A PALEY-WIENER THEOREM FOR THE SPHERICAL LAPLACE TRANSFORM ON CAUSAL SYMMETRIC SPACES OF RANK 1

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(Communicated by Roe Goodman)

ABSTRACT. We formulate and prove a topological Paley-Wiener theorem for the normalized spherical Laplace transform defined on the rank 1 causal symmetric spaces $\mathcal{M} = SO_o(1, n)/SO_o(1, n-1)$, for $n \geq 2$.

INTRODUCTION

The spherical Laplace transform on causal symmetric spaces was introduced in [FHO, §8] as a generalization of the spherical Fourier transform on Riemannian symmetric spaces defined by Helgason (see [H1, Chapter 4]). Both transforms can be expressed in terms of (integrating against) spherical functions. It was furthermore shown in [O1, §5] that the spherical functions on the Riemannian dual of a causal symmetric space can be written as an expansion in spherical functions on the causal symmetric space. The inversion formula for the spherical Laplace transform easily follows (see [O1, §6]).

One of the most important results on the spherical Fourier transform is the (topological) Paley-Wiener theorem (see [H1, Chapter 4, §7] and [H2, Chapter 3, §5] for details) generalizing the classical Paley-Wiener theorem on Euclidean spaces. In this paper we generalize these results to the normalized spherical Laplace transform on causal symmetric spaces $\mathcal{M}$ of rank 1, thereby partially solving an open problem posed by the second author in [O2, §5].

The paper is divided into two sections: in the first section we recall some results on the spherical Fourier transform on the Riemannian dual $\mathcal{M}^d$ of $\mathcal{M}$, and in the second we consider the spherical Laplace transform defined on $\mathcal{M}$. We define the Paley-Wiener space, the supposed image space of spherical Laplace transform, according to the growth and symmetry conditions satisfied by the spherical functions on $\mathcal{M}$. The Paley-Wiener theorem for the normalized spherical Laplace transform follows by using Cauchy's theorem to change the path of integration in the inversion formula and from the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}^d$.

Received by the editors December 9, 1998 and, in revised form, March 22, 1999.

2000 *Mathematics Subject Classification*. Primary 43A85, 22E30; Secondary 43A90, 33C60.

The first author was supported by a postdoc fellowship from the European Commission within the European TMR Network "Harmonic Analysis" 1998-2001 (Contract ERBFMRX-CT97-0159). The second author was supported by LEQSF grant (1996-99)-RD-A-12.

We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible; we refer the reader to [FHO], [HO], [O1] and [O2] for more details on spherical functions and the spherical Laplace transform defined on causal symmetric spaces. The spherical Laplace transform in the rank 1 case can be considered as a Laplace-Jacobi transform (see [M] for a detailed analysis of the latter transform) but we note that the Paley-Wiener theorem is new even in the rank one case.

We would like to thank H. Schlichtkrull and J.M. Unterberger for helpful discussion and comments, in particular concerning Lemma 7 and its proof.

NOTATION AND PRELIMINARIES

In the following we consider the causal symmetric (real hyperbolic) space $\mathcal{M} = G/H$ with $G = SO_o(1, n)$ and $H = SO_o(1, n-1)$ and its Riemannian dual $\mathcal{M}^d = G/K$, where $K = SO_o(n)$. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{a}$ be the abelian subalgebra of $\mathfrak{g}$ given by

$$\mathfrak{a} = \left\{ X_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

We choose the unique positive root $\alpha \in \mathfrak{a}^*$ as $\alpha(X_t) = t$. Let $\mathfrak{a}^+ = \{X_t \in \mathfrak{a} | t > 0\}$. We identify the complex dual $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}$ with $\mathbb{C}$ via the map $\mathbb{C} \ni z \mapsto z\alpha \in \mathfrak{a}_{\mathbb{C}}^*$. Let $\mathfrak{n} = \mathfrak{g}_{\alpha}$ and $\overline{\mathfrak{n}} = \mathfrak{g}_{-\alpha}$ denote the positive and negative root space respectively. Let $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, $N = \exp \mathfrak{n}$ and $\overline{N} = \exp \overline{\mathfrak{n}}$, where $\exp$ is the exponential mapping from $\mathfrak{g}$ to G . We also consider the open semigroup $S^o = HA^+H$ in G . Let finally $a_t = \exp X_t \in A$.

