POSITIVE SOLUTIONS OF A DEGENERATE ELLIPTIC EQUATION WITH LOGISTIC REACTION

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ABSTRACT. The degenerate elliptic equation $\lambda \Delta_p u + u^{q-1}(1 - u^r) = 0$ with zero Dirichlet boundary condition, where λ is a positive parameter, $2 < p < q$ and $r > 0$, is studied in three aspects: existence of maximal solution, λ -dependence of maximal solution and multiplicity of solutions.

1. INTRODUCTION AND RESULTS

Let Ω be a connected, bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with $C^{2,\alpha}$ -boundary $\partial\Omega$ for some $\alpha \in (0, 1)$. We consider the following degenerate elliptic equation:

$$(P)_{\lambda,\Omega} \begin{cases} \lambda \Delta_p u + f(u) = 0 & \text{in } \Omega, \\ u \geq 0, \neq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive parameter and Δ_p is the p -Laplace operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

with $p > 2$ and f is given by

$$f(u) = u^{q-1}(1 - u^r)$$

with $q \geq 2$ and $r > 0$. We often write ‘ $(P)_\lambda$ ’ instead of ‘ $(P)_{\lambda,\Omega}$ ’. A function $u = u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *solution of $(P)_\lambda$* if $u \geq 0$ a.e. in Ω , u does not vanish in a set of positive measure, and

$$(1.1) \quad -\lambda \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} f(u) \varphi dx = 0$$

for all $\varphi \in W_0^{1,p}(\Omega)$. A solution u of $(P)_\lambda$ is called a *maximal solution of $(P)_\lambda$* if $u \geq v$ a.e. in Ω for all solutions v of $(P)_\lambda$. Obviously, a maximal solution is decided uniquely. If a function $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies $u \geq 0$ (resp. $u \leq 0$) on $\partial\Omega$ and

$$-\lambda \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} f(u) \varphi dx \leq 0 \quad (\text{resp. } \geq 0)$$

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for all $\varphi \in W_0^{1,p}(\Omega)$ satisfying $\varphi \geq 0$ a.e. in Ω , then it is called an *upper* (resp. a *lower*) *solution of* $(P)_\lambda$.

With respect to $(P)_\lambda$, there are a few works on the *equidiffusive case* $p = q$ as follows. Let λ_1 be the first eigenvalue of $-\Delta_p$ under zero Dirichlet boundary condition. In the one-dimensional case $N = 1$, Guedda and Véron [10] have shown by phase-plane analysis that if $\lambda < 1/\lambda_1$, then $(P)_\lambda$ has a unique solution u_λ , and that a set called the *flat core of* u_λ ,

$$(1.2) \quad \mathcal{O}_\lambda = \mathcal{O}_\lambda(u_\lambda) := \{x \in \Omega; u_\lambda(x) = 1\},$$

is non-empty for sufficiently small λ . Since the length of $\mathcal{O}_\lambda$ can be indicated explicitly, we can see that as $\lambda \rightarrow 0$, $\mathcal{O}_\lambda$ spreads out toward the whole of Ω with the growth as

$$(1.3) \quad \lim_{\lambda \rightarrow 0} \lambda^{-1/p} \text{dist}(\mathcal{O}_\lambda, \partial\Omega) = C(f, p),$$

where $C(f, p) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 (F(1) - F(s))^{-1/p} ds$ and $F(s) = \int_0^s f(t)dt$. In the higher-dimensional case $N \geq 2$, phase-plane analysis is no longer useful and one has to use other methods. Constructing a suitable lower solution by using the eigenfunction for λ_1 , Kamin and Véron [12] have proved that the unique solution of $(P)_\lambda$ has a flat core for sufficiently small λ and extended the results of [10]. However, they have given only an estimate $\text{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq C\lambda^{1/p}$ as $\lambda \rightarrow 0$, where C is a constant independent of λ , without explicit information about C and any estimate of $\text{dist}(\mathcal{O}_\lambda, \partial\Omega)$ from below. In virtue of an exact estimate for $\mathcal{O}_\lambda$, García-Melián and Sabina de Lis [9] have utilized the solutions for $N = 1$, whose dependence on λ is understood well, to make upper and lower solutions and concluded that (1.3) also holds true in the case $N \geq 2$. The *subdiffusive case* $p > q$ can also be investigated in the same way as the equidiffusive case. One can observe that there exists a unique solution u_λ for every $\lambda > 0$ and that as the equidiffusive case, $\mathcal{O}_\lambda(u_\lambda)$ is nonempty for sufficiently small $\lambda > 0$ and it grows as in (1.3). See the author and Yamada [19] for $N = 1$ and [9] with its Remarks 2.2 b for $N \geq 2$. For uniqueness, see also Diaz and Saa [5].

