WEYL-HEISENBERG FRAMES FOR SUBSPACES OF $L^2(R)$

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ABSTRACT. A Weyl-Heisenberg frame

$$\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} = \{e^{2\pi i mb(\cdot)}g(\cdot - na)\}_{m,n \in \mathbb{Z}}$$

for $L^2(R)$ allows every function $f \in L^2(R)$ to be written as an infinite linear combination of translated and modulated versions of the fixed function $g \in L^2(R)$. In the present paper we find sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $\overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$, which, in general, might just be a subspace of $L^2(R)$. Even our condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R)$ is significantly weaker than the previous known conditions. The results also shed new light on the classical results concerning frames for $L^2(R)$, showing for instance that the condition $G(x) := \sum_{n \in \mathbb{Z}} |g(x - na)|^2 > A > 0$ is not necessary for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $\overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$. Our work is inspired by a recent paper by Benedetto and Li, where the relationship between the zero-set of the function G and frame properties of the set of functions $\{g(\cdot - n)\}_{n \in \mathbb{Z}}$ is analyzed.

1. PRELIMINARIES AND NOTATION

Let $\mathcal{H}$ denote a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. Let I denote a countable index set.

We say that $\{g_i\}_{i \in I} \subseteq \mathcal{H}$ is a *frame* (for $\mathcal{H}$) if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

In particular a frame for $\mathcal{H}$ is complete, i.e., $\overline{\text{span}}\{g_i\}_{i \in I} = \mathcal{H}$. In case $\{g_i\}_{i \in I}$ is not complete, $\{g_i\}_{i \in I}$ can still be a frame for the subspace $\overline{\text{span}}\{g_i\}_{i \in I}$; in that case we say that $\{g_i\}_{i \in I}$ is a *frame sequence*. The numbers A, B that appear in the definition of a frame are called *frame bounds*.

Orthonormal bases and, more generally, *Riesz bases* are frames. Recall that $\{g_i\}_{i \in I}$ is a Riesz basis for $\mathcal{H}$ if $\overline{\text{span}}\{g_i\}_{i \in I} = \mathcal{H}$ and

$$\exists A, B > 0 : \quad A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i g_i \right\|^2 \leq B \sum_{i \in I} |c_i|^2, \quad \forall \{c_i\}_{i \in I} \in \ell^2(I).$$

If $\{g_i\}_{i \in I}$ is a Riesz basis for $\overline{\text{span}}\{g_i\}_{i \in I}$, we say that $\{g_i\}_{i \in I}$ is a *Riesz sequence*.

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The present paper deals with frames having a special structure: all elements are translated and/or modulated versions of a single function. Let $L^2(R)$ denote the Hilbert space of functions on the real line which are square integrable with respect to the Lebesgue measure. First, define the following operators on functions $f \in L^2(R)$:

$$\text{Translation by } a \in R : (T_a f)(x) = f(x - a), \quad x \in R.$$

$$\text{Modulation by } b \in R : (E_b g)(x) = e^{2\pi i b x} f(x), \quad x \in R.$$

A frame for $L^2(R)$ of the form $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is called a *Weyl-Heisenberg frame* (or Gabor frame). For a collection of different papers concerning those frames we refer to the monograph [5].

Sufficient conditions for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R)$ has been known for about 10 years. The basic insight was provided by Daubechies [3]. A slight improvement was proved in [6]:

Theorem 1.1. *Let $g \in L^2(R)$ and suppose that*

$$(1) \quad \exists A, B > 0 : \quad A \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq B \text{ for a.e. } x \in R,$$

$$(2) \quad \lim_{b \rightarrow 0} \sum_{k \neq 0} \left\| \sum_{n \in \mathbb{Z}} T_{na} g T_{na + \frac{k}{b}} \bar{g} \right\|_{\infty} = 0.$$

Then there exists $b_0 > 0$ such that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a Weyl-Heisenberg frame for $L^2(R)$ for all $b \in]0; b_0[$.