Let $\eta : \mathcal{D}(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{M}^d)$ denote the Flensted-Jensen isomorphism between the commutative algebras of invariant differential operators on $\mathcal{M}$ and $\mathcal{M}^d$ respectively (mapping the Laplace-Beltrami operator Δ on $\mathcal{M}$ onto the Laplace-Beltrami operator $\Delta^d = \eta(\Delta)$ on $\mathcal{M}^d$). Let $\Pi(D)$ and $\Pi^d(D^d)$ denote the radial part (on A^+) of $D \in \mathcal{D}(\mathcal{M})$ and $D^d \in \mathcal{D}(\mathcal{M}^d)$ respectively. There exists a unique map $C_c^\infty(H \backslash S^o / H) \ni f \mapsto f^d \in C_c^\infty(K \backslash G / K)$ such that $f|_{A^+} = f^d|_{A^+}$ and $\Pi(D)f|_{A^+} = \Pi^d(\eta(D))f^d|_{A^+}$ (see [HO] or [O1, §4] for more details).

THE SPHERICAL FOURIER TRANSFORM ON $\mathcal{M}^d$

In this section we recall some well-known definitions and results for the spherical Fourier transform on a Riemannian symmetric space (see e.g. [H1, Chapter 4] and [H2, Chapter 3]).

Let $\lambda \in \mathbb{C}$. Define the Poisson kernel for $\mathcal{M}^d$ by

$$KAN \ni kan = x \mapsto a^{-(\lambda+\rho)} =: p_\lambda^d(x),$$

where $\rho = (n-1)/2$. The spherical functions on $\mathcal{M}^d$ can be written as

$$\psi_\lambda^d(a_t) = \int_K p_{-\lambda}^d(a_t k) dk = {}_2F_1 \left(\frac{1}{2}(\lambda + \rho), \frac{1}{2}(-\lambda + \rho); \rho + \frac{1}{2}; -\sinh^2 t \right),$$

for $a_t \in A$. The spherical functions are bi- K -invariant and $\Delta^d \psi_\lambda^d = (\lambda^2 - \rho^2) \psi_\lambda^d$ for all $\lambda \in \mathbb{C}$. They satisfy the following inequality, for all $\lambda \in \mathbb{C}$ and all $a_t \in A$:

$$|\psi_\lambda^d(a_t)| \leq c(1 + |t|)e^{(|\operatorname{Re} \lambda| - \rho)|t|},$$

for some constant c , and they are invariant under sign change, i.e. $\psi_{-\lambda}^d = \psi_\lambda^d$.

The spherical Fourier transform $\mathcal{F}$ on $\mathcal{M}^d$ is defined as

$$\mathcal{F}(f)(\lambda) = \int_G f(x) \psi_\lambda^d(x) dx = \int_{A^+} f(a) \psi_\lambda^d(a) \delta(a) da,$$

for $f \in C_c^\infty(K \backslash G/K)$, where $\delta(a_t) = \sinh^{2\rho} t$. Let

$$c^d(\lambda) := \int_{\overline{N}} p_\lambda^d(\overline{n}) d\overline{n} = \frac{\Gamma(2\rho)}{\Gamma(\rho)} \frac{\Gamma(\lambda)}{\Gamma(\lambda + \rho)}$$

denote the Harish-Chandra c -function for $\mathcal{M}^d$. We note that $|c^d(\lambda)|^2 = c^d(\lambda) c^d(-\lambda)$ for $\lambda \in i\mathbb{R}$. The inversion formula for $\mathcal{F}$ reads (after normalizing $d\lambda$ suitably):

$$f(x) = \int_{i\mathbb{R}} \mathcal{F}(f)(\lambda) \psi_{-\lambda}^d(x) |c^d(\lambda)|^{-2} d\lambda,$$

for all $f \in C_c^\infty(K \backslash G/K)$ and $x \in G$.