On the other hand, the structure of solution set in the *superdiffusive case* $p < q$ is essentially different from those in the other cases. For $N = 1$, using time-map, the author and Yamada [19] have shown that $(P)_\lambda$ produces a spontaneous bifurcation for λ . That is, there exists $\Lambda > 0$ such that if $\lambda > \Lambda$, then $(P)_\lambda$ has no solution; if $\lambda = \Lambda$, then $(P)_\lambda$ has a unique solution; if $\lambda < \Lambda$, then $(P)_\lambda$ has exactly two distinct solutions $\bar{u}_\lambda$ and $\underline{u}_\lambda$ satisfying $\bar{u}_\lambda > \underline{u}_\lambda$ in Ω . It also follows from our analysis that as $\lambda \rightarrow 0$, $\mathcal{O}_\lambda(\bar{u}_\lambda)$ spreads out toward the whole of Ω with (1.3) and $\underline{u}_\lambda \rightarrow 0$ uniformly in Ω . For $N \geq 2$, Guo [11] has studied the case that there exists $\beta > 0$ such that $f(0) = f(\beta) = 0$, $(\beta - x)f(x) > 0$ in $(0, \beta) \cup (\beta, +\infty)$, $\lim_{s \rightarrow 0} f(s)/s^{p-1} = 0$ and $(f(s)/s^{p-1})'' < 0$ in $(0, \beta)$ (the condition ' $f''(x) < 0$ ' in [11, Theorem 3.3] is a misprint and should be replaced by ' $(f(x)/x^{p-1})'' < 0$ '), and have found two distinct solutions. This is the case $p < q < p + 1$ in our problem and no information about the shape of solutions is given. In the present paper, we will discuss $(P)_\lambda$ in the case $2 < p < q$, $N \geq 2$, and study $(P)_\lambda$ in three aspects: (a) existence of solution, especially maximal solution; (b) λ -dependence of maximal solution; and (c) multiplicity of solutions. As for (a), we can prove the following theorem by the method of upper and lower solutions:

Theorem 1.1. *Let $2 < p < q$ and $r > 0$. Then there exists a positive number $\bar{\lambda} > 0$ such that*

- (i) *if $\lambda > \bar{\lambda}$, then $(P)_\lambda$ has no solution;*
- (ii) *if $\lambda \leq \bar{\lambda}$, then $(P)_\lambda$ has a maximal solution $\bar{u}_\lambda$;*
- (iii) *if $\lambda_1 < \lambda_2 \leq \bar{\lambda}$, then $\bar{u}_{\lambda_2} \leq \bar{u}_{\lambda_1}$;*
- (iv) *the mapping $\lambda \mapsto \bar{u}_\lambda$ is left-continuous on $(0, \bar{\lambda}]$ in $C^{1,\beta'}(\bar{\Omega})$ for any $\beta' \in (0, \beta)$, where β is the constant appearing in Proposition 2.1.*

Remark 1.1. Theorem 1.1 (i) has been obtained by Véron [21, Theorem 3] for the p -Laplace operator on a compact Riemannian manifold without boundary.

We will state our result on (b). The proof essentially consists of constructing suitable upper and lower solutions by the idea of García-Melián and Sabina de Lis [8, 9] and the one-dimensional result in [19].

Theorem 1.2. *Let $2 < p < q$ and $r > 0$. There exists a positive number $\lambda^* \in (0, \bar{\lambda}]$ such that*

- (i) *if $\lambda \leq \lambda^*$, then $\mathcal{O}_\lambda = \mathcal{O}_\lambda(\bar{u}_\lambda)$ is non-empty;*
- (ii) *if $\lambda_1 < \lambda_2 \leq \lambda^*$, then $\mathcal{O}_{\lambda_2} \subset \mathcal{O}_{\lambda_1}$;*
- (iii) *for sufficiently small $\varepsilon > 0$, there exists $\lambda \leq \lambda^*$ such that $\Omega \setminus \Omega_\varepsilon \subset \mathcal{O}_\lambda$, where $\Omega_\varepsilon := \{x \in \Omega; \text{dist}(x, \partial\Omega) < \varepsilon\}$.*

Furthermore, $\mathcal{O}_\lambda$ satisfies (1.3) as $\lambda \rightarrow 0$.

