finite measure on the disc. Since $q < p$, Hölder's Inequality implies that

$$\begin{aligned} \mu_q(S(\zeta, \delta)) &= \int_{\Phi^{-1}(S(\zeta, \delta))} |\Phi'|^q dA \\ &\leq \left(\int_{\Phi^{-1}(S(\zeta, \delta))} |\Phi'|^p dA \right)^{q/p} \left(\int_{\Phi^{-1}(S(\zeta, \delta))} 1 dA \right)^{(p-q)/p} \\ (2) \quad &= \mu_p(S(\zeta, \delta))^{q/p} A\Phi^{-1}(S(\zeta, \delta))^{(p-q)/p}. \end{aligned}$$

Since C_Φ is bounded on the Bergman spaces, Theorem 4.3 [15] implies that the measure $A\Phi^{-1}$ is Carleson. Thus (2) yields

$$(3) \quad \mu_q(S(\zeta, \delta)) \leq C \delta^{2(p-q)/p} \mu_p(S(\zeta, \delta))^{q/p}.$$

If C_Φ is bounded on D^p , Theorem 1 implies that there is a constant C such that $\mu_p(S(\zeta, \delta)) \leq C\delta^2$ for all ζ and δ as described above. Thus (3) shows that $\mu_q(S(\zeta, \delta)) \leq C\delta^2$. By a second application of Theorem 1, C_Φ is bounded on D^q .

If C_Φ is compact on D^p , then $\mu_p(S(\zeta, \delta))/\delta^2 \rightarrow 0$ uniformly on $\{\zeta : |\zeta| = 1\}$ as $\delta \rightarrow 0$. Relation (3) shows that μ_q is a compact Carleson measure. By Theorem 2, C_Φ is compact on D^q . $\square$

The following corollary is an immediate consequence of Theorems 2 and 3.

Corollary 1. 1. *If Φ is an analytic self-map of D with bounded multiplicity, then C_Φ is bounded on D^p for $1 \leq p \leq 2$.*
2. *If $\Phi \in D^p$ and $\|\Phi\|_\infty < 1$, then C_Φ is compact on D^q for $1 \leq q \leq p$.*

In [15, p. 892] MacCluer and Shapiro defined a measure ν_α on the disc by $d\nu_\alpha(z) = |\Phi'(z)|^2 (1 - |z|^2)^\alpha dA(z)$ ($\alpha > -1$). The operator C_Φ is bounded (respectively, compact) on the weighted Dirichlet space D_α if and only if the measure $\nu_\alpha \Phi^{-1}$ is α -Carleson (respectively, compact α -Carleson). Details can be found in [15].

The space D_α is said to be heavily weighted if $-1 < \alpha < 0$. If C_Φ is bounded (respectively, compact) on a heavily weighted space, then C_Φ is bounded (compact) on $D_0 = D^2$ [15]. Theorem 3 now implies that C_Φ is bounded (compact) on D^p for $1 \leq p < 2$. Shapiro has shown that compactness on D_α for $-1 < \alpha < 0$ is equivalent to the conditions $\Phi \in D_\alpha$ and $\|\Phi\|_\infty < 1$ [21].

Theorem 4. *For $1 \leq p \leq 2$, let $\alpha = 2 - p$. If C_Φ is bounded (respectively, compact) on D^p , then C_Φ is bounded (compact) on the weighted Dirichlet space D_α .*

Proof. It is enough to consider $1 \leq p < 2$.

Suppose that C_Φ is bounded on D^p and let $\alpha = 2 - p$. Then

$$(4) \quad \nu_\alpha(D) = \int_D |\Phi'(z)|^p |\Phi'(z)|^\alpha (1 - |z|^2)^\alpha dA(z).$$

Since Φ is an analytic self-map of the unit disc, the Schwarz-Pick Lemma guarantees that

$$\frac{1 - |z|^2}{1 - |\Phi(z)|^2} |\Phi'(z)| \leq 1.$$

Thus (4) implies that

$$\nu_\alpha(D) \leq \int_D |\Phi'(z)|^p (1 - |\Phi(z)|^2)^\alpha dA(z) \leq \int_D |\Phi'|^p dA.$$

Since $\Phi \in D^p$, it follows that ν_α is a finite measure on D .

