FINITE RANK BUTLER GROUPS AND TORSION-FREE MODULES OVER A DISCRETE VALUATION RING

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ABSTRACT. Fully faithful functors from isomorphism at p categories of finite rank Butler groups to torsion-free modules of finite rank over the integers localized at a prime p are constructed via categories of representations of antichains over discrete valuation rings. Descriptions and properties of modules in the images of these functors are given, including a characterization of finite representation type and a complete list of indecomposables in that case.

The two subjects of torsion-free abelian groups and p -local torsion-free abelian groups, more generally torsion-free modules over a discrete valuation ring, have traditionally had little in common. For instance, the set of types, isomorphism classes of rank-1 torsion-free abelian groups, is an uncountable lattice [6] while there are only two types in the p -local case, namely the isomorphism classes of Z_p , the integers localized at a prime p , and Q , the additive group of rationals. Pure subgroups of finite direct sums of rank-1 groups, called Butler groups, are known to be complicated [4] but p -local Butler groups are direct sums of rank-1 p -local groups [5]. On the other hand, p -local torsion-free abelian groups of finite rank can be viewed, up to quasi-isomorphism, as finite rank Butler modules over a Dedekind domain [13].

In this paper, we demonstrate the existence of a variety of fully faithful embeddings from isomorphism at p categories of finite rank Butler groups to the category $TF(Z_p)$ of p -local torsion-free abelian groups of finite rank. Under these embeddings, pure subgroups of the p -adic completion of the integers correspond to rank-1 Butler groups.

Let T be a finite lattice of types with least element τ_0 , j a non-negative integer, and $JI(T)$ the partially ordered set of join irreducible elements of T . Define $B(T, j)_p$ to be the isomorphism at p category of finite rank Butler groups G with typeset contained in T and $p^j G \leq \sum \{G(\tau) : \tau \in JI(T)\} \leq G$. Morphism sets for this category are $\text{Hom}(G, H)_p$, the localization of $\text{Hom}_Z(G, H)$ at a prime p . Isomorphism at p is the local version of near isomorphism and is strictly weaker than quasi-isomorphism [8] or [2].

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A finite lattice T of types is *p-locally free* if X_p is isomorphic to Z_p for each rank-1 group X with type $X \in T$. Our results require T to be a finite Boolean algebra, in this case $JI(T)$ is an antichain consisting of the atoms of T . The *p-rank* of a torsion-free abelian group M is the Z/pZ dimension of M/pM .

Theorem I. *Suppose T is a p -locally free finite Boolean algebra of types and j is a non-negative integer. There are uncountably many fully faithful functors F_A from $B(T, j)_p$ into $TF(Z_p)$ indexed by finite subsets A of the p -adic integers. Moreover,*

$$\text{rank } F_A(G) \geq 2(p\text{-rank } F_A(G))$$

for each G in $B(T, j)$.

The definition of F_A , including an independence condition on A , is given in Theorem 1.3. Rank 1 groups in $B(T, j)_p$ correspond under F_A to *purely indecomposable* groups, those indecomposable p -local groups of finite rank with p -rank 1. On the other hand, if $M \in TF(Z_p)$ is *co-purely indecomposable*, indecomposable with rank $-p\text{-rank} = 1$ [1], and if $3 \leq \text{rank } M$, then M is not in the image of any F_A . However, co-purely indecomposable groups can be realized as the image of Butler groups via another class of functors defined on $B(T, 0)_p$.

Theorem II. *Assume that T is a p -locally free finite Boolean algebra of types. There are uncountably many faithful functors G_A from $B(T, 0)_p$ into $TF(Z_p)$, indexed by finite subsets A of the p -adic integers. Furthermore,*

$$\text{rank } G_A(H) \leq 2(p\text{-rank } G_A(H))$$

for each H in $B(T, 0)$.

Suppose $H = (X_1 \oplus \cdots \oplus X_k)/K \in B(T, 0)$ is an indecomposable group, where K is a pure rank-1 subgroup and each X_i is a rank-1 group with type X_i an atom of T . Then $G_A(H)$ is a co-purely indecomposable p -local group. Butler groups H of this kind have been studied extensively [4]. Theorems I and II and the duality between purely indecomposable and co-purely indecomposable p -local groups given in [1] reflect the “dual” nature of rank-1 groups X that are images of a finite direct sum of rank-1 groups $X_1, \dots, X_n$ and groups of the form $H = (X_1 \oplus \cdots \oplus X_k)/K$, K a pure rank-1 subgroup.

The functors of Theorems I and II are derived from fully faithful embeddings, also denoted by F_A and G_A , from $\text{rep}(S_n, R, j)$, a category of representations of an antichain S_n over a discrete valuation ring R , into the category $TF(R)$ of finite rank torsion-free R -modules. As for images of these functors, we have (Corollary 2.6):

- (i) Each $M \in \text{image } F_A \cup \text{image } G_A$ has a fully invariant basic submodule.
- (ii) Each $M \in \text{image } F_A \cap \text{image } G_A$ is a free R -module.
- (iii) If $M \in \text{image } F_A$, then M is a homomorphic image of a finite direct sum of purely indecomposable modules with rank ≤ 2 and endomorphism ring R .
- (iv) If $M \in \text{image } G_A$, then M is a pure submodule of a finite direct sum of purely indecomposable modules with rank ≤ 2 and endomorphism ring R .