Remark 1.2. From the last assertion of Theorem 1.2, we can see that the growth order of maximal solution of $(P)_\lambda$ when $\lambda \rightarrow 0$ is same as that of case $p \geq q$.

To mention (c), we define the functional Φ on $W_0^{1,p}(\Omega)$ corresponding with $(P)_\lambda$:

$$(1.4) \quad \Phi(u) = \frac{\lambda}{p} \|\nabla u\|_p^p - \int_\Omega \bar{F}(u) dx,$$

where $\bar{F}(u) = \int_0^u \bar{f}(s) ds$ and $\bar{f}(s) := f(s)$ in $[0, 1]$, $:= 0$ in $\mathbb{R} \setminus [0, 1]$. By the Mountain Pass Theorem (cf. [1, 17]) for Φ , we can find a distinct solution from $\bar{u}_\lambda$ for small λ ($< \bar{\lambda}$), and consequently deduce the multiplicity of solutions. At this time, it plays an important role that $\Phi(\bar{u}_\lambda)$ becomes negative if λ is sufficiently small. In other words, the larger $\mathcal{O}_\lambda(\bar{u}_\lambda)$ spreads out, the more $\Phi(\bar{u}_\lambda)$ decreases.

Theorem 1.3. *Let $2 < p < q$ and $r > 0$. There exists a positive number $\Lambda \in (0, \bar{\lambda}]$ such that if $\lambda < \Lambda$, then $(P)_\lambda$ has another solution $u_\lambda \leq \bar{u}_\lambda$, $\neq \bar{u}_\lambda$.*

Remark 1.3. We expect that a solution distinct from $\bar{u}_\lambda$ exists for all $\lambda \in (0, \bar{\lambda})$. Theorem 1.3, whose proof directly utilizes the growth of flat hat, gives a partial result for this problem. It will be discussed in the forthcoming paper [18]. (See also Remark 2.2.) In connection with multiplicity for the p -Laplace operator, we can refer to Ambrosetti, Garcia Azorero and Peral [2], Drábek and Pohozaev [7].

2. PROOFS OF RESULTS

The following proposition is fundamental in this paper.

Proposition 2.1. *Let u be a solution of $(P)_\lambda$. Then $u \in C_0^{1,\beta}(\bar{\Omega}) \cap C^{2,\alpha}(\bar{\Omega}_\varepsilon)$ for some $\beta \in (0, 1)$ and sufficiently small $\varepsilon > 0$, and $0 < u(x) \leq 1$ for all $x \in \Omega$.*

Proof of Proposition 2.1. Let u be any solution of $(P)_\lambda$. Putting $\varphi = (u - 1)^+ := \max\{u - 1, 0\} \in W_0^{1,p}(\Omega)$ in (1.1), we have

$$\lambda \int_{\Omega} |\nabla(u - 1)^+|^p dx = \int_{\{u > 1\}} f(u)(u - 1) dx \leq 0.$$

Hence $(u - 1)^+ = 0$ a.e. in Ω ; so $u(x) \leq 1$ a.e. in Ω . This boundedness and a regularity result of Lieberman [13, Theorem 1] deduce that $u \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$. Thus, it follows from Vázquez's maximum principle [20, Theorem 5] that $0 < u (\leq 1)$ in Ω and

$$(2.1) \quad \frac{\partial u}{\partial \mathbf{n}}(x) < 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ denotes an outer normal at $\partial\Omega$. By (2.1) and the fact that $|\nabla u| \in C^{0,\beta}(\overline{\Omega})$, there exists $\varepsilon_0 > 0$ such that $|\nabla u| \geq \delta > 0$ in Ω_{ε_0} for some $\delta > 0$. Therefore, since the equation of $(P)_\lambda$ in Ω_{ε_0} becomes a strictly elliptic one, we can conclude from classical theory that $u \in C^{2,\alpha}(\overline{\Omega_\varepsilon})$ for all $\varepsilon \in (0, \varepsilon_0)$. $\square$

Lemma 2.1. *For sufficiently small $\lambda > 0$, there exists a maximal solution $\bar{u}_\lambda$ such that $\mathcal{O}_\lambda$ is non-empty and*

$$(2.2) \quad \limsup_{\lambda \rightarrow 0} \lambda^{-1/p} \text{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq C(f, p).$$