The proof of Theorem 1.1 is based on the following identity, valid for all continuous functions f with compact support whenever g satisfies (1):

$$\begin{aligned} (3) \quad & \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 \\ &= \frac{1}{b} \int |f(x)|^2 G(x) dx \\ &+ \frac{1}{b} \sum_{k \neq 0} \int \overline{f(x)} f(x - k/b) \sum_{n \in \mathbb{Z}} g(x - na) \overline{g(x - na - k/b)} dx. \end{aligned}$$

An estimate of the second term in (3) now shows that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is actually a frame for all values of b for which

$$(4) \quad \sum_{k \neq 0} \left\| \sum_{n \in \mathbb{Z}} T_{na} g T_{na + \frac{k}{b}} \bar{g} \right\|_{\infty} < A.$$

A more recent result can be found in [4]: in Theorem 2.3 it is proved that if (1) is satisfied and there exists a constant $D < A$ such that

$$(5) \quad \sum_{k \neq 0} \sum_{n \in \mathbb{Z}} |g(x - na) g(x - na - \frac{k}{b})| \leq D \text{ for a.e. } x \in R,$$

then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(R)$ with bounds $\frac{A-D}{b}, \frac{B+D}{b}$. The reader should observe that [4] does not provide us with a generalization of the results in [3], [6] in a strict sense: there are cases where (5) is satisfied but (4) is not, and vice versa. The main point is that other conditions (that are easy to check) for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame can be derived from (5); cf. Theorem 2.4 in [4].

Define the *Fourier Transform* $\mathcal{F}(f) = \hat{f}$ of $f \in L^1(R)$ by

$$\hat{f}(y) = \int f(x) e^{-2\pi i y x} dx.$$

As usual we extend the Fourier Transform to an isometry from $L^2(R)$ onto $L^2(R)$. We denote the inverse Fourier transformation of $g \in L^2(R)$ by $\mathcal{F}^{-1}g$ or $\check{g}$. It is important to observe the following comutator relations, valid for all $a \in R$:

$$\mathcal{F}T_a = E_{-a}\mathcal{F}, \quad \mathcal{F}E_a = T_a\mathcal{F}.$$

We need a result from [2]. The basic insight was provided by Benedetto and Li [1], who treated the case $a = 1$.

Theorem 1.2. *Let $g \in L^2(R)$. Then $\{T_{na}g\}_{n \in \mathbb{Z}}$ is a frame sequence with bounds A, B if and only if*

$$0 < aA \leq \sum_{n \in \mathbb{Z}} |\hat{g}(\frac{x+n}{a})|^2 \leq aB \text{ for a.e. } x \text{ for which } \sum_{n \in \mathbb{Z}} |\hat{g}(\frac{x+n}{a})|^2 \neq 0.$$

In that case $\{T_{na}g\}_{n \in \mathbb{Z}}$ is a Riesz sequence if and only if the set of x for which $\sum_{n \in \mathbb{Z}} |\hat{g}(\frac{x+n}{a})|^2 = 0$ has measure zero.

Theorem 1.2 leads immediately to an equivalent condition to (1). Define the function G and its kernel N_G by

$$G : R \rightarrow [0, \infty], \quad G(x) := \sum_{n \in \mathbb{Z}} |g(x - na)|^2, \\ N_G = \{x \in R \mid G(x) = 0\}.$$

Corollary 1.3. *$\{E_{\frac{n}{a}}g\}_{n \in \mathbb{Z}}$ is a frame sequence with bounds A, B if and only if*

$$0 < \frac{A}{a} \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq \frac{B}{a} \text{ for a.e. } x \in R - N_G.$$

In that case $\{E_{\frac{n}{a}}g\}_{n \in \mathbb{Z}}$ is a Riesz sequence iff N_G has measure zero.

Proof. The inequality

$$0 < \frac{A}{a} \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq \frac{B}{a} \text{ for a.e. } x \in R - N_G$$

holds if and only if

$$(6) \quad 0 < \frac{A}{a} \leq \sum_{n \in \mathbb{Z}} |g([x - n]a)|^2 \leq \frac{B}{a} \text{ for a.e. } x \in R - N_G.$$

By Theorem 1.2, (6) is equivalent to $\{T_{\frac{n}{a}}\check{g}\}_{n \in \mathbb{Z}}$ being a frame sequence with bounds A, B . Applying the Fourier transformation this is equivalent to $\{E_{\frac{n}{a}}g\}_{n \in \mathbb{Z}}$ being a frame sequence with bounds A, B . $\square$

2. THE RESULTS

From now on we concentrate on Weyl-Heisenberg frames $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$. Our first result gives a sufficient condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame sequence. It can be considered as a “subspace version” of a result by Ron and Shen; cf. [7]. Our condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R)$ is significantly weaker than the conditions mentioned in section 1.