Theorems I and II allow the translation of known results for isomorphism at p categories of Butler groups directly into the image of F_A and the image of G_A . A full subcategory $\mathcal{C}$ of $TF(Z_p)$ has *finite representation type* if there are only finitely many isomorphism classes of indecomposables in $\mathcal{C}$ and *wild mod p representation type* if for each Z_p -algebra Λ that is finitely generated and free as a Z_p -module,

there is a ring homomorphism from $\text{End } M$ onto $\Lambda/p\Lambda$. Representation types of categories of Z_p -representations are computed in [3]. This yields:

Corollary III. *The full subcategory of $TF(Z_p)$ consisting of p -local groups in the image of F_A , respectively the image of G_A , has*

- (a) *finite representation type if and only if $|A| \leq 2$;*
- (b) *wild mod p representation type if and only if $|A| \geq 3$.*

In Section 2, a complete list of isomorphism classes of indecomposable groups in the image of F_A and the image of G_A is given for the case that $|A| = 2$ (Corollary 2.4). Indecomposable groups in the image of F_A have either p -rank 1 and rank ≤ 3 or else p -rank 2 and rank 4 while indecomposables in the image of G_A are either isomorphic to Z_p or have p -rank 2 and rank 3.

Computations of representation type and classification of indecomposables for quasi-homomorphism categories in $TF(R)$ are given in terms of dimensions of splitting fields in [9, Theorem 5.11]. These results are obtained by embedding the quasi-homomorphism subcategory QTF_K of modules with a fixed splitting field K as a full subcategory of finite dimensional modules over a finite dimensional Q -algebra. By way of comparison with the results in this paper, let $n = [K : Q]$. Then QTF_K has finite representation type if and only if $n \leq 3$. Moreover, if $n = 3$, then a strongly indecomposable reduced M in QTF_K has p -rank 1 and rank ≤ 2 or p -rank 2 and rank 3.

The results of Section 1 carry over to torsion-free modules of finite rank over almost maximal valuation domains with appropriate modifications of definitions and proofs. For relevant results, see [7, Chapter 14].

1. TORSION-FREE MODULES OVER DISCRETE VALUATION RINGS AND REPRESENTATIONS

Let R be a discrete valuation ring (commutative domain), with prime p and quotient field Q , and denote the p -adic completion of R by R^* . The following fundamental properties and definitions of torsion-free R -modules can be found, for example, in [1] or [9]. If M is a torsion-free reduced R -module of finite rank, then M contains a *basic submodule* B , i.e. B is a finitely generated free R -module that is pure in M with M/B divisible. The *p -rank of M* is the R/pR -dimension of M/pM , in fact $p\text{-rank } M = \text{rank } B$ for a basic submodule B of M . Moreover, M is a pure submodule of M^* , the p -adic completion of M , M^* is a free R^* -module with rank the rank of B , and M^*/M is divisible. Thus, we may assume that M is a pure submodule of $M^* = B^* = B \otimes_R R^*$. Every R -endomorphism of M extends uniquely to an R^* -endomorphism of M^* .

Suppose M is a finite rank torsion-free R -module with $p\text{-rank } M = n$ and $\text{rank } M = n + k$. Then M is free if and only if $k = 0$ and divisible if and only if $n = 0$. If $n = 1$, then $M = N \oplus D$, where D is divisible and N is *purely indecomposable*, reduced and indecomposable with p -rank 1. On the other hand, if $k = 1$, then $M = L \oplus F$, where F is a free R -module and L is *co-purely indecomposable*, indecomposable with $\text{rank } L = p\text{-rank } L + 1$. Basic properties of purely and co-purely indecomposable groups are given in [1] or [9].

For an example of a purely indecomposable module, let $t \in R^* \setminus R$ and let R_t denote the pure submodule of R^* generated by $\{1, t\}$. Then Z_p is a basic submodule of R_t , $R_t^* = R^*$, $p\text{-rank } R_t = 1$, and $\text{rank } R_t = 2$.

The following proposition is well known.

Proposition 1.1. *Suppose M and N are two purely indecomposable R -modules with $\text{rank } M = \text{rank } N$.*

- (a) [1] *The following are equivalent:*
 - (i) *M is isomorphic to N ;*
 - (ii) *M is quasi-isomorphic to N ;*
 - (iii) *$\text{Hom}_R(M, N) \neq 0$.*
- (b) [14] *Let $M = R_s$ and $N = R_t$ for $s, t \in R^* \setminus R$. Then M and N are isomorphic if and only if there are elements a, b, c, d in Q with $t = (c + ds)/(a + bs)$.*

The properties of purely indecomposables given in Proposition 1.1(a) are analogous to those of rank-1 torsion-free abelian groups. Proposition 1.2 gives analogs of the pure fully invariant subgroups $G(\tau)$, for G a torsion-free abelian group and τ a type, where an element t of R^* plays the role of a type τ . Let M be a torsion-free module of finite rank with a basic submodule B and assume $B \leq M \leq M^* = R^* \otimes_R B$. For $t \in R^*$, define $M(t) = \{m \in M : tm \in M\}$. Notice that if $t \in R$, then $M(t) = M$.