Proof. Take $R > 0$ and $x_0 \in \Omega$ satisfying $B_R(x_0) \subset \Omega$, where $B_R(x_0)$ is the ball with radius R and center at x_0 . To obtain a lower solution of $(P)_{\lambda,\Omega}$, we will construct a lower solution v_{R,x_0} of $(P)_{\lambda,B_R(x_0)}$. It suffices to find a radially symmetric one, i.e., $v(\rho) = v_{R,x_0}(x)$ satisfying

$$(2.3) \quad \begin{cases} \lambda(\rho^{N-1}|v_\rho|^{p-2}v_\rho)_\rho + \rho^{N-1}f(v) \geq 0 & \text{in } (0, R), \\ v_\rho(0) = v(R) = 0, \end{cases}$$

where $\rho = |x - x_0|$. By a change of variable $\xi = g(\rho)$ such that

$$\xi = g(\rho) = \begin{cases} \frac{R^{1-\theta} - \rho^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1, \\ \log \frac{R}{\rho} & \text{if } \theta = 1, \end{cases}$$

where $\theta := (N - 1)/(p - 1)$, (2.3) can be rewritten as follows:

$$(2.4) \quad \begin{cases} \lambda(|w_\xi|^{p-2}w_\xi)_\xi + g^{-1}(\xi)^{p\theta}f(w) \geq 0 & \text{in } (0, T), \\ w(0) = w_\xi(T) = 0, \end{cases}$$

where $w(\xi) = v(g^{-1}(\xi))$ and $T = +\infty$ if $\theta \geq 1$, $= \frac{R^{1-\theta}}{1-\theta}$ if $\theta < 1$. In order to find a function w satisfying (2.4), we take any $b \in (0, T)$ and consider the following auxiliary boundary value problem:

$$(2.5) \quad \begin{cases} \lambda(|\phi_\xi|^{p-2}\phi_\xi)_\xi + g^{-1}(b)^{p\theta}f(\phi) = 0 & \text{in } (0, b), \\ \phi(0) = \phi(b) = 0. \end{cases}$$

A change of scale $\xi = b\eta$ gives

$$(2.6) \quad \begin{cases} \lambda(|\psi_\eta|^{p-2}\psi_\eta)_\eta + \{bg^{-1}(b)^\theta\}^p f(\psi) = 0 & \text{in } (0, 1), \\ \psi(0) = \psi(1) = 0, \end{cases}$$

where $\psi(\eta) = \phi(b\eta)$. Take λ sufficiently small as

$$\lambda \leq \left\{ \frac{bg^{-1}(b)^\theta}{2C(f, p)} \right\}^p.$$

Then, we already know from [19, Theorem 3.3] that (2.6) has a solution ψ such that $\psi(x) = 1$ in $[C_{\lambda, b}/b, 1 - C_{\lambda, b}/b]$, $0 \leq \psi(x) < 1$ otherwise, where

$$(2.7) \quad C_{\lambda, b} = \frac{C(f, p)}{g^{-1}(b)^\theta} \lambda^{1/p} \quad (\leq b/2).$$

Thus, (2.5) also has a solution ϕ such that $\phi(x) = 1$ in $[C_{\lambda, b}, b - C_{\lambda, b}]$ and $0 \leq \phi(x) < 1$ otherwise. Using ϕ , we construct a function w satisfying (2.4) as follows: $w = \phi$ in $[0, C_{\lambda, b})$, $= 1$ in $[C_{\lambda, b}, T)$. Indeed, since g^{-1} is monotone decreasing,

$$\lambda(|w_\xi|^{p-2}w_\xi)_\xi + g^{-1}(\xi)^{p\theta}f(w) = \{g^{-1}(\xi)^{p\theta} - g^{-1}(b)^{p\theta}\}f(\phi) \geq 0 \quad \text{in } [0, C_{\lambda, b})$$

and the boundary conditions are obviously satisfied. Therefore $v(\rho) = w(g(\rho))$ satisfies (2.3); hence the function

$$(2.8) \quad v_{R, x_0}(x) = \begin{cases} 1 & \text{if } 0 \leq |x - x_0| \leq g^{-1}(C_{\lambda, b}), \\ \phi(g(|x - x_0|)) & \text{if } g^{-1}(C_{\lambda, b}) < |x - x_0| \leq R \end{cases}$$