Let $L^2(R - N_G)$ denote the set of functions in $L^2(R)$ that vanishes at N_G .

Theorem 2.1. *Let $g \in L^2(R)$, $a, b > 0$ and suppose that*

$$(7) \quad A := \inf_{x \in [0, a] - N_G} \left[\sum_{n \in Z} |g(x - na)|^2 - \sum_{k \neq 0} \left| \sum_{n \in Z} g(x - na) \overline{g(x - na - \frac{k}{b})} \right| \right] > 0,$$

$$(8) \quad B := \sup_{x \in [0, a]} \sum_{k \in Z} \left| \sum_{n \in Z} g(x - na) \overline{g(x - na - \frac{k}{b})} \right| < \infty.$$

Then $\{E_{mb}T_{na}g\}_{m,n \in Z}$ is a frame for $L^2(R - N_G)$ with bounds $\frac{A}{b}, \frac{B}{b}$.

Proof. First, observe that $\overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n \in Z} \subseteq L^2(R - N_G)$. Now consider a function $f \in L^2(R - N_G)$ which is bounded and has support in a compact set. The Heil-Walnut argument (3) is valid under the assumption (8) and it gives that

$$(3) \quad \begin{aligned} & \sum_{m,n \in Z} |\langle f, E_{mb}T_{na}g \rangle|^2 \\ &= \frac{1}{b} \int |f(x)|^2 \sum_{n \in Z} |g(x - na)|^2 dx \\ &+ \frac{1}{b} \sum_{k \neq 0} \int \overline{f(x)} f(x - k/b) \sum_{n \in Z} g(x - na) \overline{g(x - na - k/b)} dx. \end{aligned}$$

We want to estimate the second term above. For $k \in Z$, define

$$H_k(x) := \sum_{n \in Z} T_{na}g(x) \overline{T_{na+k/b}g(x)}.$$

First, observe that

$$\begin{aligned} & \sum_{k \neq 0} |T_{-k/b}H_k(x)| \\ &= \sum_{k \neq 0} |T_{-k/b} \sum_{n \in Z} T_{na}g(x) \overline{T_{na+k/b}g(x)}| \\ &= \sum_{k \neq 0} \left| \sum_{n \in Z} T_{na-k/b}g(x) \overline{T_{na}g(x)} \right| \\ &= \sum_{k \neq 0} \left| \sum_{n \in Z} T_{na+k/b}g(x) \overline{T_{na}g(x)} \right| \\ &= \sum_{k \neq 0} \left| \sum_{n \in Z} \overline{T_{na+k/b}g(x)} T_{na}g(x) \right| \\ &= \sum_{k \neq 0} |H_k(x)|. \end{aligned}$$

Now, by a slight modification of the argument in [4], Theorem 2.3,

$$\begin{aligned}
& \left| \sum_{k \neq 0} \int \overline{f(x)} f(x - k/b) \sum_{n \in \mathbb{Z}} g(x - na) \overline{g(x - na - k/b)} dx \right| \\
& \leq \sum_{k \neq 0} \int |f(x)| \cdot |T_{k/b} f(x)| \cdot |H_k(x)| dx \\
& = \sum_{k \neq 0} \int |f(x)| \sqrt{|H_k(x)|} \cdot |T_{k/b} f(x)| \sqrt{|H_k(x)|} dx \\
& \leq \sum_{k \neq 0} \left(\int |f(x)|^2 |H_k(x)| dx \right)^{1/2} \left(\int |T_{k/b} f(x)|^2 |H_k(x)| dx \right)^{1/2} \\
& \leq \left(\sum_{k \neq 0} \int |f(x)|^2 |H_k(x)| dx \right)^{1/2} \cdot \left(\sum_{k \neq 0} \int |T_{k/b} f(x)|^2 |H_k(x)| dx \right)^{1/2} \\
& = \left(\int |f(x)|^2 \sum_{k \neq 0} |H_k(x)| dx \right)^{1/2} \cdot \left(\int |f(x)|^2 \sum_{k \neq 0} |T_{-k/b} H_k(x)| dx \right)^{1/2} \\
& = \int |f(x)|^2 \sum_{k \neq 0} |H_k(x)| dx.
\end{aligned}$$