Proposition 1.2. *Assume M and N are finite rank reduced torsion-free R -modules and $t \in R^* \setminus R$.*

- (a) *$M(t)$ is a pure submodule of M and if $f: M \rightarrow N$ is an R -homomorphism, then $f(M(t)) \leq N(t)$.*
- (b) *$R_t M(t) = \text{Hom}(R_t, M)R_t$, the image of the evaluation map $\text{Hom}(R_t, M) \otimes_R R_t \rightarrow M$.*
- (c) *$M(t) \cap M(s) \leq M(t+s)$.*
- (d) *$M(t)(s) = M(ts) \cap M(t)$.*

Proof. (a) Suppose $m \in M$ and $p^i m \in M(t)$. Then $p^i t m \in p^i M^* \cap M = p^i M$ so that $tm \in M$. Thus, $m \in M(t)$ and so $M(t)$ is pure in M . Recall that $f: M \rightarrow N$ extends to an R^* -homomorphism f^* from M^* to N^* . Hence, if $m \in M$ with $tm \in M$, then $f(tm) = f^*(tm) = tf^*(m) = tf(m) \in N$. This shows that $f(M(t)) \leq N(t)$.

(b) Let $m \in M(t)$. Then $R_t m \leq M$ since $R_t = \langle R \oplus Rt \rangle_*$ and M is pure in M^* . Define $f: R_t \rightarrow M$ by $f(y) = ym$, an R -homomorphism with $m \in \text{image } f$. Consequently, $M(t) \leq \text{Hom}(R_t, M)R_t$ and $R_t M(t) \leq \text{Hom}(R_t, M)R_t$.

Conversely, let $0 \neq f: R_t \rightarrow M$. Then f is a monomorphism, otherwise $\ker f$ is a pure cyclic submodule of R_t and image f is divisible, a contradiction to the assumption that M is reduced. Hence, image $f = R_t m \leq M$ for $m = f(1) \in M$. In particular, $m \in M(t)$ and image $f \leq R_t M(t)$ as desired.

(c) If $m \in M$ and tm and sm are in M , then $(t+s)m$ is in M .

(d) Let $m \in M(t)(s)$. Then $m \in M(t)$ with $sm \in M(t)$ and so $tsm \in M$. This shows that $M(t)(s) \leq M(st) \cap M(t)$. Conversely, if $m \in M(st) \cap M(t)$, then $stm \in M$ and $tm \in M$. Hence, $m \in M(t)(s)$. $\square$

Let S be a finite partially ordered set and $\text{rep}(S, R, j)$ the category with objects $(U_0, U_i : i \in S)$, where U_0 is a finitely generated free R -module, each U_i is an R -summand of U_0 , U_i is contained in U_s if $i \leq s$ in S , and $p^j U_0 \leq \sum \{U_i : i \in S\} \leq U_0$. A morphism $f: U \rightarrow V$ is an R -homomorphism $f: U_0 \rightarrow V_0$ such that $f(U_i) \leq V_i$ for each i .

The category of torsion-free R -modules of finite rank is denoted by $TF(R)$. An additive functor $F: C \rightarrow D$ is *fully faithful* if $F: \text{Hom}(U, V) \rightarrow \text{Hom}(F(U), F(V))$ is an isomorphism for each U, V in C .

Theorem 1.3. *Let $S_k = \{1, \dots, k\}$ be an antichain and $A = \{t_1, \dots, t_k\}$ a subset of $R^* \setminus R$ such that $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$ is a Q -independent set for each $1 \leq i \leq k$. For each non-negative integer j , there is a fully faithful embedding $F_A: \text{rep}(S_k, R, j) \rightarrow TF(R)$ with image consisting of those M with fully invariant basic submodule B such that $M/(B + R_{t_1}M(t_1) + \dots + R_{t_k}M(t_k))$ is torsion, each $M(t_i)$ is contained in B , and $p^j B \leq M(t_1) + \dots + M(t_k) \leq B$.*

Proof. Let $U = (U_0, U_i : 1 \leq i \leq k)$ and define

$$M = F_A(U) = \langle U_0 + U_1 t_1 + \dots + U_k t_k \rangle_* \leq U_0^* = R^* \otimes_R U_0.$$

Given a representation morphism $f: U \rightarrow V$, let $F_A(f)$ be the restriction of the R^* -homomorphism $1 \otimes f: U_0^* \rightarrow V_0^*$ to $F_A(U)$. It follows that $F_A(f): F_A(U) \rightarrow F_A(V)$ and that F_A is a faithful functor.

The free module U_0 is a basic submodule of M . This is because U_0 is a pure submodule of U_0^* , hence of M , and M/U_0 is divisible since U_0^*/U_0 is divisible and M is a pure submodule of U_0^* . Moreover, for each i , U_i is contained in $M(t_i)$. To see that $M(t_i) = U_i$, assume $m \in M(t_i)$. Then $m = q(u_0 + u_1 t_1 + \dots + u_k t_k)$ with $t_i m = q'(u'_0 + u'_1 t_1 + \dots + u'_k t_k) = q(u_0 t_i + u_1 t_1 t_i + \dots + u_k t_k t_i)$ for some $q, q' \in Q$ and $u_j, u'_j \in U_j$. Choose an R -basis for U_0 , necessarily an R^* -basis for U_0^* , and write each u_i in terms of this basis, recalling that $U_i \leq U_0$. Now equate coefficients and use the assumption that the set $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$ is Q -independent to see that $m \in U_i$.

We now have that $B = U_0$ is a basic submodule of M and $U_i = M(t_i) \leq B$. Then $M/(B + R_{t_1}M(t_1) + \dots + R_{t_k}M(t_k))$ is torsion since $U_i = M(t_i) \leq R_{t_i}M(t_i)$ by Proposition 1.2(b) and $p^j U_0 = p^j B \leq M(t_1) + \dots + M(t_k) = U_1 + \dots + U_k \leq U_0 = B$. By Proposition 1.2(a), each $M(t_i)$ is fully invariant in M so that B is also fully invariant in M .