is a lower solution of $(P)_{\lambda, B_R(x_0)}$. Now, we define $\tilde{v}_{R, x_0}(x) = v_{R, x_0}(x)$ in $B_R(x_0)$, $= 0$ in $\Omega \setminus B_R(x_0)$. Then, one can observe that $\tilde{v}$ is a lower solution of $(P)_{\lambda, \Omega}$. Taking the function $u \equiv 1$ as an upper solution, we obtain a maximal solution $\bar{u}_\lambda$ of $(P)_\lambda$ such that $\tilde{v}_{R, x_0}(x) \leq \bar{u}_\lambda(x) \leq 1$ for all $x \in \Omega$ by Diaz's book [4, Theorem 4.14] (see also Deuel and Hess [3], and Puel [15, Theorem 4.2]). In particular, it follows from (2.8) that $\bar{u}_\lambda(x) = 1$ in $B_{g^{-1}(C_{\lambda, b})}(x_0)$. By the arbitrariness of x_0 satisfying $B_R(x_0) \subset \Omega$ and the uniqueness of maximal solution, it holds that $\bar{u}_\lambda(x) = 1$ in $\Omega \setminus \Omega_{R'}$, where $R' = R'(\lambda, b) = R - g^{-1}(C_{\lambda, b})$. Thus $\text{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq R'$. It follows from (2.7) and l'Hospital's theorem that $R'(\lambda, b) = R^\theta C_{\lambda, b} + o(\lambda^{1/p})$ as $\lambda \rightarrow 0$; so we obtain

$$\limsup_{\lambda \rightarrow 0} \lambda^{-1/p} \text{dist}(\mathcal{O}_\lambda, \partial\Omega) \leq \lim_{\lambda \rightarrow 0} \lambda^{-1/p} R'(\lambda, b) = \left\{ \frac{R}{g^{-1}(b)} \right\}^\theta C(f, p).$$

Passing to the limit as $b \rightarrow 0$, we conclude (2.2). $\square$

Proof of Theorem 1.1. Define

$$\bar{\lambda} = \sup\{\lambda > 0; (P)_\lambda \text{ has a solution}\}.$$

Since Lemma 2.1 implies $\bar{\lambda} > 0$, we will show $\bar{\lambda} < +\infty$ to see (i). Suppose that there exists a sequence $\{\lambda_m\}_{m=1}^\infty$ such that $\lambda_m \rightarrow \infty$ as $m \rightarrow \infty$ and $(P)_{\lambda_m}$ has a solution $u_m = u_{\lambda_m}$. Putting $\lambda = \lambda_m$ and $u = \varphi = u_m$ in (1.1), we have $\lambda_m \|\nabla u_m\|_p^p = \int_\Omega u_m f(u_m) dx$. Since $sf(s) \leq s^p$ for $s \in [0, 1]$ if $p < q$, it follows that $\lambda_m \|\nabla u_m\|_p^p \leq \|u_m\|_p^p$. Combining this inequality and the Poincaré inequality, we obtain $C\lambda_m \|u_m\|_p^p \leq \|u_m\|_p^p$, where C is a positive constant. Since $\|u_m\|_p^p > 0$, the inequality is a contradiction for sufficiently large m . Next, we will prove (ii) and (iii). Consider the case $\lambda < \bar{\lambda}$. From the definition of $\bar{\lambda}$, for $\lambda < \bar{\lambda}$ there exists $\mu \in (\lambda, \bar{\lambda}]$ such that $(P)_\mu$ has a solution u_μ . By an easy calculation, u_μ is a lower solution of $(P)_\lambda$. Since $u \equiv 1$ is an upper solution of $(P)_\lambda$, it follows from [4, Theorem 4.14] that $(P)_\lambda$ admits a maximal solution $\bar{u}_\lambda$ satisfying $\bar{u}_\lambda \geq u_\mu$. (Note that the same arguments give the proof of (iii).) The case $\lambda = \bar{\lambda}$ is treated as follows. Let $\{\lambda_m\}_{m=1}^\infty$ be a positive increasing sequence satisfying $0 < \lambda_m < \bar{\lambda}$ and $\lambda_m \rightarrow \bar{\lambda}$

as $m \rightarrow \infty$, and let $\bar{u}_m$ be the maximal solution of $(P)_{\lambda_m}$. From [13, Theorem 1], we know that $\{\bar{u}_m\}$ is uniformly bounded in $C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$. Thus, Ascoli-Arzelà's theorem assures that there exist $u_{\bar{\lambda}}$ and a subsequence of $\{\bar{u}_m\}$, still denoted by $\{\bar{u}_m\}$, such that $\bar{u}_m \rightarrow u_{\bar{\lambda}}$ in $C^{1,\beta'}(\bar{\Omega})$ for each $\beta' \in (0, \beta)$. It is easy to see that $u_{\bar{\lambda}} \geq 0$ in Ω and that $u_{\bar{\lambda}}$ satisfies (1.1). To observe that $u_{\bar{\lambda}} \not\equiv 0$, we assume $u_{\bar{\lambda}} \equiv 0$. Since $\{\bar{u}_m\}$ converges to 0 uniformly in Ω as $m \rightarrow \infty$, it follows from $p < q$ that for sufficiently large m