Note that $\sum_{k \neq 0} |H_k(x)| = \sum_{k \neq 0} |\sum_{n \in \mathbb{Z}} T_{na} g(x) \overline{T_{na+k/b} g(x)}|$ is a periodic function with period a . By (3) and the assumption (7) we now have

$$\begin{aligned}
& \sum_{m, n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 \\
& \geq \frac{1}{b} \int |f(x)|^2 \left[\sum_{n \in \mathbb{Z}} |g(x - na)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g(x - na) \overline{g(x - na - \frac{k}{b})} \right| \right] dx \\
& \geq \frac{A}{b} \|f\|^2.
\end{aligned}$$

Similary, by (3) and (8),

$$\begin{aligned}
& \sum_{m, n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 \\
& \leq \frac{1}{b} \int |f(x)|^2 \left[\sum_{n \in \mathbb{Z}} |g(x - na)|^2 + \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g(x - na) \overline{g(x - na - \frac{k}{b})} \right| \right] dx \\
& = \frac{1}{b} \int |f(x)|^2 \sum_{k \in \mathbb{Z}} \left| \sum_{n \in \mathbb{Z}} g(x - na) \overline{g(x - na - \frac{k}{b})} \right| dx \\
& \leq \frac{B}{b} \|f\|^2.
\end{aligned}$$

Since those two estimates holds on a dense subset of $L^2(R - N_G)$, they hold on $L^2(R - N_G)$. Thus $\{E_{mb} T_{na} g\}_{m, n \in \mathbb{Z}}$ is a frame for $L^2(R - N_G)$ with the desired bounds. $\square$

The advantage of Theorem 2.1 compared to the results in section 1 is that we compare the functions $\sum_{n \in \mathbb{Z}} |g(x - na)|^2$ and $\sum_{k \neq 0} |H_k(x)|$ *pointwise* rather than assuming that the supremum of $\sum_{k \neq 0} |H_k(x)|$ is smaller than the infimum of $\sum_{n \in \mathbb{Z}} |g(x - na)|^2$. It is easy to give concrete examples where Theorem 2.1 shows that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(R)$ but where the conditions in section 1 are not satisfied:

Example. Let $a = b = 1$ and define

$$g(x) = \begin{cases} 1+x & \text{if } x \in [0, 1[, \\ \frac{1}{2}x & \text{if } x \in [1, 2[, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in [0, 1[$ we have

$$G(x) = \sum_{n \in \mathbb{Z}} |g(x - n)|^2 = g(x)^2 + g(x+1)^2 = \frac{5}{4}(x+1)^2$$

and

$$\sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g(x - n) \overline{g(x - n - k)} \right| = (1+x)^2,$$

so by Theorem 2.1 $\{E_m T_n g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(R)$ with bounds $A = \frac{1}{4}, B = 9$. But $\inf_{x \in R} G(x) = \frac{5}{4}$ and

$$\sum_{k \neq 0} \left\| \sum_{n \in \mathbb{Z}} T_n g T_{n+k} \bar{g} \right\|_\infty = 4,$$

so the condition (4) is not satisfied. (5) is not satisfied either.

Remark. It is well known that G being bounded below is a necessary condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R)$; cf. [3]. Theorem 2.1 shows that this condition is not necessary for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame sequence. However, it is implicit in (7) that G has to be bounded below on $R - N_G$ in order for Theorem 2.1 to work, and an easy modification of the proof in [3] shows that this is actually a necessary condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(R - N_G)$. We shall later give examples of frame sequences for which G is not bounded below on $R - N_G$.

In case g has support in an interval of length $\frac{1}{b}$ an equivalent condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame sequence can be given. First, observe that by (3) this condition on g implies that for all continuous functions f with compact support, we have

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{na}g \rangle|^2 = \frac{1}{b} \int |f(x)|^2 G(x) dx.$$

It is not hard to show that this actually holds for all $f \in L^2(R)$; cf. [6].

Corollary 2.2. *Suppose that $g \in L^2(R)$ has compact support in an interval I of length $|I| \leq 1/b$. Then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence with bounds A, B if and only if*

$$0 < bA \leq \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq bB, \quad \text{for a.e. } x \in R - N_G.$$

In that case $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is actually a frame for $L^2(R - N_G)$.

