SEMIDIRECT SUM OF GROUPS IN WHICH ENDOMORPHISMS ARE GENERATED BY INNER AUTOMORPHISMS

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ABSTRACT. An I–E group is a group G in which every endomorphism is finitely generated by its inner automorphisms. In this paper a characterization for a semidirect sum of I–E groups to be an I–E group is obtained and some well-known results are generalized. We then use this characterization to prove that a semidirect sum of finite I–E groups will again be an I–E group if the normal semidirect summand is unique and fully invariant. Conditions for a group to be an I–E group are also given.

1. INTRODUCTION

A group G is called an I–E group if all endomorphisms of G are generated by inner automorphisms. $\text{Inn}(G)$ and $\text{End}(G)$ denote the set of all inner automorphisms and endomorphisms of G , respectively. The group operation is denoted additively even when G is not necessarily abelian. Consequently, we use semidirect sum instead of semidirect product. Herein we use the right-hand mapping convention: $a(fg) = (af)g$ for all $a \in G$. The terminology used in this paper follows Meldrum [18] and Robinson [21].

The study of I–E groups can be traced back to at least two different origins. In 1963, L. Fuchs [9] had raised the following question: *For which abelian groups G do their automorphism groups $\text{Aut}(G)$ generate the endomorphism ring $\text{End}(G)$?* R. S. Pierce [20], R. W. Stringall [23], H. Freedman [7] and F. Castagna [3] gave certain results on both the positive and the negative sides of this question. During the same period, A. Fröhlich [8] had shown that finite simple groups are I–E groups. The next step in this direction was taken by J. J. Malone and C. G. Lyons; they showed that a dihedral group D_n of order $2n$ is an I–E group only if n is odd [12, 13]. In a further step, J. J. Malone had shown that the generalized quaternion groups Q_n are not I–E groups [15]. These investigations were recently generalized by C. G. Lyons and G. Mason in [14]; they proved that dicyclic groups of order $4n$ with n odd are I–E groups. Next Y. Fong and J. D. P. Meldrum proved that symmetric groups S_n with $n > 4$ are I–E groups [5, 6]. In [22], G. Saad, M. J. Thomsen, and S. A. Syskin claimed alternating groups A_n with $n \neq 4$, and special linear groups $SL(n, q)$, except $SL(2, 3)$, are all I–E groups. For a detailed history of I–E groups, refer either to [16] or [22].

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The above results primarily determine some concrete examples for which a group G is I-E. Recently, questions concerning the structure of I-E groups have been considered. J. J. Malone and G. Mason [17] have shown: *a semidirect sum of cyclic groups of relatively prime order is I-E when the cyclic normal semidirect summand is the commutator subgroup*. C. G. Lyons and G. L. Peterson [10] then made the following improvement: *a semidirect sum of cyclic groups of relatively prime order is I-E*. Observing that a finite abelian group is I-E if and only if it is cyclic, S. A. Syskin [25] proves: *a semidirect sum of two I-E groups of relatively prime orders is an I-E group*.

Their results depend heavily on [10, Theorem 2.1], which characterizes when a semidirect product of I-E groups of relatively prime orders is an I-E group. In this paper, not assuming G to be finite, this characterization theorem is generalized and, at the same time, a fairly concise proof is provided in Theorem 2.1. We then use this result, in Theorem 2.11, to prove that a semidirect sum of I-E groups will be an I-E group if the normal semidirect summand is fully invariant and is a unique minimal normal subgroup. Some conditions to ensure that a group will be an I-E group are also given in Proposition 2.10 and in Theorem 3.4. Moreover, we briefly discuss the direct sum of I-E groups. Using Theorem 3.4, we generalize another result by C. G. Lyons and G. L. Peterson in Corollary 3.5. Examples are provided to illustrate and delimit our results.

The near-ring generated by the group of inner automorphisms $\text{Inn}(G)$ is denoted by $I(G)$, and $E(G)$ will denote the near-ring generated by the endomorphisms $\text{End}(G)$. For a group G and its subgroups H and K , the centralizer of H in K will be denoted as $C_K(H)$. Let $I(G, H) = \{f \in I(G) \mid Gf \subseteq H\}$ and $E(G, H) = \{f \in E(G) \mid Gf \subseteq H\}$. Moreover $\mathcal{J}_2(N)$ denotes the $\mathcal{J}_2$ radical of the near-ring N . Details about the $\mathcal{J}_2$ radical can be found in [18].

2. SEMIDIRECT SUMS OF I-E GROUPS

Recall the $I(G)$ -subgroups [18, p. 157] are equivalent to normal subgroups of G , and the $E(G)$ -subgroups are equivalent to the fully invariant subgroups of G . Therefore a necessary condition for G to be an I-E group is that each normal subgroup must be fully invariant. It can easily be shown that this condition is equivalent for a group to be an I-E group when G is finitely generated abelian. Unfortunately, this condition is not sufficient in general. For example, the group A_4 satisfies this condition but fails to be an I-E group [22]. The following result characterizes the I-E property for a group, which is a semidirect sum of a fully invariant subgroup H and a subgroup K , in terms of the behavior of the projection map.