It remains to prove that $\nu_\alpha \Phi^{-1}(S(\zeta, \delta)) \leq C\delta^{2+\alpha}$ for all $|\zeta| = 1$ and $0 < \delta \leq 2$. By the argument given in the previous paragraph,

$$\nu_\alpha \Phi^{-1}(S(\zeta, \delta)) \leq \int_{\Phi^{-1}(S(\zeta, \delta))} |\Phi'(z)|^p (1 - |\Phi(z)|^2)^\alpha dA(z).$$

For $z \in \Phi^{-1}(S(\zeta, \delta))$, $1 - |\Phi(z)| < \delta$. Thus

$$\nu_\alpha \Phi^{-1}(S(\zeta, \delta)) \leq (2\delta)^\alpha \int_{\Phi^{-1}(S(\zeta, \delta))} |\Phi'|^p dA = (2\delta)^\alpha \mu_p(S(\zeta, \delta)).$$

Since C_Φ is bounded on D^p , Theorem 1 implies that $\mu_p(S(\zeta, \delta)) < C\delta^2$. It follows that the measure $\nu_\alpha \Phi^{-1}$ is α -Carleson, as required.

The assertion about compactness is proved in a similar way. $\square$

The following theorem was first suggested by Shapiro [23].

Theorem 5. *If C_Φ is compact on D^p for some $1 \leq p \leq 2$, then C_Φ is compact on the Hardy spaces.*

Proof. Since C_Φ is compact on D^p , Theorem 3 implies that C_Φ is compact on D^1 . Theorem 4 now implies that C_Φ is compact on the weighted Dirichlet space $D_1 = H^2$. As noted in [25], compactness on H^2 is equivalent to compactness on H^q for $0 < q < \infty$. $\square$

Corollary 2. 1. *If Φ is an inner function, then C_Φ is not compact on D^p for any $1 \leq p \leq 2$.*
 2. *If C_Φ is compact on D^p , then the angular derivative exists nowhere on the circle.*
 3. *If C_Φ is compact on D^p , then C_Φ is compact on A_α^q for $0 < q < \infty$ and $\alpha > -1$.*
 4. *Suppose that Φ' is bounded. The operator C_Φ is compact on D^p for all $1 \leq p \leq 2$ if and only if $\|\Phi\|_\infty < 1$.*

Proof. The first three assertions follow from Theorem 5 and the well-known results about compactness on the Hardy spaces and the weighted Bergman spaces [15].

Suppose that Φ' is bounded and $\|\Phi\|_\infty = 1$. It follows that the angular derivative is finite at some point on the circle, and thus C_Φ is not compact on D^p .

The remaining implication follows from Corollary 1. $\square$

There exist inner functions Φ with no finite angular derivative at any point on the circle. Thus the converse of the second assertion in the corollary is false.

Shapiro [21] has shown that there are functions Φ with $\|\Phi\|_\infty = 1$ such that C_Φ is compact on D^2 , the Dirichlet space. Theorem 3 implies that C_Φ is compact on D^p for $1 \leq p \leq 2$. Thus the condition $\|\Phi\|_\infty < 1$ is not necessary for C_Φ to be compact on the Dirichlet-type spaces D^p . In comparison, MacCluer proved that C_Φ is compact on S^p ($1 \leq p < \infty$) if and only if $\Phi \in S^p$ and $\|\Phi\|_\infty < 1$ [14].