Let $R > 0$. Let $C_R^\infty(K \backslash G/K) := \{f \in C_c^\infty(K \backslash G/K) \mid f(a_t) = 0 \text{ for } t > R\}$ and define the Paley-Wiener space $\mathcal{H}_R(\mathbb{C})$ as the space of even holomorphic functions g on $\mathbb{C}$ of exponential type R , i.e. satisfying the estimate

$$\sup_{\lambda \in \mathbb{C}} e^{-R|\operatorname{Re} \lambda|} (1 + |\lambda|)^N |g(\lambda)| < \infty,$$

for all $N \in \mathbb{N} \cup \{0\}$. Furthermore denote by $\mathcal{H}(\mathbb{C})$ the union of the spaces $\mathcal{H}_R(\mathbb{C})$, $R > 0$.

Theorem 1 (The Paley-Wiener Theorem). *The Fourier transform is a topological linear isomorphism from $C_c^\infty(K \backslash G/K)$ onto $\mathcal{H}(\mathbb{C})$. More precisely it is a topological linear isomorphism from $C_R^\infty(K \backslash G/K)$ onto $\mathcal{H}_R(\mathbb{C})$ for all $R > 0$.*

THE SPHERICAL LAPLACE TRANSFORM ON $\mathcal{M}$

We define spherical functions on $\mathcal{M}$ according to [O1, Definition 4.1]:

Definition 2. An H -bi-invariant continuous function $\varphi : S^\circ \rightarrow \mathbb{C}$ is called a spherical function if there exists a character χ of $\mathcal{D}(\mathcal{M})$ such that (in the sense of distributions) $D\varphi = \chi(D)\varphi$ for all $D \in \mathcal{D}(\mathcal{M})$.

Define the Poisson kernel for $\mathcal{M}$ (and the open orbit HAN) by

$$HAN \ni han = x \mapsto a^{-(\lambda+\rho)} =: p_\lambda(x)$$

and $p_\lambda \equiv 0$ on $G \backslash HAN$. We can construct spherical functions φ_λ as follows:

$$\varphi_\lambda(x) := \int_H p_{-\lambda}(xh) dh,$$

for $x \in S^\circ$, whenever this integral exists (see [O1, Theorem 4.10]). Using the calculations in [FHO, §10] we see that the integral converges for $x \in S^\circ$ and $\operatorname{Re} \lambda < 1 - \rho$ and we get the following explicit formula for φ_λ :

$$\varphi_\lambda(a_t) = c(\lambda) (2 \cosh t)^{\lambda-\rho} {}_2F_1 \left(\frac{1}{2}(-\lambda + \rho), \frac{1}{2}(-\lambda + \rho + 1); 1 - \lambda; \cosh^{-2} t \right),$$

where

$$c(\lambda) := \int_{\overline{N}} p_\lambda(\overline{n}) d\overline{n} = 2^{2\rho-1} \Gamma(\rho) \frac{\Gamma(-\lambda - \rho + 1)}{\Gamma(-\lambda + 1)}.$$

We note that $\Delta \varphi_\lambda = (\lambda^2 - \rho^2) \varphi_\lambda$ when defined.

Let φ_λ^o denote the normalized spherical function $\varphi_\lambda^o(x) = c(\lambda)^{-1}\varphi_\lambda(x)$. We have the following uniform growth estimate on φ_λ^o (due to Helgason (rank one) and Gangolli):

Lemma 3. *Fix $\sigma > 0$. There exists a constant c_σ such that*

$$|\varphi_\lambda^o(a_t)| \leq c_\sigma e^{(\operatorname{Re}\lambda - \rho)t},$$

for $\operatorname{Re}\lambda \leq 0$ and all $t \in]\sigma, \infty[$.

Sketch of the proof. Consider a formal power series solution to the differential equation $\Pi(\Delta)\varphi_\lambda^o = \delta(t)^{-1} \frac{\partial}{\partial t}(\delta(t) \frac{\partial \varphi_\lambda^o}{\partial t}) = (\lambda^2 - \rho^2)\varphi_\lambda^o$ of the form $e^{(\lambda - \rho)t} \sum_{n=0}^{\infty} e^{-nt} \Gamma_n(\lambda)$, with $\Gamma_n(\lambda)$ to be determined (and $\Gamma_0 \equiv 1$). Substitution into the differential equation gives a recurrence formula that defines $\Gamma_n(\lambda)$ uniquely for $\lambda \notin \frac{1}{2}\mathbb{N}$. We can estimate $\Gamma_n(\lambda)$ as follows: There exist constants $c, \varkappa > 0$ such that

$$|\Gamma_n(\lambda)| < c(1+n)^\varkappa,$$

for $\operatorname{Re}\lambda \leq 0$ and all $n \in \mathbb{N}$. The above estimate on φ_λ^o follows easily. See [BS, §7-9] for a complete proof in a more general setup. $\square$