To see that F is a full functor, let $g: F_A(U) \rightarrow F_A(V)$ and let f be the restriction of g to U_0 , a basic submodule of $F_A(U)$. Now $f: U_0 \rightarrow V_0$ with $f(U_i) = f(F_A(U)(t_i)) \leq F_A(V)(t_i) = V_i$ for each i , by Proposition 1.2(a). Thus, $f: U = (U_0, U_i) \rightarrow V = (V_0, V_i)$ is a representation morphism with $F_A(f) = g$, as desired.

Finally, let M be in $TF(R)$ satisfying the above conditions. Then $U = (B, M(t_i)) \in \text{rep}(S_k, R, j)$ since $M(t_i)$ is pure in, hence a summand of, the finitely generated free R -module B and $p^j B \leq M(t_1) + \dots + M(t_k) \leq B$. Moreover, $F_A(U) = \langle B + M(t_1)t_1 + \dots + M(t_k)t_k \rangle_* = \langle B + R_{t_1}M(t_1) + \dots + R_{t_k}M(t_k) \rangle_* = M$, as desired. $\square$

Corollary 1.4. *Let $F_A: \text{rep}(S_k, R, j) \rightarrow TF(R)$ be the functor given in Theorem 1.3.*

- (a) *If $F_A(U) = M$, with $U = (U_0, U_i)$, then p -rank $M = \text{rank } U_0$ and $\text{rank } M = \text{rank } U_0 + \text{rank } U_1 + \dots + \text{rank } U_k \geq 2(p\text{-rank } M)$.*
- (b) *If $U = (U_0, U_i)$ with $\text{rank } U_0 = 1$, then $M = F_A(U)$ is purely indecomposable with $\text{End } M = R$.*
- (c) *If $F_A(U) = M$ is co-purely indecomposable, then $\text{rank } M = 2$ and M is isomorphic to R_t for some $t \in A$.*

Proof. (a) By Theorem 1.3, U_0 is a basic submodule of M so that $\text{rank } U_0 = p\text{-rank } M$. Moreover, $U_1 t_1 + \cdots + U_k t_k = U_1 t_1 \oplus \cdots \oplus U_k t_k$ by the independence condition on A and $\text{rank } U_0 \leq \text{rank } U_1 + \cdots + \text{rank } U_k$. Hence, $\text{rank } M = \text{rank } U_0 + \text{rank } U_1 + \cdots + \text{rank } U_k \geq 2(p\text{-rank } M)$.

(b) is a consequence of (a) and the fact that $R = \text{End } U \cong \text{End } F_A(U)$, since F_A is a fully faithful functor.

(c) Since M is co-purely indecomposable, $\text{rank } M = p\text{-rank } M + 1$. Apply (a) to see that $\text{rank } M = 2$, whence $p\text{-rank } M = 1$ and M is isomorphic to $\langle 1, t \rangle_* = R_t$ for some $t \in A$. $\square$

Define $\Sigma(S_{k+1}, R)$ to be the full subcategory of $\text{rep}(S_{k+1}, R, 0)$ with objects $V = (V_0, V_1, \dots, V_k, V_\#)$ such that $V_0 = V_1 \oplus \cdots \oplus V_k$, $V_\#$ is a pure submodule of V_0 , and $V_\# \cap V_i = 0$ for each $1 \leq i \leq k$.

Lemma 1.5 ([3]). *There is a category equivalence from $\text{rep}(S_k, R, 0)$ to $\Sigma(S_{k+1}, R)$. The functor is given by $(U_0, U_i) \rightarrow (U_1 \oplus \cdots \oplus U_k, U_1, \dots, U_k, U_\#)$, where $U_\#$ is the kernel of the epimorphism $U_1 \oplus \cdots \oplus U_k \rightarrow U_0$ induced by the inclusion of the U_i 's in U_0 .*

Let $V = (V_0, V_1, \dots, V_k, V_\#) \in \Sigma(S_{k+1}, R)$. Given a subset $A = \{t_1, \dots, t_k\}$ of R^* , define $V_\#(A) = \{(t_1 v_1, \dots, t_k v_k) \in V_0^* = V_1^* \oplus \cdots \oplus V_k^* : (v_1, \dots, v_k) \in V_\# \leq V_0 = V_1 \oplus \cdots \oplus V_k\}$. Let $M \in TF(R)$ with a basic submodule $B = B_1 \oplus \cdots \oplus B_k$. Then $M \leq M^* = B^* = B_1^* \oplus \cdots \oplus B_k^*$. Define $M_A = \{(b_1^*, \dots, b_k^*) \in M : (t_1 b_1^*, \dots, t_k b_k^*) \in M\}$ and $M_A(A) = \{(t_1 b_1^*, \dots, t_k b_k^*) : (b_1^*, \dots, b_k^*) \in M_A\}$, a submodule of M with $\text{rank } M_A = \text{rank } M_A(A)$. Clearly, M_A and $M_A(A)$ depend on a choice of basic submodule B and a decomposition $B = B_1 \oplus \cdots \oplus B_k$.