$$C\|\bar{u}_m\|_p^p \leq \|\nabla \bar{u}_m\|_p^p = \frac{1}{\lambda_m} \int_{\Omega} \bar{u}_m f(\bar{u}_m) dx \leq \frac{C}{2} \|\bar{u}_m\|_p^p,$$

which contradicts to $\|\bar{u}_m\|_p^p > 0$. Therefore, $u_{\bar{\lambda}}$ is a solution of $(P)_{\bar{\lambda}}$. We have to show the maximality of $u_{\bar{\lambda}}$. Suppose that $u_{\bar{\lambda}}$ is not maximal. Then, $(P)_{\bar{\lambda}}$ has a maximal solution $v_{\bar{\lambda}} \geq u_{\bar{\lambda}}$, ($\neq u_{\bar{\lambda}}$) and there exists $x_0 \in \Omega$ such that $u_{\bar{\lambda}}(x_0) < v_{\bar{\lambda}}(x_0)$. By (iii), since $\bar{u}_m$ decreases toward $u_{\bar{\lambda}}$ as $m \rightarrow \infty$, it holds that $u_{\bar{\lambda}}(x_0) \leq \bar{u}_m(x_0) < v_{\bar{\lambda}}(x_0)$ for sufficiently large m . On the other hand, it follows from (iii) and the fact $\lambda_m < \bar{\lambda}$ that $v_{\bar{\lambda}}(x_0) \leq \bar{u}_m(x_0)$. These inequalities contradict each other; so $u_{\bar{\lambda}}$ is maximal, which can be written as $\bar{u}_{\bar{\lambda}}$. Finally, one can observe (iv) in the similar way as the proof for maximality of $u_{\bar{\lambda}}$. $\square$

Proof of Theorem 1.2. The existence of λ^* satisfying (i) is directly induced from Lemma 2.1 and (ii) follows from (iii) of Theorem 1.1. From the proof of Lemma 2.1, (iii) is obvious for sufficiently small $\varepsilon > 0$ such that $\Omega \setminus \Omega_{\varepsilon} \neq \emptyset$. It remains to show (1.3), i.e., growth-order of $\mathcal{O}_{\lambda}$ as $\lambda \rightarrow 0$ near $\partial\Omega$.

Take any $x_0 \in \partial\Omega$. Let $a > 0$ (resp. $R > 0$) be sufficiently small (resp. large) such that the annulus $A := \{x \in \mathbb{R}^N; a < |x - y_0| < R\}$, where $y_0 := x_0 + a\mathbf{n}$ and $\mathbf{n}$ denotes the outer normal at x_0 , satisfies $\Omega \subset A$. Define $\tilde{u}_{\lambda}$ by $\tilde{u}_{\lambda} := \bar{u}_{\lambda}$ in Ω , $= 0$ in $A \setminus \Omega$. Then $\tilde{u}_{\lambda}$ is a lower solution of $(P)_{\lambda,A}$; so a maximal solution $\bar{v}_{\lambda,A}$ of $(P)_{\lambda,A}$ exists, in particular

$$(2.9) \quad \bar{u}_{\lambda}(x) \leq \bar{v}_{\lambda,A}(x) \quad \text{in } \Omega.$$

From the maximality, $\bar{v}_{\lambda,A}$ is radially symmetric on A ; hence $v(\rho) = \bar{v}_{\lambda,A}(x)$ satisfies

$$(2.10) \quad \begin{cases} \lambda(\rho^{N-1}|v_{\rho}|^{p-2}v_{\rho})_{\rho} + \rho^{N-1}f(v) = 0 & \text{in } (a, R), \\ v(a) = v(R) = 0, \end{cases}$$

where $\rho = |x - y_0|$. As in the proof of Lemma 2.1, we introduce a change of variable

$$\xi = h(\rho) = \begin{cases} \frac{\rho^{1-\theta} - a^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1, \\ \log \frac{\rho}{a} & \text{if } \theta = 1, \end{cases}$$

where $\theta := (N-1)/(p-1)$; then (2.10) can be rewritten as

$$\begin{cases} \lambda(|w_{\xi}|^{p-2}w_{\xi})_{\xi} + h^{-1}(\xi)^{p\theta}f(w) = 0 & \text{in } (0, T), \\ w(0) = w(T) = 0, \end{cases}$$