Theorem 2.1. *Let G be semidirect sum of a fully invariant subgroup H and a subgroup K of G . Suppose H and K are both I-E groups. Then the following are equivalent:*

- (1) G is an I-E group.
- (2) $I(G, H) = E(G, H)$.
- (3) *The projection map $p: G \rightarrow K$ is in $I(G)$ and $p\alpha(1 - p) \in I(G)$ for all $\alpha \in \text{End}(G)$.*

Proof. (1) implies (2) and (2) implies (3) are clear. We want to show that (3) implies (1).

Let $\alpha \in \text{End}(G)$ and write $\alpha = (1 - p)\alpha + p\alpha$ where 1 denotes the identity map of G . Since H is fully invariant, $H\alpha \subseteq H$. Thus

$$\alpha|_H = (1 - p)\alpha|_H \in \text{End}(H).$$

Since H is an I-E group, we have $(1 - p)\alpha|_H = \tilde{r}$ for some $\tilde{r} \in I(H)$. Because $\tilde{r} = \sum_{i=1}^n \varepsilon_i \rho_{h_i}$ where each ρ_{h_i} is an inner-automorphism induced by $h_i \in H$ and $\varepsilon_i \in \{1, -1\}$ for $i \in \{1, 2, \dots, n\}$, we may view $\tilde{r} = r|_H$ where $r = \sum_{i=1}^n \varepsilon_i \rho_{h_i} \in I(G)$. Note that $1 - p \in I(G)$ by our hypothesis and hence $(1 - p)r \in I(G)$. It follows that $(1 - p)\alpha = (1 - p)r \in I(G)$.

Now, we need to show $p\alpha \in I(G)$. Write $p\alpha = p\alpha(1 - p) + p\alpha p$. It suffices to show that $p\alpha p \in I(G)$. Note that $p\alpha p$ is an endomorphism of G and $Gp\alpha p \subseteq K$. Therefore $p\alpha p|_K \in \text{End}(K)$.

By using an argument similar to that used above to prove $(1 - p)\alpha \in I(G)$, we obtain $p\alpha p \in I(G)$. Therefore $p\alpha \in I(G)$ and $E(G) = I(G)$ as desired. $\square$

The following example shows that in Theorem 2.1 the requirement $p\alpha(1 - p) \in I(G)$ is indeed necessary.

Example 2.2. Let $G = A_5 \oplus \mathbb{Z}_5$. Observe that both A_5 and $\mathbb{Z}_5$ are I-E groups and that A_5 is a fully invariant subgroup of G . In fact, A_5 is the commutator subgroup of G . From [25, Theorem 3], we know the projection map $p: G \rightarrow \mathbb{Z}_5$ is in $I(G)$. However, the order of $E(G)$ is $59^{59} \cdot 5 \cdot 60^{240}$ and the order of $I(G)$ is $59^{59} \cdot 5$ by using the results presented in [24]. Therefore G is not an I-E group.

Corollary 2.3. *Let G be semidirect sum of a fully invariant subgroup H and a subgroup K of G . Let $p: G \rightarrow K$ be the projection map. Suppose H and K are both I-E groups and $\{p\} \cup \mathcal{J}_2(E(G)) \subseteq I(G)$. Then G is an I-E group.*

Proof. It is routine to verify that $p\alpha(1 - p)\beta$ is nilpotent for all $\alpha, \beta \in E(G)$. Since $\mathcal{J}_2(E(G))$ contains all nil right $E(G)$ -subgroups [18, Corollary 5.24], $\mathcal{J}_2(E(G))$ must contain $p\alpha(1 - p)E(G)$, and in particular contains all the elements $p\alpha(1 - p)$ for all $\alpha \in \text{End}(G)$. By Theorem 2.1, G is an I-E group. $\square$

A 0-symmetric near-ring N is called *2-primal* if the prime radical $\mathcal{P}_0(N)$ is equal to the completely prime radical $\mathcal{P}_2(N)$. Examples for 2-primal near-rings are abundant. Let G be finite dihedral group with order not divisible by 4 or the generalized quaternion group. Then $E(G)$ is 2-primal. For more details, please refer to [1, 2].

Corollary 2.4. *Let G be semidirect sum of a fully invariant subgroup H and a subgroup K of G . Let $p: G \rightarrow K$ be the projection map. Suppose H and K are both I-E groups and $E(G)$ is 2-primal. If $\{p\} \cup \mathcal{P}_0(E(G)) \subseteq I(G)$, then G is an I-E group.*

Proof. Since $\mathcal{P}_0(E(G))$ contains all the nilpotent elements including $p\alpha(1 - p)$ for all $\alpha \in \text{End}(G)$, the assertion follows from Theorem 2.1. $\square$

The utility of Corollary 2.3 and Corollary 2.4 can be readily demonstrated by considering $E(S_3)$ where S_3 is the symmetric group of order 6. Observe that $\{p\} \cup \mathcal{J}_2(E(S_3)) \subseteq I(S_3)$ and $E(S_3)$ is 2-primal with $\{p\} \cup \mathcal{P}_0(E(S_3)) \subseteq I(S_3)$ [11]. Hence $E(S_3)$ illustrates both Corollary 2.3 and 2.4. As a corollary, we obtain one of the main results of C. G. Lyons and G. L. Peterson [10].