Theorem 1.6. *Let $A = \{t_1, \dots, t_k\}$ be a subset of $R^* \setminus R$ such that $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$ is a Q -independent set for each $1 \leq i \leq k$. There is a faithful embedding*

$$G_A : \text{rep}(S_k, R, 0) \rightarrow TF(R)$$

that is full for all those $U \in \text{rep}(S_k, R, 0)$ with no rank-1 summands. The image of G_A consists of all those M such that $M/(B + M_A(A))$ is torsion, where M_A is contained in a basic submodule $B = B_1 \oplus \cdots \oplus B_k$ of M with each B_i fully invariant in M .

Proof. In view of Lemma 1.5, it suffices to define a functor $G_A : \Sigma(S_{k+1}, R) \rightarrow TF(R)$. Let $V = (V_0, V_i, V_\#) \in \Sigma(S_{k+1}, R)$ and define

$$M = G_A(V) = \langle V_0 + V_\#(A) \rangle_* \leq V_0^* = R^* \otimes_R V_0.$$

Given a representation morphism $f : V \rightarrow V'$, extend $f : V_0 \rightarrow V'_0$ to an R^* -homomorphism $f^* : V_0^* \rightarrow V'^*_0$ and define $G_A(f)$ to be the restriction of f^* to $G_A(V)$. It follows that $G_A(f) : G_A(V) \rightarrow G_A(V')$ and G_A is a faithful functor.

Observe that $B = V_0 = V_1 \oplus \cdots \oplus V_k = B_1 \oplus \cdots \oplus B_k$ is a basic submodule of M . Moreover, $V_\# \leq M_A$ since $V_\#(A) \leq M$. Conversely, $M_A \leq V_\#$. This can be seen by choosing an R -basis of V_0 consisting of a basis for each V_i . This is then an R^* -basis of $V_0^* = V_1^* \oplus \cdots \oplus V_k^*$. Write $m \in M_A$ in terms of that basis and, just as in the proof of Theorem 1.3, equate coefficients and use the independence conditions on A to conclude that $m \in V_\#$. Thus, $V_\# = M_A$ is contained in $B = V_0$, and $M/(B + M_A(A)) = \langle V_0 + V_\#(A) \rangle_*/(V_0 + V_\#(A))$ is torsion.

To see that G_A is full, assume that $V = (V_0, V_i, V_\#)$ and $V' = (V'_0, V'_i, V'_\#) \in \Sigma(S_{k+1}, R)$ have no rank-1 summands and let $g : M = G_A(V) \rightarrow M' = G_A(V')$

be an R -homomorphism. Extend g to an R^* -homomorphism $g^*: M^* = V_0^* \rightarrow (M')^* = (V_0')^*$. Then $g: M_A = V_\# \rightarrow M'_A = V'_\#$. Let $\pi_i: V_0 \rightarrow V_i$ be a projection of V_0 onto the summand V_i . Since V has no rank-1 summands, each $V_i/\pi_i(V_\#)$ must be torsion, otherwise there is a cyclic summand of V_i that gives rise to a rank-1 representation summand of V . Consequently, $g: V_i \rightarrow V'_i$ for each i . But $V_0 = V_1 \oplus \cdots \oplus V_k$, so g induces a representation morphism $f: V \rightarrow V'$ with $G_A(f) = g$, as desired.

Finally, let M be as described in the statement of the theorem. Then $V = (B, B_i, M_A) \in \Sigma(S_{k+1}, R)$, noting that M_A is pure in B and $M_A \cap B_i = 0$ for each i since A is a subset of $R^* \setminus R$. Hence, $G_A(V) = \langle B + M_A(A) \rangle_* = M$. $\square$

Corollary 1.7. *Let $G_A: \text{rep}(S_k, R, 0) \rightarrow TF(R)$ be the functor given in Theorem 1.6.*

(a) *If $M = G_A(U)$ with $U = (U_0, U_i)$, then $p\text{-rank } M = \text{rank } U_1 + \cdots + \text{rank } U_k$ and $\text{rank } M \leq 2(p\text{-rank } M)$. If U has no rank-1 summands, then $\text{rank } M = 2(p\text{-rank } M) - \text{rank } U_0$.*

(b) *Let $U = (U_0, U_i) \in \text{rep}(S_k, Z_p, 0)$ with $k \geq 2$ with $U_0 = R^{k-1} = R \oplus \cdots \oplus R$, $U_1 = R \oplus 0 \oplus \cdots \oplus 0, \dots, U_{k-1} = 0 \oplus 0 \oplus \cdots \oplus R$, and $U_k = R(1, 1, \dots, 1)$. Then $M = G_A(U)$ is co-purely indecomposable with $\text{End } M = R$.*

(c) *If $M = G_A(U)$ is purely indecomposable, then M is isomorphic to R .*

Proof. (a) is a consequence of Lemma 1.5 and Theorem 1.6. Specifically, $\text{rank } V_0 = p\text{-rank } M$, where $V = (U_1 \oplus \cdots \oplus U_k, U_1, \dots, U_k, U_\#) = (V_0, V_1, \dots, V_k, V_\#)$, $\text{rank } M = \text{rank } V_0 + \text{rank } V_\#(A)$, and $\text{rank } V_\#(A) \leq \text{rank } V_0$. Finally, if U , hence V , has no rank-1 summands, then $\text{rank } M = \text{rank } V_0 + \text{rank } V_\#(A) = \text{rank } V_0 + \text{rank } V_\# = \text{rank } V_0 + \text{rank } V_0 - \text{rank } U_0 = 2(p\text{-rank } M) - \text{rank } U_0$.

