RANDOM ITERATIONS OF HOLOMORPHIC MAPS IN COMPLEX BANACH SPACES

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ABSTRACT. Conditions guaranteeing the uniform convergence to constant maps of random iterations of holomorphic contractions on unbounded domains in complex Banach spaces are established.

1. INTRODUCTION

The fundamental problem which arises in several areas of mathematics and applications is to understand the behaviour of high random iterations $f_1 \circ f_2 \circ \cdots \circ f_n$ or $f_n \circ f_{n-1} \circ \cdots \circ f_1$ and iterations $f^{[n]} = f \circ f \circ \cdots \circ f$ (n times), $n \in \mathbb{N}$, where f, f_n , $n \in \mathbb{N}$, are self-maps of domains in some spaces (e.g. see [1]–[26] and references therein). Special cases of this type of iterations appear, for example, in fixed point theory, the theory of differential equations, functional equations, numerical analysis, perturbation theory, fractal geometry, computer graphics, population biology, etc.

It is a well-known problem to determine conditions guaranteeing the uniform convergence to constant maps of random iterations $f_1 \circ f_2 \circ \cdots \circ f_n$, $n \in \mathbb{N}$, where f_n , $n \in \mathbb{N}$, are holomorphic self-maps of domains in complex spaces. The solution of this problem is known for an important but special class of domains in finite-dimensional complex spaces $\mathbb{C}$ and $\mathbb{C}^n$ and appeared in [1, 2, 8, 9, 10, 11, 15, 26] (see also references therein). The proofs in the papers cited above are crucially based on: the Riemann mapping theorem, the adaptation of Henrici's technique, the Wolff-Denjoy theorem, the properties of the modulus of equicontinuity of maps or the inequality $K_G(z, w) \geq K_{B(z, R)}(z, w) = \omega(0, \|w - z\|/R) \geq \|w - z\|/R$, $z, w \in G$, when G is bounded, where K_G is the Kobayashi distance of G , ω denotes the Poincaré distance on the unit disc in $\mathbb{C}$ and $G \subset B(z, R) = \{w \in \mathbb{C}^n : \|w - z\| < R\}$. However, the situation is quite different if random iterations are studied on unbounded domains contained in infinite-dimensional complex spaces and, in such a case, it remains an open question whether random iterations of holomorphic self-maps converge uniformly to constant maps. In the present paper we give conditions guaranteeing the uniform convergence to constant maps of random iterations of holomorphic self-maps of domains in complex Banach spaces under a framework where the space is not assumed to be finite-dimensional or the domain to be bounded. Theorem 2.1

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of this paper includes the fixed point theorem of C. J. Earle and R. S. Hamilton [6] and the result from W. J. Zhang and F. J. Ren [26, Theorem 2.1]. The methods of proving Theorem 2.1 differ from those given by the above-mentioned authors. For details, see Section 4.

2. STATEMENT OF RESULT

The statement of the theorem requires a definition. Let E be a complex Banach space. Let G be a nonempty domain (an open and connected set) contained in E . We say that a set $D \subset E$ lies strictly inside G if $\text{dist}[D, E \setminus G] > 0$.

We show

Theorem 2.1. *Let G be a nonempty domain contained in a complex Banach space E . Let a subset D of G be bounded, and suppose that D lies strictly inside G . Let $\mathcal{H}_{G,D} = \{f \in \text{Hol}(G, E) : f(G) \subset D\}$, and suppose that one of the following cases holds:*

- (a) G is convex.
- (b) D is star-shaped (in particular, convex).
- (c) D is connected and contained in a compact subset of G .

Then, for each sequence $\{f_n\}$ from $\mathcal{H}_{G,D}$, there exists a point $w \in G$ such that the sequence $\{F_n\}$, $F_n = f_1 \circ f_2 \circ \cdots \circ f_n$, $n \in \mathbb{N}$, converges to the constant map $x \mapsto w$ uniformly on G .

Maps belonging to the family $\mathcal{H}_{G,D}$ are called holomorphic contractions.