We define the normalized spherical Laplace transform $\mathcal{L}^o$ on $\mathcal{M}$ as

$$\mathcal{L}^o(f)(\lambda) = \int_{A^+} f(a) \varphi_\lambda^o(a) \delta(a) da,$$

for $f \in C_c^\infty(A^+) \cong C_c^\infty(H \backslash S^o / H)$ and $\operatorname{Re}\lambda < 1 - \rho$. From the explicit formula for φ_λ^o , we see that the function $\lambda \mapsto \mathcal{L}^o(f)(\lambda)$ extends to a meromorphic function on $\mathbb{C}$ with at most poles for $\lambda \in \mathbb{N}$.

Let $R > r > 0$ and define $C_{r,R}^\infty(A^+) := \{f \in C_c^\infty(A^+) | f(a_t) = 0 \text{ for } 0 < t < r \text{ and } t > R\}$. We equip $C_{r,R}^\infty(A^+)$ with the natural Fréchet space topology and $C_c^\infty(A^+)$ with the inductive limit topology. From Lemma 3 we get the following uniform growth estimate on the normalized spherical Laplace transform acting on $C_{r,R}^\infty(A^+)$:

Lemma 4. *Let $R > r > 0$ and let $N \in \mathbb{N} \cup \{0\}$. There exists a constant $c > 0$, only depending on r and R , such that, for all $f \in C_{r,R}^\infty(A^+)$*

$$\sup_{\operatorname{Re}\lambda \leq 0} e^{-r\operatorname{Re}\lambda} (1 + |\lambda|^2)^N |\mathcal{L}^o f(\lambda)| \leq c \sum_{n=0}^N \|(\Delta + \rho^2)^n f\|_\infty < \infty.$$

Proof. Since $\mathcal{L}^o(\Delta f)(\lambda) = (\lambda^2 - \rho^2)\mathcal{L}^o(f)(\lambda)$ and $\lambda^2 = \lambda^2 - \rho^2 + \rho^2$, we easily get

$$\begin{aligned} (1 + |\lambda|^2)^N |\mathcal{L}^o(f)(\lambda)| &= \sum_{n=0}^N \binom{N}{n} |\lambda|^{2n} |\mathcal{L}^o(f)(\lambda)| \\ &= \sum_{n=0}^N \binom{N}{n} |\mathcal{L}^o((\Delta + \rho^2)^n f)(\lambda)| \leq c e^{r\operatorname{Re}\lambda} \sum_{n=0}^N \binom{N}{n} \|(\Delta + \rho^2)^n f\|_\infty, \end{aligned}$$

for $\operatorname{Re}\lambda \leq 0$ and all $f \in C_{r,R}^\infty(A^+)$, where $c > 0$ is a constant only depending on r and R . $\square$

Using the correspondence between (the radial parts of) invariant differential operators on $\mathcal{M}$ respectively on $\mathcal{M}^d$ (see also [O1, Theorem 5.9]), we get

$$\psi_\lambda^d(a) = c^d(\lambda) \varphi_\lambda^o(a) + c^d(-\lambda) \varphi_{-\lambda}^o(a),$$

for $a \in A^+$ and $\lambda \notin \mathbb{Z} \setminus \{0\}$ (or use [O1, p.63, Eq (17) & p.64, Eq (22)] and the hypergeometric expressions for the spherical functions), which is the Harish-Chandra expansion formula for ψ_λ^d . Let $f \in C_c^\infty(A^+)$. We see that $\mathcal{L}^o f$ satisfies the following functional equation:

$$c^d(\lambda)\mathcal{L}^o(f)(\lambda) + c^d(-\lambda)\mathcal{L}^o(f)(-\lambda) = \mathcal{F}(f^d)(\lambda),$$

for $\lambda \notin \mathbb{Z} \setminus \{0\}$. The inversion formula for the normalized spherical Laplace transform is now an easy consequence of the inversion formula for the spherical Fourier transform (see also [M, p. 993]):

Theorem 5 (The Inversion Formula). *Let $f \in C_c^\infty(A^+)$. Then*

$$f(a) = 2 \int_{i\mathbb{R}} \mathcal{L}^o(f)(\lambda) \psi_{-\lambda}^d(a) c^d(-\lambda)^{-1} d\lambda,$$

for all $a \in A^+$.