(b) Apply the functor of Lemma 1.5 to see that $U \rightarrow V = (V_0, V_i)$, where

$$V = (R^k = R \oplus \cdots \oplus R, R \oplus 0 \cdots \oplus 0, \dots, 0 \oplus \cdots \oplus R, R(1, \dots, 1)) \in \Sigma(S_{k+1}, R).$$

By (a), $M = G_A(V) = \langle R^k, R(t_1, \dots, t_k) \rangle_*$ has $p\text{-rank } k$, $\text{rank } k + 1$, and endomorphism ring isomorphic to $\text{End } V = R$. In particular, M is co-purely indecomposable.

(c) As a consequence of (a), $\text{rank } M = p\text{-rank } M = 1$, noting that U_0 must be non-zero. Thus, M is isomorphic to R , as desired. $\square$

The next proposition demonstrates that co-purely indecomposables (purely indecomposables) are generators (co-generators), for modules in $TF(R)$.

Proposition 1.8. *Suppose $M \in TF(R)$ with $p\text{-rank } n$ and $\text{rank } n + k$ for $n, k \geq 1$.*

(a) *There is an exact sequence*

$$0 \rightarrow F \rightarrow C_1 \oplus \cdots \oplus C_k \rightarrow M \rightarrow 0$$

where F is a free R -module and each C_i has $p\text{-rank } n$ and $\text{rank } n + 1$.

(b) *The module M is a pure submodule of some $N = N_1 \oplus \cdots \oplus N_m$ with M/N divisible and $p\text{-rank } N_i = 1$ for each i .*

Proof. (a) Let B be a basic submodule of M of rank n and extend B to a maximal free submodule $K = B \oplus Rm_1 \oplus \cdots \oplus Rm_k$ of M . Then M is the purification of K in M^* and M/B is isomorphic to Q^k . Define C_i to be the purification of $B \oplus Rm_i$ in M . Then B is a basic submodule of C_i with $C_i/B = Q$ so that C_i has rank $n + 1$ and $p\text{-rank } n$. Now $N = C_1 + \cdots + C_k$ is a submodule of M containing B . In fact, $N = M$, as N/B is a k -dimensional Q -space contained in M/B . Inclusion induces an exact sequence $0 \rightarrow F \rightarrow C = C_1 \oplus \cdots \oplus C_k \rightarrow M \rightarrow 0$.

Now $p\text{-rank } C = kn$ and $\text{rank } C = k(n+1)$. Thus, $p\text{-rank } F = kn - n = (k-1)n$ and $\text{rank } F = k(n+1) - (n+k) = (k-1)n$, whence F is a free R -module.

(b) [1, Proposition 4.4.a]. $\square$

2. BUTLER GROUPS AND p -LOCAL GROUPS

A *type* is an isomorphism class of a torsion-free abelian group of rank 1. The set of all types form a lattice [6]. A *Butler group* G is a pure subgroup of a finite direct sum $C_1 \oplus \cdots \oplus C_m$ of rank 1 torsion-free abelian groups C_i [5]. If G is a Butler group, then *typeset* G , the set of types of pure rank-1 subgroups of G , is contained in the lattice of types T generated by the types of C_i 's. In particular, typeset G is finite if G is a Butler group.

Let T be a finite sublattice of the lattice of all types with least element τ_0 and $B(T)$ the category of finite rank Butler groups with typeset contained in T . Let $JI(T)$ be the partially ordered set of join irreducible elements of T and fix a prime p . The lattice T is *p -locally free* if $X_p \cong Z_p$ for each rank-1 group X with type $X \in T$. For $j \geq 0$, let $B(T, j)$ be the full subcategory of $B(T)$ consisting of those G with $p^j G$ contained in $\Sigma\{G(\tau) : \tau \in JI(T)\}$. The isomorphism at p category of $B(T, j)$ is denoted by $B(T, j)_p$; the objects are groups in $B(T, j)$ and morphism sets are $\text{Hom}(G, H)_p = \text{Hom}(G, H) \otimes Z_p$; see [3].

Up to isomorphism at p , there is essentially no loss of generality in considering $B(T, j)$, since if G is in $B(T)$ and has no rank-1 summand of type τ_0 , then G is isomorphic at p to a group $H \in B(T, j)$ for some $j \geq 0$ [3].

Lemma 2.1 ([15]). *If T is a p -locally free finite lattice of types, then there is a category equivalence $H : B(T, j)_p \rightarrow \text{rep}(JI(T)^{op}, Z_p, j)$ given by $H(G) = (G_p, G(\tau)_p : \tau \in JI(T))$.*

Theorems I and II of the introduction follow from the next theorem.

Theorem 2.2. *Let T be a p -locally free finite Boolean algebra of types with atoms $\tau_1, \dots, \tau_k$ and $A = \{t_1, \dots, t_k\}$ a subset of $Z_p^* \setminus Z_p$ such that $\{1, t_j, t_i t_j : 1 \leq j \leq k\}$ is a Q -independent set for each i .*

(a) *For each non-negative integer j , there is a fully faithful embedding*

$$F_A : B(T, j)_p \rightarrow TF(Z_p)$$

given by $F_A(G) = \langle G_p + G(\tau_1)_p t_1 + \cdots + G(\tau_k)_p t_k \rangle_ \leq G_p^*$.*

(b) *There is a faithful embedding*

$$G_A : B(T, 0)_p \rightarrow TF(Z_p)$$

that is full for groups H with no rank-1 summands given by

$$G_A(H) = \langle H(\tau_1)_p \oplus \cdots \oplus H(\tau_k)_p + H_{\#p}(A) \rangle_* \leq H(\tau_1)_p^* \oplus \cdots \oplus H(\tau_k)_p^*$$

where $H_{\#}$ is the kernel of $H(\tau_1) \oplus \cdots \oplus H(\tau_k) \rightarrow H \rightarrow 0$.