3. PROOF OF THEOREM 2.1

Let $\{f_n\}$ be an arbitrary and fixed sequence from $\mathcal{H}_{G,D}$ and let $F_n = f_1 \circ f_2 \circ \cdots \circ f_n$, $n \in \mathbb{N}$.

Denote by $H^\infty(G)$ the space of holomorphic maps $g : G \rightarrow \mathbb{C}$ such that $\sup\{|g(x)| : x \in G\} < +\infty$. Then $H^\infty(G)$ is a normed vector space with respect to the norm $\|g\| = \sup\{|g(x)| : x \in G\}$. Denote by $H^\infty(G)^*$ its topological dual space.

The evaluation map $\varphi : G \rightarrow H^\infty(G)^*$ is defined by

$$\varphi(x)(g) = g(x) \text{ for } x \in G \text{ and } g \in H^\infty(G).$$

We prove that the map $\varphi : G \rightarrow H^\infty(G)^*$ is holomorphic in G and

$$(3.1) \quad \varphi'(x)(v)(g) = g'(x)(v) \quad \text{for } (x, v) \in G \times E, \quad g \in H^\infty(G).$$

Indeed, let $x \in G$ be arbitrary and fixed and let B be a balanced neighborhood of zero of the form $B = \{v \in E : x + \lambda v \in G, |\lambda| \leq 1\}$. If $v \in B$, then $x + \{\lambda v : |\lambda| \leq 1\}$ is a compact subset of G and hence there exists $R > 1$ such that $x + \{\lambda v : |\lambda| \leq R\} \subset G$. By holomorphy, $\varphi(x + v)(g) = g(x + v) = \sum_{m=0}^{\infty} P_{m,x}(v)(g)$, $g \in H^\infty(G)$, where, for every positive integer m , by the Cauchy integral formula,

$$P_{m,x}(v)(g) = \frac{1}{2\pi i} \int_{|\lambda|=R} \frac{g(x + \lambda v)}{\lambda^{m+1}} d\lambda, \quad g \in H^\infty(G).$$

From the above we obtain $|P_{m,x}(v)(g)| \leq \|g\| R^{-m}$, $g \in H^\infty(G)$, where, by definition, $\|g\| = \sup\{|g(y)| : y \in G\} < +\infty$ and, consequently, $\|P_{m,x}(v)\| \leq R^{-m}$,

$m \in \mathbb{N}$. Now, let us observe that

$$\begin{aligned} & \left\| \varphi(x+v) - \sum_{m=0}^n P_{m,x}(v) \right\| \\ &= \sup \left\{ \left| \left[\varphi(x+v) - \sum_{m=0}^n P_{m,x}(v) \right] (g) \right| : \|g\| \leq 1, g \in H^\infty(G) \right\} \\ &= \sup \left\{ \left| \sum_{m=n+1}^{\infty} P_{m,x}(v)(g) \right| : \|g\| \leq 1, g \in H^\infty(G) \right\} \\ &\leq \sum_{m=n+1}^{\infty} \sup \{ |P_{m,x}(v)(g)| : \|g\| \leq 1, g \in H^\infty(G) \} \\ &= \sum_{m=n+1}^{\infty} \|P_{m,x}(v)\| \leq \sum_{m=n+1}^{\infty} R^{-m}. \end{aligned}$$

Hence $\varphi(x+v) = \sum_{m=0}^{\infty} P_{m,x}(v)$ for each $v \in B$. Thus φ is holomorphic and, moreover, the formula $\varphi'(x)(v)(g) = P_{1,x}(v)(g) = g'(x)(v)$ for $(x, v) \in G \times E$, $g \in H^\infty(G)$, holds and (3.1) is thus proved.

Let $\alpha_G : G \times E \rightarrow \mathbb{R}_+$ be a map of the form

$$(3.2) \quad \alpha_G(x, v) = \|\varphi'(x)(v)\|, \quad (x, v) \in G \times E,$$

and let ρ_G be the integrated form of α_G , i.e.

$$(3.3) \quad \rho_G(a, b) = \inf_{\gamma} L(\gamma), \quad L(\gamma) = \int_0^1 \alpha_G[\gamma(t), \gamma'(t)] dt,$$

for all $a, b \in G$, where the infimum is taken over all curves $\gamma : \langle 0, 1 \rangle \rightarrow G$ with piecewise continuous derivatives, such that $\gamma(0) = a, \gamma(1) = b$. The map ρ_G is pseudometric on G . We call ρ_G the Carathéodory-Reiffen-Finsler pseudometric on G . For details, see [6] and [14].