Proof. Suppose that g has support in an interval I of length $|I| \leq \frac{1}{b}$. If $0 < bA \leq G(x) \leq bB$ for a.e. $x \in R - N_G$, it follows from Theorem 2.1 that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence with the desired bounds. Now suppose that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence with bounds A, B . Then, for every interval I of length $|I| = 1/b$ and every function $f \in L^2(I)$,

$$\sum_{m,n} |\langle f, E_{mb}T_{na}g \rangle|^2 = \frac{1}{b} \int_R |f(x)|^2 G(x) dx \leq B \|f\|^2.$$

But this is clearly equivalent to

$$G(x) = \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq Bb \text{ a.e.}$$

To prove the lower bound for G we proceed by way of contradiction. Suppose that for some $\epsilon > 0$ we have $0 < G(x) \leq (1 - \epsilon)Ab$ on a set of positive measure. In this case there is a set Δ of positive measure and supported in an interval of length $\leq \frac{1}{b}$ so that $0 < G(x) \leq (1 - \epsilon)Ab$ on Δ . Then, for any function $f \in L^2(R)$ supported on Δ , we have

$$\begin{aligned} \sum_{m,n} |\langle f, E_{mb}T_{na}g \rangle|^2 &= \frac{1}{b} \int_R |f(x)|^2 G(x) dx \\ &\leq \frac{(1 - \epsilon)Ab}{b} \int_R |f(x)|^2 dx = (1 - \epsilon)A \|f\|^2. \end{aligned}$$

Since $G(x) > 0$ on Δ , there is a $k \in \mathbb{Z}$ so that $\chi_\Delta T_{ka}g$ is not the zero function. With $\Delta' := \Delta \cap \text{Supp}(T_{ka}g)$ we have

$$f := \chi_{\Delta'} T_{ka}g \in \overline{\text{span}}\{E_{mb}T_{ka}g\}_{m \in \mathbb{Z}} \subseteq \overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}},$$

so the above calculation shows that the lower bound for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is at most $(1 - \epsilon)A$, which is a contradiction. Thus

$$G(x) \geq bA \text{ for a.e. } x \in R - N_G.$$

In case the condition in Corollary 2.2 is satisfied, it follows from Theorem 2.1 that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(R - N_G)$. $\square$

For functions g with the property that the translates $T_{na}g, n \in \mathbb{Z}$, have disjoint support we can give an equivalent condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame sequence. Define the function

$$\tilde{G}(x) : R \rightarrow [0, \infty], \quad \tilde{G}(x) = \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2.$$

Proposition 2.3. *Let $g \in L^2(R)$, $a, b > 0$ and suppose that*

$$(9) \quad \text{supp}(g) \cap \text{supp}(T_{na}g) = \emptyset, \quad \forall n \in \mathbb{Z} - \{0\}.$$

Then $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence with bounds A, B if and only if there exist $A, B > 0$ such that

$$bA \leq \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq bB \text{ for a.e. } x \in R - N_{\tilde{G}}.$$

In that case, $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a Riesz sequence iff $N_{\tilde{G}}$ has measure zero.

Proof. Because of the support condition (9), it is clear that $\{E_{mb}g\}_{m \in \mathbb{Z}}$ is a frame sequence iff $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence, in which case the sequences have the same frame bounds. But by Corollary 1.3 $\{E_{mb}g\}_{m \in \mathbb{Z}}$ is a frame sequence with bounds A, B iff

$$bA \leq \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq bB \text{ for a.e. } x \in R - N_{\tilde{G}}.$$

Also, $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a Riesz sequence iff $\{E_{mb}g\}_{m \in \mathbb{Z}}$ is a Riesz sequence, which, by Corollary 1.3, is the case iff $N_{\tilde{G}}$ has measure zero. $\square$

We are now ready to show that G being bounded below on $R - N_G$ (by a positive number) is not a necessary condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame sequence.

Example. Let $a, b > 0$ and suppose that $\frac{1}{ab} \notin N$. Chose $\epsilon > 0$ such that

$$[0, \epsilon] + na \cap [\frac{1}{b}, \frac{1}{b} + \epsilon] = \emptyset, \quad \forall n \in \mathbb{Z}.$$

This implies that $\epsilon < \min(a, \frac{1}{b})$. Define

$$g(x) := \begin{cases} x & \text{if } x \in [0, \epsilon], \\ \sqrt{1 - (x - \frac{1}{b})^2} & \text{if } x \in [\frac{1}{b}, \frac{1}{b} + \epsilon], \\ 0 & \text{otherwise.} \end{cases}$$

Then the condition (9) in Proposition 2.3 is satisfied. Also, for $x \in [0, \epsilon]$,

$$\tilde{G}(x) = \sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 = g(x)^2 + g(x+1)^2 = 1$$

and for $x \in [\epsilon, \frac{1}{b}]$, we have $\tilde{G}(x) = 0$. Thus, by Proposition 2.3 $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence. But for $x \in [0, \epsilon]$,

$$G(x) = \sum_{n \in \mathbb{Z}} |g(x - na)|^2 = x^2.$$

Thus G is not bounded below by a positive number on $R - N_G$. By the remark after Theorem 2.1 this implies that $\overline{\text{span}}\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \neq L^2(R - N_G)$.