Proof. The join irreducibles of a finite Boolean algebra T is an antichain S_k consisting of the atoms of T . The functor F_A is the composite of $H : B(T, j)_p \rightarrow \text{rep}(S_k, Z_p, j)$ given in Lemma 2.1 and the functor $F_A : \text{rep}(S_k, Z_p, j) \rightarrow TF(Z_p)$ given in Theorem 1.3. The functor G_A is the composite of $H : B(T, 0)_p \rightarrow \text{rep}(S_k, Z_p, 0)$ given in Lemma 2.1 and the functor $G_A : \text{rep}(S_k, Z_p, 0) \rightarrow TF$ given in Theorem 1.6. $\square$

Corollary III of the introduction follows immediately from Theorem 2.2 and the following:

Theorem 2.3 ([3]). *If T is a p -locally free finite Boolean algebra of types with k atoms, then $B(T, j)_p$ has:*

- (a) *wild modulo p representation type if and only if $k \geq 3$, $j \geq 0$,*
- (b) *finite representation type if and only if $k \leq 2$, $j \geq 0$.*

The following is a complete list of indecomposables in the image of F_A and the image of G_A for the case that $|A| = 2$, obtained from a complete list of indecomposables in $\text{rep}(S_2, Z_p, j)$ given in [3].

Corollary 2.4. *Let T be a finite p -locally free Boolean algebra of types with atoms τ_1, τ_2 , and let $A = \{t_1, t_2\}$ be a subset of $Z_p^* \setminus Z_p$ such that $\{1, t_1, t_2, t_2 t_1, t_2^2, t_1^2\}$ is a Q -independent set.*

- (a) *If $G \in B(T, j)$ and $M = F_A(G)$ is indecomposable, then M is isomorphic to:*
 - (i) Z_p *if M has p -rank 1 and rank 1,*
 - (ii) $\langle Z_p, Z_p t_1 \rangle_*$ *or* $\langle Z_p, Z_p t_2 \rangle_*$ *if M has p -rank 1 and rank 2,*
 - (iii) $\langle Z_p, Z_p t_1, Z_p t_2 \rangle_*$ *if M has p -rank 1 and rank 3.*
 - (iv) $\langle B, Z_p t_1 \oplus 0, 0 \oplus Z_p t_2 \rangle_* \leq B^*$ *if M has p -rank 2 and rank 4, where*

$$B = Z_p \oplus Z_p + Z_p(1, 1)/p^j \leq Q \oplus Q.$$

- (b) *If $H \in B(T, 0)$ and $M = G_A(H)$ is indecomposable, then M is isomorphic to*

- (i) Z_p *if M has p -rank 1 and rank 1,*
- (ii) $\langle Z_p \oplus Z_p + Z_p(t_1, t_2) \rangle_* \leq Z_p^* \oplus Z_p^*$ *if M has p -rank 2 and rank 3.*

Corollary 2.5. *Assume the notation of Theorem 2.2.*

- (a) *If $M = F_A(G)$ for G in $B(T, j)$, then $\text{rank } G = p\text{-rank } M$ and $\text{rank } M = p\text{-rank } M + \text{rank } G(\tau_1) + \cdots + \text{rank } G(\tau_k)$.*
- (b) *If $G \in B(T, j)$ has rank 1, then $F_A(G)$ is purely indecomposable.*
- (c) *If $H \in B(T, 0)$ and $M = G_A(H)$, then $p\text{-rank } M = \text{rank } H(\tau_1) + \cdots + \text{rank } H(\tau_k)$ and $\text{rank } M \leq 2(p\text{-rank } M)$. If H has no rank-1 summands, then $\text{rank } M = 2(p\text{-rank } M) - \text{rank } H$.*
- (d) *If $H = (X_1 \oplus \cdots \oplus X_k)/(1, 1, \dots, 1)_*$, where X_i is a subgroup of Q containing 1, $1/p \notin X_i$, and type $X_i = \tau_i$ is an atom of T for each i , then $G_A(H)$ is co-purely indecomposable.*

Proof. (a)–(d) follow from Theorem 2.2 and Corollaries 1.4 and 1.7. □

As a consequence of Proposition 1.8 and the following corollary, each M in the image of F_A is both a pure submodule and a homomorphic image of a finite direct sum of modules with p -rank 1 and endomorphism ring R and each M in the image of G_A is both a pure submodule and a torsion-free homomorphic image of a finite direct sum of modules with $\text{rank} - p\text{-rank} = 1$ and endomorphism ring R .

Corollary 2.6. *Assume the notation of Theorems 1.3 and 1.6.*

- (a) *If $M = F_A(G)$, then there is an epimorphism $N_1 \oplus \cdots \oplus N_m \rightarrow M \rightarrow 0$ of R -modules such that each N_i is a purely indecomposable module with $\text{rank} \leq 2$ and $\text{End } N_i = R$.*

(b) If $M = G_A(H)$, then M is a pure submodule of $N_1 \oplus \cdots \oplus N_m$ such that each N_i is a purely indecomposable module with $\text{rank} \leq 2$ and $\text{End } N_i = R$.