We prove that if one of assumptions (a)–(c) holds, then

$$(3.4) \quad M_D = \sup\{\rho_G(x, y) : x, y \in D\} < +\infty.$$

Indeed, in case (a), the set G is convex and D lies strictly inside G , and then $\text{co}(D)$ (co stands for the convex hull operator) also lies strictly inside G , and $M_D \leq M_{\text{co}(D)}$. Let $\varepsilon > 0$ be such that $u + B(\varepsilon) \subset G$ for all $u \in M_{\text{co}(D)}$, where $B(\varepsilon) = \{x \in E : \|x\| \leq \varepsilon\}$. Let $g \in H^\infty(G)$ with $\|g\| \leq 1$. If $h \in E \setminus \{0\}$, then $\varepsilon h / \|h\| \in B(\varepsilon)$ and, for $u \in \text{co}(D)$, by the Cauchy integral formula,

$$|g'(u)(h/\|h\|)| = \left| (2\pi i)^{-1} \int_{|\lambda|=\varepsilon} g(u + \lambda h/\|h\|) \lambda^{-2} d\lambda \right| \leq \sup_{v \in G} |g(v)| \leq 1.$$

Therefore

$$(3.5) \quad |g'(u)(h)| \leq \|h\| \quad \text{for all } h \in E \text{ and } u \in \text{co}(D).$$

For any $x, y \in \text{co}(D)$, there exists $\gamma : \langle 0, 1 \rangle \rightarrow \text{co}(D)$ defined by the formula $\gamma(t) = tx + (1-t)y$. By (3.3), we obtain

$$\begin{aligned} (3.6) \quad L(\gamma) &= \int_0^1 \alpha_G[\gamma(t), \gamma'(t)] dt = \int_0^1 \|\varphi'[tx + (1-t)y](y-x)\| dt \\ &\leq \sup\{\|\varphi'[tx + (1-t)y](y-x)\| : t \in \langle 0, 1 \rangle\}. \end{aligned}$$

However,

$$(3.7) \quad \begin{aligned} \|\varphi'(u)(v)\| &= \sup\{|\varphi'(u)(v)(g)| : g \in H^\infty(G), \|g\| \leq 1\} \\ &= \sup\{|g'(u)(v)| : g \in H^\infty(G), \|g\| \leq 1\}, \quad (u, v) \in G \times E. \end{aligned}$$

From (3.5)–(3.7) we obtain $L(\gamma) \leq \sup\{|g'[tx + (1-t)y](y-x)| : t \in (0,1), g \in H^\infty(G), \|g\| \leq 1\} \leq \|y-x\|$. By our assumptions and the definition of ρ_G , if $x, y \in \text{co}(D)$, then $\rho_G(x, y) \leq L(\gamma) \leq \|y-x\| \leq \text{diam}(\text{co}(D))$ and thus $M_D < +\infty$.

(b) Let D be star-shaped with respect to $s \in D$. For any $x_i \in D$, there exist $\gamma_i : \langle 0, 1 \rangle \rightarrow D$ defined by the formulae $\gamma_i(t) = ts + (1-t)x_i$, $i = 1, 2$. By the assumptions and the definition of ρ_G , if $x_1, x_2 \in D$, then $\rho_G(x_1, x_2) \leq \sum_{i=1}^2 L(\gamma_i) \leq \sum_{i=1}^2 \|s - x_i\|$ and thus $M_D \leq \sup\{L(\gamma_1) + L(\gamma_2) : x_1, x_2 \in D\} \leq 2 \text{diam}(D) < +\infty$.