For $ab > 1$ it is even possible to construct an orthonormal sequence having all the features of the above example. For example, let $a = 2, b = 1$ and

$$g(x) := \begin{cases} x & \text{if } x \in [0, 1], \\ \sqrt{2x - x^2} & \text{if } x \in [1, 2], \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 = 1, \quad \forall x,$$

it follows by Proposition 2.3 that $\{E_m T_{2n} g\}_{m,n \in \mathbb{Z}}$ is a Riesz sequence with bounds $A = B = 1$, which implies that $\{E_m T_{2n} g\}_{m,n \in \mathbb{Z}}$ is an orthonormal sequence. But $G(x) = \sum_{n \in \mathbb{Z}} |g(x - 2n)|^2$ is not bounded below on $R - N_G$.

G being bounded above is still a necessary condition for $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ to be a frame sequence (repeat the argument in Corollary 2.2). $\tilde{G}$ also has to be bounded above:

Proposition 2.4. *If $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence with upper bound B , then*

$$\sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq B \quad a.e.$$

Proof. If $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame sequence, then $\{\mathcal{F}^{-1}E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} = \{T_{mb}E_{na}\tilde{g}\}_{m,n \in \mathbb{Z}}$ is a frame sequence with the same bounds. In particular the sequence $\{T_{mb}\tilde{g}\}_{m,n \in \mathbb{Z}}$ has the upper frame bound B . By Theorem 1.2 (or, more precisely, the proof of it in [2]) it follows that

$$\sum_{m \in \mathbb{Z}} |g(\frac{x+m}{b})|^2 \leq B \quad \text{for a.e. } x.$$

It follows that $\sum_{m \in \mathbb{Z}} |g(x + \frac{m}{b})|^2 \leq B \quad a.e.$ □

Remark. Recall that a wavelet frame for $L^2(R)$ has the form

$$\{\frac{1}{a^{n/2}}g(\frac{x}{a^n} - mb)\}_{m,n \in \mathbb{Z}},$$

where $a > 1, b > 0$ and $g \in L^2(R)$ are fixed.

As well as Weyl-Heisenberg frames, wavelet frames play a very important role in applications. The theory for the two types of frames was developed at the same time, with the main contribution due to Daubechies. Several results for Weyl-Heisenberg frames have counterparts for wavelet frames. For example, Theorem 5.1.6 in [6] gives sufficient conditions for $\{\frac{1}{a^{n/2}}g(\frac{x}{a^n} - mb)\}_{m,n \in \mathbb{Z}}$ to be a frame based on a calculation similar to (3).

Also our results for Weyl-Heisenberg frames have counterparts for wavelet frames. The ideas in the proof of Theorem 2.1 can be used to modify [6], Theorem 5.1.6, which leads to the following:

Theorem 2.5. *Let $a > 1, b > 0$ and $g \in L^2(R)$ be given. Let*

$$N := \left\{ \gamma \in [1, a] \left| \sum_{n \in \mathbb{Z}} |\hat{g}(a^n \gamma)|^2 = 0 \right. \right\}$$

and suppose that

$$\begin{aligned} A &:= \inf_{|\gamma| \in [1, a] - N} \left[\sum_{n \in \mathbb{Z}} |\hat{g}(a^n \gamma)|^2 - \sum_{k \neq 0} \sum_{n \in \mathbb{Z}} |\hat{g}(a^n \gamma) \hat{g}(a^n \gamma + k/b)| \right] > 0, \\ B &:= \sup_{|\gamma| \in [0, a]} \sum_{k, n \in \mathbb{Z}} |\hat{g}(a^n \gamma) \hat{g}(a^n \gamma + k/b)| < \infty. \end{aligned}$$

Then $\{\frac{1}{a^{n/2}}g(\frac{x}{a^n} - mb)\}_{m,n \in \mathbb{Z}}$ is a frame sequence with bounds $\frac{A}{b}, \frac{B}{b}$.

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