where $w(\xi) = v(h^{-1}(\xi))$ and $T = h(R)$. It is easy to see that w is a lower solution of

$$(2.11) \quad \begin{cases} \lambda(|\phi_{\xi}|^{p-2}\phi_{\xi})_{\xi} + h^{-1}(b)^{p\theta}f(\phi) = 0 & \text{in } (0, b), \\ \phi(0) = 0, \phi(b) = 1, \end{cases}$$

for any $b \in (0, T)$. Thus, (2.11) has a maximal solution $\bar{\phi}$ such that

$$(2.12) \quad w(\xi) \leq \bar{\phi}(\xi) \quad \text{in } (0, b).$$

In fact, we know from [19, Theorem 3.3] that $0 < \bar{\phi}(\xi) < 1$ in $(0, D_{\lambda,b})$, $\bar{\phi}(\xi) = 1$ otherwise, where $D_{\lambda,b} = C(f, p)\lambda^{1/p}/h^{-1}(b)^\theta$ ($\leq b/2$). Hence, it follows from (2.9) and (2.12) that $\bar{u}_\lambda(x) \leq \phi(h(|x - y_0|)) < 1$ if $x \in \Omega$ and $a < |x - y_0| < h^{-1}(D_{\lambda,b})$. This means that $\text{dist}(x_0, \mathcal{O}_\lambda) \geq h^{-1}(D_{\lambda,b}) - a$ for each $x_0 \in \partial\Omega$. Making $a > 0$ (resp. $R > 0$) sufficiently small (resp. large), one can get an uniform estimate $\text{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq h^{-1}(D_{\lambda,b}) - a$. Since $h^{-1}(D_{\lambda,b}) - a = a^\theta D_{\lambda,b} + o(\lambda^{1/p})$ as $\lambda \rightarrow 0$, it is possible to obtain that

$$\liminf_{\lambda \rightarrow 0} \lambda^{-1/p} \text{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq \left\{ \frac{a}{h^{-1}(b)} \right\}^\theta C(f, p).$$

Passing to the limit $b \rightarrow 0$, we have

$$(2.13) \quad \liminf_{\lambda \rightarrow 0} \lambda^{-1/p} \text{dist}(\mathcal{O}_\lambda, \partial\Omega) \geq C(f, p);$$

so combining (2.13) and (2.2) of Lemma 2.1, we conclude (1.3). $\square$

Remark 2.1. From (2.13) and more delicate analyses of (2.2), we can see

$$\lim_{\lambda \rightarrow 0} \lambda^{-1/p} \sup_{x \in \partial\Omega} \text{dist}(x, \mathcal{O}_\lambda) = C(f, p),$$

which implies that $\mathcal{O}_\lambda$ uniformly spreads out toward the whole of Ω as the order of $\lambda^{1/p}$.

Proof of Theorem 1.3. In virtue of Proposition 2.1, it is well known that u is a solution of $(P)_\lambda$ if and only if u is a critical point of the C^1 -functional Φ , defined by (1.4) (cf. Rabinowitz's book [17, Proposition B.10]). We will check all conditions of the Mountain Pass Theorem (cf. [1, 17]). Take any $q^* \in (p, p^*)$, where $p^* := Np/(N - p)$ if $p < N$, $:= +\infty$ if $p \geq N$, and fix it. Since $p < q$, for any $\delta > 0$ there exists $C_\delta > 0$ such that $|\bar{f}(s)| \leq \delta s^{p-1} + C_\delta s^{q^*-1}$. First, it is easy to see that Φ satisfies the Palais-Smale condition. Indeed, let $\{u_n\}$ be any sequence in $W_0^{1,p}(\Omega)$ such that $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then, it follows from the boundedness of $\bar{F}$ that $\{\|\nabla u_n\|_p\}$ is bounded; namely $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, a result of Dinca, Jebelean and Mawhin [6, Lemma 2.1] yields the assertion. In addition, the Sobolev inequality assures that there exist constants $\gamma, \rho > 0$ such that $\Phi(u) \geq \gamma$ if $\|\nabla u\|_p = \rho$, because

$$\begin{aligned} \Phi(u) &\geq \frac{\lambda}{p} \|\nabla u\|_p^p - \frac{\delta}{p} \|u\|_p^p - \frac{C_\delta}{q^*} \|u\|_{q^*}^{q^*} \\ &\geq \left(\frac{\lambda - C_1 \delta}{p} - \frac{C_2 C_\delta}{q^*} \|\nabla u\|_p^{q^*-p} \right) \|\nabla u\|_p^p \geq \gamma > 0, \end{aligned}$$