(c) Suppose that either $M = F_A(G)$ or else $M = G_A(H)$ and let S be the pure submodule of R^* generated by $\{1, t_1, \dots, t_k\}$. Then $S \otimes_R M \cong S^n$ for $n = p\text{-rank } M$.

(d) Suppose that $M = F_A(G) = G_A(H)$. Then M is a free R -module.

Proof. (a) Write $M = F_A(U) = \langle U_0 + U_1 t_1 + \cdots + U_k t_k \rangle_*$ as in Theorem 1.3. Then $M = N$ for $N = U_0 + R_{t_1} U_1 + \cdots + R_{t_k} U_k$. This is because N/U_0 is torsion-free divisible with $\text{rank} = \text{rank } M/U_0$, recalling that U_0 is a basic submodule of M and $R_{t_i} U_i/U_i \cong Q \otimes_R U_i$.

(b) Write $M = G_A(V) = \langle V_0 + V_{\#}(A) \rangle_* \leq V_0^*$ as in Theorem 1.6. Then M embeds as a pure submodule of $N = R_{t_1} V_1 \oplus \cdots \oplus R_{t_k} V_k$, recalling that $V_0 = V_1 \oplus \cdots \oplus V_k$ and N is pure in $N^* = V_0^*$.

(c) Let $M = F_A(U) = \langle U_0 + U_1 t_1 + \cdots + U_k t_k \rangle_* \leq U_0^* = R^* \otimes_R U_0$ as in Theorem 1.3. Then $S \otimes_R U_0$ is pure in U_0^* , since R is a discrete valuation ring, U_0 is pure in U_0^* , and S is torsion free. Clearly, $M \leq S \otimes_R U_0$ so that $S \otimes_R M = S \otimes_R U_0 \cong S^n$ for $n = \text{rank } U_0 = p\text{-rank } M$, recalling that U_0 is a basic submodule of M . A similar argument shows that if $M = G_A(V) = \langle V_0 + V_{\#}(A) \rangle_* \leq V_0^*$ as in Theorem 1.6, then $S \otimes_R M = S \otimes_R V_0$ with V_0 a basic submodule of M .

(d) Note that if $M = F_A(U)$, then $M(t_i) \neq 0$ for each t_i in A . On the other hand, if $M = G_A(H)$, then $M(t_i) = 0$ unless $A = \{t_1\}$. In this case, $M = F_A(U)$ is a finite direct sum of copies of R and modules of p -rank 1 and rank 2, since $U \in \text{rep}(S_1, R, j)$ is a direct sum of rank-1 representations $(R, 0)$ and (R, R) . On the other hand, R_t is not in the image of G_A by Corollary 1.7. This shows that M is a free R -module. $\square$

REFERENCES

- [1] D. Arnold, *A duality for torsion-free modules of finite rank over a discrete valuation ring*, Proc. London Math. Soc. (3) **24** (1972), 204–216. MR **45**:1895
- [2] ———, *Finite Rank Torsion-free Abelian Groups and Rings*, Lecture Notes in Math. 931, Springer-Verlag, New York, 1982. MR **84d**:20002
- [3] D. Arnold and M. Dugas, *Representation type of posets and finite rank Butler groups*, Colloq. Math. **74** (1997), 299–320. MR **98m**:16016
- [4] D. Arnold and C. Vinsonhaler, *Finite rank Butler groups: a survey of recent results*, Abelian Groups, Lecture Notes in Pure and Applied Math., Marcel Dekker, New York, 1993, 17–41. MR **94c**:20089
- [5] M. C. R. Butler, *A class of torsion-free abelian groups of finite rank*, Proc. London Math. Soc. (3) **15** (1965), 680–698. MR **36**:1532
- [6] L. Fuchs, *Infinite abelian groups*, Vol. II, Academic Press, New York, 1973. MR **50**:2362
- [7] L. Fuchs and L. Salce, *Modules over valuation domains*, Lecture Notes in Pure and Applied Math. 97, Marcel Dekker, New York, 1985. MR **86h**:13008
- [8] L. Lady, *Nearly isomorphic torsion-free abelian groups*, J. Algebra **35** (1975), 235–238.
- [9] ———, *Splitting fields for torsion-free modules over discrete valuation rings*, I, J. Alg. **49** (1977), 261–275. MR **58**:22039
- [10] ———, *Extension of scalars for torsion-free modules over Dedekind domains*, Symposia Math., Vol. 23, Academic Press, London/New York, 1979, 287–305. MR **82i**:13011
- [11] ———, *Splitting fields for torsion-free modules over discrete valuation rings*, II, J. Alg. **66** (1980), 281–306. MR **83g**:16056
- [12] ———, *Splitting fields for torsion-free modules over discrete valuation rings*, III, **66** (1980), 307–320. MR **83g**:16056
- [13] ———, *A seminar on splitting rings for torsion-free modules over Dedekind domains*, Abelian Groups, Lecture Notes in Math. 1006, Springer-Verlag, New York, 1983, 1–48. MR **85f**:13007

- [14] F. Richman, *A class of rank-2 torsion-free groups*, Studies on Abelian Groups, Paris, 1968, 327–333. MR **39**:5695
- [15] ———, *Isomorphism of Butler groups at a prime*, Contemporary Mathematics, American Mathematical Society, vol. 171, 1994, 333–357. MR **95h**:20068

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