(c) In this case, there exist $a, b \in \overline{D}$ such that $M_D = \sup\{\rho_G(x, y) : x, y \in D\} = \rho_G(a, b)$. Let $\varepsilon > 0$ be such that $u + B(\varepsilon) \subset G$ for all $u \in D$, where $B(\varepsilon) = \{x \in E : \|x\| \leq \varepsilon\}$. Since $\overline{D}$ is connected, there exist $k \in \mathbb{N}$ and $\gamma_i(t) = tu_i + (1-t)u_{i+1}$, $i = 1, \dots, k$, such that $u_i \in \overline{D}$ for $i = 1, \dots, k+1$, $a = u_1$ and $b = u_{k+1}$, $u_i, u_{i+1} \in U_i$ for $i = 1, \dots, k-1$, and $U_i \cap U_j \neq \emptyset$ iff $|i-j| \leq 1$, $u_i \in U_{i-1} \cap U_i$ for $i = 2, \dots, k-1$ and $U_i = u_i + B(\varepsilon) \subset G$ for $i = 1, \dots, k$; then $M_D \leq k \cdot \text{diam}(D) < +\infty$.

We prove that

$$(3.8) \quad \|f_n(x) - f_n(y)\| \leq \text{diam}(D) \cdot \|\varphi(x) - \varphi(y)\| \quad \text{for all } x, y \in G.$$

Indeed, suppose that $x, y \in G$. By the Hahn-Banach theorem, there exists $l \in E^*$ such that

$$(3.9) \quad l[f_n(x) - f_n(y)] = \|f_n(x) - f_n(y)\| \quad \text{and} \quad |l(u)| \leq \|u\| \quad \text{for all } u \in E.$$

Let g_n be the map defined by the formula

$$(3.10) \quad g_n(u) = (1/\text{diam}(D))(l \circ f_n)(u), \quad u \in G.$$

We may assume without loss of generality that $0 \in D$ to obtain

$$(3.11) \quad g_n \in H^\infty(G), \quad \|g_n\| \leq 1,$$

and, moreover,

$$(3.12) \quad |g_n(x) - g_n(y)| = |[\varphi(x) - \varphi(y)](g_n)| \leq \|\varphi(x) - \varphi(y)\| \|g_n\|.$$

From (3.12), using (3.9)–(3.11), we get (3.8).

Let $G, G_1 \subset E$ be domains and let $k : G \rightarrow E$ be a holomorphic map such that $k(G) \subset G_1$. Then, by using (3.2) for the domains G and G_1 ,

$$(3.13) \quad \alpha_{G_1}[k(x), k'(x)(v)] \leq \alpha_G(x, v), \quad (x, v) \in G \times E.$$

Indeed, then k induces a map $H^\infty(k) : H^\infty(G_1) \rightarrow H^\infty(G)$ by the formula $H^\infty(k)(h) = h \circ k$, $h \in H^\infty(G_1)$, and $H^\infty(k)$ is linear continuous and $\|H^\infty(k)\| \leq 1$. Therefore the adjoint map $H^\infty(k)^* : H^\infty(G)^* \rightarrow H^\infty(G_1)^*$ is also linear continuous and $\|H^\infty(k)^*\| \leq 1$. Let $\varphi : G \rightarrow H^\infty(G)^*$ and $\psi : G_1 \rightarrow H^\infty(G_1)^*$ be the respective evaluation maps. Then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H^\infty(G)^* \\ k \downarrow & & \downarrow H^\infty(k)^* \\ G_1 & \xrightarrow{\psi} & H^\infty(G_1)^* \end{array}$$

commutes. Consequently,

$$\begin{aligned}\alpha_{G_1}[k(x), k'(x)(v)] &= \|\psi'(k(x))(k'(x)(v))\| = \|H^\infty(k)^*(\varphi'(x)(v))\| \\ &\leq \|\varphi'(x)(v)\| = \alpha_G(x, v), \quad (x, v) \in G \times E,\end{aligned}$$

i.e. (3.13) holds.