Theorem 6. *Suppose $\|\Phi\|_\infty = 1$. If C_Φ is bounded on D^p and the angular derivative of Φ exists at no point of the circle, then C_Φ is compact on D^q for $1 \leq q < p$.*

Proof. As in the proof of Theorem 3, the hypotheses imply that μ_q is a finite measure and

$$(5) \quad \frac{\mu_q(S(\zeta, \delta))}{\delta^2} \leq \left(\frac{\mu_p(S(\zeta, \delta))}{\delta^2} \right)^{q/p} \left(\frac{A\Phi^{-1}(S(\zeta, \delta))}{\delta^2} \right)^{(p-q)/p}.$$

Since the angular derivative of Φ exists nowhere on the unit circle, the measure $A\Phi^{-1}$ is compact Carleson [15]. Thus the second expression on the right in (5) tends to 0 uniformly on $\{\zeta : |\zeta| = 1\}$ as $\delta \rightarrow 0$. Because C_Φ is bounded on D^p , the first expression on the right is bounded. Thus the measure μ_q is compact Carleson, and Theorem 2 implies the result. $\square$

If Φ is bounded multiplicity, then C_Φ is compact on the Hardy spaces if and only if the angular derivative of Φ exists nowhere on the circle [15]. Theorem 7 states the analogous result for the D^p spaces, $1 \leq p < 2$.

Theorem 7. *Suppose that Φ is of bounded multiplicity and $\|\Phi\|_\infty = 1$. The following are equivalent:*

1. *The operator C_Φ is compact on D^p for all $p, 1 \leq p < 2$.*
2. *C_Φ is compact on the Hardy spaces.*
3. *The angular derivative of Φ exists at no point on the circle.*

Proof. It suffices to prove that the third assertion implies the first.

Since Φ has bounded multiplicity, C_Φ is bounded on D^2 . Theorem 6 now implies the result. $\square$

In [11], Jovovic and MacCluer point out that a univalent function mapping D onto a non-tangential approach region will not induce a compact composition operator on the Dirichlet space. Thus univalence and non-existence of the angular derivative are not sufficient to imply that C_Φ is compact on D^2 .

SECTION 2

In this section we present a brief discussion of inner functions which induce bounded composition operators on the space D^p .

If Φ is a finite Blaschke product, then Φ' is bounded, and thus Φ induces a composition operator on D^p for $1 \leq p \leq 2$. D. J. Newman and H. S. Shapiro proved that if Φ is inner and $\Phi \in D^2$, then Φ is a finite Blaschke product [16]. If Φ is an infinite Blaschke product or a singular inner function, then Φ need not belong to D^1 [3, 20]. Since $D^p \subset D^1$ for $1 \leq p \leq 2$, this shows that such an inner function may fail to induce a bounded composition operator on any of the Dirichlet-type spaces.

As a positive example, let $\Lambda(z) = e^{-(1+z)/(1-z)}$ for $z \in D$. It is clear that $\Lambda \notin D^2$. Let $w \in \Lambda(D) = D \setminus \{0\}$ and denote the preimages of w by $z_n(w)$, $n \in \mathbf{Z}$. For $1 \leq p < 2$,

$$N_p(\Lambda, w) = \sum_{n \in \mathbf{Z}} \frac{|1 - z_n(w)|^{2(2-p)}}{(2|w|)^{2-p}}.$$

There is a natural number N and a positive constant C_1 independent of w such that $C_1 < |n| |1 - z_n(w)|$ for $|w| > 1/2$ and $|n| > N$. It follows that $N_{3/2}(\Lambda, w) = \infty$ for all $|w| > 1/2$, and thus $\Lambda \notin D^p$ for $3/2 \leq p \leq 2$.

By similar reasoning there is a positive constant C_2 and a natural number N such that $|n| |1 - z_n(w)| < C_2$ for all $|n| > N$ and all $w \in \Lambda(D)$. It follows that $N_p(\Lambda, w) < C |w|^{p-2}$ for $1 \leq p < 3/2$. Thus μ_p is a finite measure. A calculation shows that μ_p is Carleson. By Theorem 1 C_Λ is bounded on D^p if and only if $1 \leq p < 3/2$.