All the above suggests the following definition of the Paley-Wiener space, the supposed image space of the normalized spherical Laplace transform:

Definition 6. Let $R > r > 0$. We define the Paley-Wiener space $PW_{r,R}(\mathbb{C})$ as the space of meromorphic functions g on $\mathbb{C}$, with at most poles for $\lambda \in \mathbb{N}$, such that

(i)

$$\sup_{\operatorname{Re} \lambda \leq 0} e^{-r \operatorname{Re} \lambda} (1 + |\lambda|)^N |g(\lambda)| < \infty,$$

for all $N \in \mathbb{N} \cup \{0\}$, and

(ii) the c^d -weighted average $P^{\text{av}} g(\lambda) := c^d(\lambda)g(\lambda) + c^d(-\lambda)g(-\lambda)$ extends to a function in $\mathcal{H}_R(\mathbb{C})$.

Furthermore denote by $PW(\mathbb{C})$ the union of the spaces $PW_{r,R}(\mathbb{C})$ over all $R > r > 0$.

We define a Fréchet space topology on $PW_{r,R}(\mathbb{C})$ by means of the seminorms

$$\sigma_{r,N}(g) = \sup_{\operatorname{Re} \lambda \leq 0} e^{-r \operatorname{Re} \lambda} (1 + |\lambda|)^N |g(\lambda)|$$

and

$$\sigma_{R,N}(g) = \sup_{\lambda \in \mathbb{C}} e^{-R|\operatorname{Re} \lambda|} (1 + |\lambda|)^N |P^{\text{av}} g(\lambda)|.$$

We furthermore equip the space $PW(\mathbb{C})$ with the inductive limit topology.

We remark that $P^{\text{av}} \mathcal{L}^o$ acts injectively on $C_c^\infty(A^+)$, since $P^{\text{av}} \mathcal{L}^o(f) = \mathcal{F}(f^d) = 0$ implies $f = f^d = 0$ on A^+ for any $f \in C_c^\infty(A^+)$, by the injectivity of the spherical Fourier transform is injective. But we will need that P^{av} is injective on $PW(\mathbb{C})$. The following lemma and its proof was communicated to us by H. Schlichtkrull.

Lemma 7. *Let g be a meromorphic function on $\mathbb{C}$ that satisfies item (i) of Definition 6. Assume that $P^{\text{av}} g = 0$. Then $g = 0$.*

Proof. Let $g^1(\lambda) = g(\lambda)/c^d(-\lambda)$. Then $P^{\text{av}} g(\lambda) = 2c^d(\lambda)c^d(-\lambda)\operatorname{avg}^1(\lambda)$, where

$$\operatorname{avg}^1(\lambda) := \frac{1}{2}[g^1(\lambda) + g^1(-\lambda)]$$

is the average of g^1 over the Weyl group ± 1 . It follows from the assumption $\text{P}^{\text{av}}g = 0$ that $\text{av}g^1 = 0$. Let

$$\gamma(s) = \int_{\mathbb{R}} g^1(i\lambda) e^{-is\lambda} d\lambda, \quad s \in \mathbb{R},$$

denote the Euclidean Fourier transform of $g^1(i \cdot)$. It follows from (i) and [H1, Proposition IV.7.2] that (i) is satisfied by g^1 as well. In particular, $g^1(i \cdot) \in L^1(\mathbb{R})$. The condition (i) implies that g is holomorphic in an open set containing $\{z \in \mathbb{C} \mid \text{Re} z \leq 0\}$. Moreover, the standard argument with Cauchy's theorem shows that γ is supported on $[r, \infty[$. On the other hand, the average $\text{av}\gamma$ of γ is the Fourier transform of $\text{av}g^1(i \cdot)$, which vanishes; hence $\text{av}\gamma$ vanishes as well. The support condition now implies that $\gamma = 0$. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R})$, we conclude that g^1 , and therefore also g , vanishes. $\square$