Suppose that $a \in G$. The set $f_n(G) - f_n(G)$ is absorbed by $B(\varepsilon)$ (since $f_n(G) - f_n(G)$ is bounded) and thus there exists $\delta > 0$ such that $\delta[f_n(G) - f_n(G)] \subset B(\varepsilon)$. Consequently, the map k_n defined by the formula

$$k_n(x) = f_n(x) + \delta[f_n(x) - f_n(a)], \quad x \in G,$$

is a holomorphic map of G into G , $k_n(a) = f_n(a)$ and $k'_n(a) = (1 + \delta)f'_n(a)$. Thus, by (3.13), we have $\alpha_G[k_n(a); k'_n(a)v] \leq \alpha_G(a; v)$ for all $v \in E$ and, consequently,

$$(3.14) \quad \alpha_G[f_n(a); f'_n(a)(v)] \leq \lambda \cdot \alpha_G(a; v) \quad \text{for all } v \in E,$$

where $\lambda = (1 + \delta)^{-1} < 1$.

Let ρ_G be the integrated form of α_G defined by (3.3). Then $f_n \circ \gamma$ is a piecewise C^1 -curve in G from $f_n(a)$ to $f_n(b)$ and (3.14) implies

$$(3.15) \quad \alpha_G[(f_n \circ \gamma)(t); (f_n \circ \gamma)'(t)] \leq \lambda \cdot \alpha_G[\gamma(t); \gamma'(t)] \quad \text{for all } t \in \langle 0, 1 \rangle.$$

Integrating (3.15) on both sides with respect to $t \in \langle 0, 1 \rangle$, by (3.3), we have

$$(3.16) \quad \rho_G[f_n(a), f_n(b)] = \inf_{\gamma} L(f_n \circ \gamma) \leq \lambda \cdot \inf_{\gamma} L(\gamma) = \lambda \cdot \rho_G(a, b)$$

for all $a, b \in G$.

From (3.2) and (3.3) we get the estimate

$$(3.17) \quad \|\varphi(x) - \varphi(y)\| \leq L(\gamma)$$

for each piecewise C^1 -curve γ from x to y in G . Thus from (3.8) and (3.3) we get

$$(3.18) \quad \|f_n(x) - f_n(y)\| \leq \text{diam}(D) \cdot \rho_G(x, y) \quad \text{for all } x, y \in G.$$

Suppose that $a \in G$. For any $n, m \in \mathbb{N}$ and all $b \in G$, by (3.18), (3.16) and (3.4), since the class of holomorphic maps has the composition property, we obtain

$$(3.19) \quad \|F_n(a) - F_n(b)\| \leq \text{diam}(D) \cdot \lambda^{n-1} \cdot \rho_G[f_n(a), f_n(b)] \leq \text{diam}(D) \cdot M_D \cdot \lambda^{n-1},$$

$$(3.20) \quad \begin{aligned}\|F_n(a) - F_{n+m}(b)\| &\leq \text{diam}(D) \cdot \lambda^{n-1} \cdot \rho_G[f_n(a), (f_n \circ f_{n+1} \circ \cdots \circ f_{n+m})(b)] \\ &\leq \text{diam}(D) \cdot M_D \cdot \lambda^{n-1}.\end{aligned}$$

Finally, from (3.20) and (3.4) it follows that, for $b = a$, the sequence $\{F_n(a)\}$ is a Cauchy sequence and since $F_n(a) + B(\varepsilon) \subset G$, there exists $w \in G$ such that $\{F_n(a)\}$ converges to w . Furthermore, inequalities (3.19) imply $\|F_n(b) - w\| \leq \|F_n(a) - w\| + \text{diam}(D) \cdot M_D \cdot \lambda^{n-1}$, $b \in G$. Consequently,

$$\forall \eta > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \forall b \in G \quad \|F_n(b) - w\| < \eta,$$

i.e. the sequence $\{F_n\}$ converges to w uniformly on G . $\square$

4. CONCLUDING REMARKS

4.1. If $H^\infty(G)$ contains an injective map g , then ρ_G is metric. Indeed, by the definition of φ , we have $|g(x) - g(y)| = |[\varphi(x) - \varphi(y)](g)| \leq \|\varphi(x) - \varphi(y)\| \|g\|$. Thus φ is injective. A consequence of (3.17) and (3.3) is the inequality $\|\varphi(x) - \varphi(y)\| \leq \rho_G(x, y)$ and, by the injectivity of φ , we get the assertion.