where C_1 and C_2 are positive constants, provided that $\delta \in (0, \lambda/C_1)$ and $\|\nabla u\|_p = \rho$ is sufficiently small. Next, clearly $\Phi(0) = 0$ and we will show that the maximal solution $\bar{u}_\lambda$ of $(P)_\lambda$ satisfies $\Phi(\bar{u}_\lambda) < 0$ for sufficiently small $\lambda > 0$. Since $\lambda \|\nabla \bar{u}_\lambda\|_p^p = \int_\Omega \bar{u}_\lambda f(\bar{u}_\lambda) dx$, $\Phi(\bar{u}_\lambda)$ can be expressed as

$$\Phi(\bar{u}_\lambda) = \left(\frac{1}{p} - \frac{1}{q} \right) \|\bar{u}_\lambda\|_q^q - \left(\frac{1}{p} - \frac{1}{q+r} \right) \|\bar{u}_\lambda\|_{q+r}^{q+r}$$

with use of $\overline{F}(\overline{u}_\lambda) = F(\overline{u}_\lambda)$, due to Proposition 2.1. Noting that $\|\overline{u}_\lambda\|_\tau^\tau = |\mathcal{O}_\lambda| + \int_{\Omega \setminus \mathcal{O}_\lambda} |\overline{u}_\lambda|^\tau$ for any $\tau \geq 1$, we have

$$\begin{aligned} \Phi(\overline{u}_\lambda) &= \int_{\Omega \setminus \mathcal{O}_\lambda} \left\{ \left(\frac{1}{p} - \frac{1}{q} \right) |\overline{u}_\lambda|^q - \left(\frac{1}{p} - \frac{1}{q+r} \right) |\overline{u}_\lambda|^{q+r} \right\} dx - \left(\frac{1}{q} - \frac{1}{q+r} \right) |\mathcal{O}_\lambda| \\ &\leq C|\Omega \setminus \mathcal{O}_\lambda| - \left(\frac{1}{q} - \frac{1}{q+r} \right) |\mathcal{O}_\lambda| \\ &= C|\Omega| - \left(C + \frac{1}{q} - \frac{1}{q+r} \right) |\mathcal{O}_\lambda|. \end{aligned}$$

Take $\varepsilon > 0$ so sufficiently small that

$$|\Omega_\varepsilon| < \frac{\frac{1}{q} - \frac{1}{q+r}}{C + \frac{1}{q} - \frac{1}{q+r}} |\Omega|.$$

We see from (iii) of Theorem 1.2 that there exists $\Lambda \in (0, \bar{\Lambda}]$ such that if $\lambda < \Lambda$, then $|\mathcal{O}_\lambda| > |\Omega \setminus \Omega_\varepsilon|$. Thus, if $\lambda \in (0, \Lambda)$, then

$$\begin{aligned} \Phi(\overline{u}_\lambda) &< C|\Omega| - \left(C + \frac{1}{q} - \frac{1}{q+r} \right) |\Omega \setminus \Omega_\varepsilon| \\ &= \left(C + \frac{1}{q} - \frac{1}{q+r} \right) |\Omega_\varepsilon| - \left(\frac{1}{q} - \frac{1}{q+r} \right) |\Omega| \\ &< 0. \end{aligned}$$

Therefore, all conditions for the Mountain Pass Theorem hold; so we obtain a solution u_λ of $(P)_\lambda$, which is distinct from $\overline{u}_\lambda$ and satisfies $\Phi(u_\lambda) > 0$. $\square$

Remark 2.2. In connection with multiplicity of solutions, we have known a number of results on the linear diffusion case. Rabinowitz [16] has studied the case $p = 2 < q$ (for example, equations like $\lambda \Delta u + u^2(1 - u) = 0$) by combining critical point theory and the Leray-Schauder degree theory, and proved there exists $\Lambda > 0$ such that if $\lambda > \Lambda$, then $(P)_\lambda$ has no solution and if $\lambda < \Lambda$, then $(P)_\lambda$ has at least two distinct solutions (see also Ambrosetti and Rabinowitz [1], and Rabinowitz [17]). Particularly, when Ω is a ball, Ouyang and Shi [14] have obtained a precise global bifurcation diagram and concluded that there exist *exactly* two solutions for small λ by using a bifurcation theorem of Crandall and Rabinowitz.

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