As a generalization of this example let F be defined by

$$(6) \quad F(z) = \int_{\{x: |x|=1\}} \frac{1+xz}{1-xz} d\mu(x)$$

where μ is a convex combination of finitely many point masses on the circle. For $z \in D$, define $\Psi(z) = e^{-F(z)}$ and $h(z) = (F(z) - 1)/(F(z) + 1)$. Then h is an analytic self-map of the disc and $\Psi = \Lambda \circ h$. Since h is rational and $|h(e^{i\theta})| = 1$ for $0 \leq \theta < 2\pi$, h is a finite Blaschke product. It follows that C_Ψ is a bounded operator on D^p for $1 \leq p < 3/2$.

We close this section by noting a connection between the Besov spaces B_γ and the spaces D^p . A function f analytic in D belongs to B_γ ($0 < \gamma < 1$) if

$$\int_0^{2\pi} \int_0^1 |f(re^{i\theta})| (1-r)^{1/\gamma-2} dr d\theta < \infty.$$

A result of Hardy and Littlewood [9, p. 84] implies that $B_\gamma \subset A^{2\gamma}$. Several authors have studied conditions on inner functions Φ sufficient to imply that $\Phi' \in B_\gamma$, and thus, $\Phi \in D^{2\gamma}$ [1, 2, 3, 8, 18, 20].

SECTION 3

The proofs of Theorems 1 and 2 will be presented in this section. The proofs depend upon Theorem 4.3 in [15], which states that for a finite positive measure ν , the identity map $I : A^p \rightarrow L^p(\nu)$ is bounded (respectively, compact) if and only if ν is Carleson (respectively, compact Carleson).

We may assume that Φ is not constant. Unless otherwise indicated, all integrals in this section are taken over the unit disc D .

For the proof of Theorem 1, assume that C_Φ is bounded on D^p . Let $f \in D^p$ with $f(0) = 0$. The hypothesis implies that

$$\int |f' \circ \Phi|^p |\Phi'|^p dA \leq C \int |f'|^p dA.$$

By a change of variable (see [7], p. 36) and the definition of μ_p ,

$$\int |f' \circ \Phi|^p |\Phi'|^p dA = \int |f'|^p d\mu_p.$$

Let $g \in A^p$ and define $f(z) = \int_0^z g(w) dw$. The argument above shows that

$$(7) \quad \int |g|^p d\mu_p \leq C \int |g|^p dA$$

and it follows that μ_p is a Carleson measure.

For the converse, suppose that μ_p is a Carleson measure. Then (7) holds for every $g \in A^p$. It follows that

$$\int |(f \circ \Phi)'|^p dA = \int |f'|^p d\mu_p \leq C \int |f'|^p dA \leq C \|f\|_{D^p}^p$$

for every $f \in D^p$. Since evaluation at $\Phi(0)$ is a bounded linear functional on D^p , the argument shows that C_Φ is a bounded operator on D^p . The proof of Theorem 1 is complete. $\square$

For the proof of Theorem 2, note that $C_\Phi : D^p \rightarrow D^p$ is compact if and only if $\|f_n \circ \Phi\|_{D^p} \rightarrow 0$ for any bounded sequence (f_n) in D^p with $f_n \rightarrow 0$ uniformly on compact subsets.

Suppose that C_Φ is compact on D^p . Let (g_n) be a bounded sequence in A^p with $g_n \rightarrow 0$ uniformly on compact subsets, and define $f_n(z) = \int_0^z g_n(w) dw$. The hypothesis implies $\|g_n\|_{L^p(\mu_p)} = \|(f_n \circ \Phi)'\|_{A^p} \rightarrow 0$ as $n \rightarrow \infty$ and thus μ_p is compact Carleson.

For the converse, assume that μ_p is compact Carleson. Suppose $\|f_n\|_{D^p} \leq 1$ and $f_n \rightarrow 0$ uniformly on compact subsets. Since $I : A^p \rightarrow L^p(\mu_p)$ is compact, it follows that $\|(f_n \circ \Phi)'\|_{A^p} = \|f_n'\|_{L^p(\mu_p)} \rightarrow 0$. Since $|f_n(\Phi(0))| \rightarrow 0$, the argument yields $\|f_n \circ \Phi\|_{D^p} \rightarrow 0$. Thus C_Φ is compact on D^p . $\square$

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