Theorem 8 (The Paley-Wiener Theorem). *The normalized spherical Laplace transform $\mathcal{L}^\circ$ is a topological linear isomorphism from $C_c^\infty(A^+)$ onto $PW(\mathbb{C})$. More precisely it is a topological linear isomorphism from $C_{r,R}^\infty(A^+)$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.*

Proof. By the Paley-Wiener theorem for the spherical Fourier transform, Lemma 4 and the open mapping theorem for Fréchet spaces, it only remains to show that the normalized spherical Laplace transform maps $C_{r,R}^\infty(A^+)$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.

Consider the wave packet $\mathcal{I}g \in C^\infty(A^+)$ of $g \in PW_{r,R}(\mathbb{C})$ defined by

$$\mathcal{I}g(a) = 2 \int_{i\mathbb{R}} g(\lambda) \psi_{-\lambda}^d(a) c^d(-\lambda)^{-1} d\lambda,$$

for $a \in A^+$. By Cauchy's theorem we get, for $0 < t < r$ and $\mu < 0$,

$$\begin{aligned} \mathcal{I}g(a_t) &= 2 \int_{i\mathbb{R}} g(\lambda) \psi_{-\lambda}^d(a_t) c^d(-\lambda)^{-1} d\lambda \\ &= 2 \int_{i\mathbb{R}} g(\lambda + \mu) \psi_{-\lambda-\mu}^d(a_t) c^d(-\lambda - \mu)^{-1} d\lambda \\ &\rightarrow 0 \quad \text{for } \mu \rightarrow -\infty, \end{aligned}$$

since $|\psi_{-\lambda}^d(a_t) c^d(-\lambda)^{-1}| \leq c(1 + |\lambda|)^\rho (1 + t) e^{(|\text{Re}\lambda| - \rho)t}$ for $\text{Re}\lambda \geq 0$, for some constant $c > 0$. An easy calculation shows that

$$\begin{aligned} \mathcal{I}g(a_t) &= 2 \int_{i\mathbb{R}} g(\lambda) \psi_{-\lambda}^d(a_t) c^d(-\lambda)^{-1} d\lambda \\ &= \sum_{w=\pm 1} \int_{i\mathbb{R}} g(w\lambda) \psi_{-w\lambda}^d(a_t) c^d(-w\lambda)^{-1} d\lambda \\ &= \sum_{w=\pm 1} \int_{i\mathbb{R}} c^d(w\lambda) g(w\lambda) \psi_{-w\lambda}^d(a_t) |c^d(\lambda)|^{-2} d\lambda \\ &= \int_{i\mathbb{R}} \left(\sum_{w=\pm 1} c^d(w\lambda) g(w\lambda) \right) \psi_{-\lambda}^d(a_t) |c^d(\lambda)|^{-2} d\lambda \\ &= \int_{i\mathbb{R}} \text{P}^{\text{av}} g(\lambda) \psi_{-\lambda}^d(a_t) |c^d(\lambda)|^{-2} d\lambda, \end{aligned}$$

which we recognize as the inverse Fourier transform of $P^{\text{av}}g \in \mathcal{H}_R(\mathbb{C})$; whence $\mathcal{I}g(a_t) = 0$ for all $t > R$ by the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}^d$.

Since $P^{\text{av}}\mathcal{L}^o f = \mathcal{F}f^d$ for all $f \in C_c^\infty(A^+)$, the above also yields

$$P^{\text{av}}\mathcal{L}^o \mathcal{I}g = \mathcal{F}(\mathcal{I}g)^d = P^{\text{av}}g,$$

for all $g \in PW(\mathbb{C})$; hence Lemma 7 implies that $\mathcal{L}^o \mathcal{I}g = g$ for all $g \in PW(\mathbb{C})$ and we conclude that $\mathcal{L}^o$ maps $C_c^\infty(A^+)$ onto $PW(\mathbb{C})$. $\square$

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