4.2. The conclusion of Theorem 2.1 remains valid if assumptions (a)–(c) are replaced by more general assumption (3.4).

4.3. We derive a short proof of Theorem 2.1 in the special case when G is a nonempty bounded domain. In this case, there exists $R > 0$ such that, for any $a \in G$, we have $G \subset B(a, R) = \{x \in E; \|x - a\| < R\}$. Moreover, we get

$$\rho_{B(a,R)}(a, x) = \omega[0, \|x - a\|/R] = \sum_{n=0}^{+\infty} [1/(2n+1)] [\|x - a\|/R]^{2n+1} \geq \|x - a\|/R,$$

$x \in B(a, R)$, where $\omega(\cdot, \cdot)$ is a Poincaré metric on $\Delta \times \Delta$, $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Consequently, for all $a, b \in G$, we have

$$(4.1) \quad \rho_G(a, b) \geq \rho_{B(a,R)}(a, b) \geq \|b - a\|/R.$$

From (4.1) and (3.16) we obtain

$$\begin{aligned} \|F_n(a) - F_n(b)\| &\leq R \cdot \lambda^{n-1} \cdot \rho_G[f_n(a), f_n(b)] \leq R \cdot M_D \cdot \lambda^{n-1}, \\ \|F_n(a) - F_{n+m}(b)\| &\leq R \cdot \lambda^{n-1} \cdot \rho_G[f_n(a), (f_n \circ f_{n+1} \circ \cdots \circ f_{n+m})(b)] \\ &\leq R \cdot M_D \cdot \lambda^{n-1}. \end{aligned}$$

Further reasoning is analogous. Theorem 2.1 includes the result from [26, Theorem 2.1].

4.4. One of the most celebrated theorems of the fixed point theory of holomorphic maps is the theorem of C. J. Earle and R. S. Hamilton [6] which states that each holomorphic map $f : G \rightarrow G$, where G is a bounded domain in E and $f(G)$ lies strictly inside G , has a unique fixed point $w \in G$, and $f^{[n]}(a) \rightarrow w$ for any $a \in G$. Their result depends on the fact that $\text{diam}(G) < +\infty$ and one can define a metric on G so that if f maps G into G and is holomorphic, then f is nonexpansive, and if f maps G strictly inside G , then f is a strict contraction in this metric.

Theorem 2.1 includes the Earle-Hamilton fixed point theorem [6]. Indeed, maintaining the notation of the previous sections, suppose $G \subset E$ to be a bounded or unbounded domain, $f : G \rightarrow G$ holomorphic and $f(G)$ bounded and lying strictly inside G . If $b = f(a)$, then from (3.19) it follows that

$$\|f^{[n]}(a) - f^{[n+1]}(a)\| \leq \text{diam}(f(G)) \cdot \lambda^n \cdot \rho_G(a, f(a)).$$

Consequently, $\{w_n\}$, $w_n = f^{[n]}(a)$, $n \in \mathbb{N}$, is a Cauchy sequence and thus converges to some $w \in G$. Since $w_n = f(w_{n-1})$, therefore, by the continuity of f , we obtain $f(w) = w$. If $w' \in G$ and $f(w') = w'$, then, by (3.16), we have $\rho_G(w, w') = \rho_G(f(w), f(w')) \leq \lambda \cdot \rho_G(w, w')$, which implies $\rho_G(w, w') = 0$. Furthermore, from (3.18) we then get $\|w - w'\| = \|f(w) - f(w')\| \leq \text{diam}(f(G)) \cdot \rho_G(w, w') = 0$. Thus $w = w'$.

4.5. More recent important results in fixed point theory, concerning the Denjoy-Wolff Theorem, are established by C.-H. Chu and P. Mellon [3], S. Reich and D. Shoikhet [17] and R. C. Sine [18].

4.6. By means of a different set of ideas, I. N. Baker and P. J. Rippon [1, 2], J. Dye, M. A. Khamsi and S. Reich [4], J. M. Dye and S. Reich [5], J. Gill [8]–[11], L. Lorentzen [15] and S. Reich [16] showed the convergence of random iterations of various types of contractions.

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