Corollary 2.5 ([10, Theorem 2.1]). *Suppose that G is the semidirect sum of a normal subgroup H and a subgroup K where $(|H|, |K|) = 1$ and H, K are both I-E groups. Then the following are equivalent:*

- (1) G is an I-E group.
- (2) The projection map $p : G \rightarrow K \in I(G)$.

Proof. Let π be the set of prime factors of $|H|$. The hypothesis that $(|H|, |K|) = 1$ implies that G is a Hall π -separable group and H is the unique Hall π -subgroup of G . From Theorem 9.1.6 in [21] and the fact that the homomorphic image of a π -subgroup is a π -subgroup, we see that H is a fully invariant subgroup of G . Observe that $p\alpha(1-p) = 0$ for all $\alpha \in \text{End}(G)$ when the order of H and K are relatively prime. By applying Theorem 2.1 above, we obtain the result. $\square$

The following example shows that Theorem 2.1 is a proper generalization of the Lyons-Peterson result [10, Theorem 2.1].

Example 2.6. Let $G = A_8 \oplus PSL(3, 4)$ where A_8 is the alternating group of degree 8 and $PSL(3, 4)$ is the projective special linear group of order $20160 = 8!/2$. Note that $PSL(3, 4)$ is not isomorphic to A_8 ; for it can be demonstrated that $PSL(3, 4)$ has no elements of order 15, unlike A_8 which has $(1, 2, 3, 4, 5)(6, 7, 8)$, an element of order 15. Let $p : G \rightarrow PSL(3, 4)$ be the projection map. Observe that both A_8 and $PSL(3, 4)$ are fully invariant subgroups of G . Hence $p\alpha(1-p) = 0 \in I(G)$ for all $\alpha \in \text{End}(G)$. The restriction map $p|_{PSL(3,4)}$ is an endomorphism of $PSL(3, 4)$ which is an I-E group. Therefore $p|_{PSL(3,4)} \in I(PSL(3, 4))$. On the other hand, $p|_{A_8} = 0 \in I(A_8)$ and so $p \in I(G)$. Hence G is an I-E group by Theorem 2.1. Note that the orders of A_8 and $PSL(3, 4)$ are not relatively prime. In fact, they have the same order.

Let G be a semidirect sum of a normal subgroup H and a subgroup K . From [19, Lemma 4.1], we know that K being an I-E group is necessary for G to be an I-E group. In Example 2.7, we exhibit a group G such that H is fully invariant in G and K is an I-E group with $(|H|, |K|) = 1$. Moreover, the projection map $p : G \rightarrow K$ is in $I(G)$, but G is not an I-E group. Therefore assuming that H is an I-E group in Corollary 2.5 is not superfluous. However, Example 2.8 shows that, in general, G can be an I-E group with H fully invariant, but H is not an I-E group.

Example 2.7. We first quote a result from [22, Theorem 16]: Let G have a minimal normal subgroup H of order p^n for some prime p and $n \geq 1$ such that $C_G(H) = H$ and G/H is cyclic of order q prime to p . Then $I(G) = E(G)$ if and only if $n = 1$.

Let $H \cong \bigoplus_{i=1}^n \mathbb{Z}_2$ where $n \neq 1$ and $K = \mathbb{Z}_3$. Let G be the semidirect sum of H with K . G is not an I-E group. Here it can be seen that H is a fully invariant subgroup of G and K is an I-E group. But $H \cong \bigoplus_{i=1}^n \mathbb{Z}_2$ is not an I-E group when $n \neq 1$. Note that the projection map $p : G \rightarrow K$ is in $I(G)$ by [25, Theorem 2].

Example 2.8. Let G be the symmetric group S_4 of degree 4 and consider S_4 as the semidirect sum of the alternating group A_4 and the cyclic group $\mathbb{Z}_2$. S_4 is an I-E group but the fully invariant subgroup A_4 is not an I-E group.

Using the condition of relatively prime on the order of a subgroup and its index, Proposition 2.10 shows that the I-E property can be lifted from a maximal normal subgroup. Note that Proposition 2.10 can be deduced from [25, Theorem 2]. We will provide a detailed constructive proof for easier reference and hopefully motivate

some clue to improve this result. The following lemma was quoted from [25] which we will use in Proposition 2.10.

Lemma 2.9 ([25, Theorem 1]). *Let G be a finite group with a unique minimal normal subgroup H . Assume that G is a semidirect sum of H and a subgroup K . Then the projection map $p: G \rightarrow K$ is in $I(G)$.*

Proposition 2.10. *Let G be a finite group with a maximal normal subgroup H such that the order of H is coprime to its index. If H is an I-E group, then G is an I-E group.*

Proof. By using the Schur–Zassenhaus theorem, we know the complement K of H in G exists. Since G/H is simple, it is an I-E group. From Corollary 2.5, we can conclude that G is an I-E group if the projection map $p: G \rightarrow K$ is in $I(G)$.

Note that the centralizer of H in K , denoted $C_K(H)$, is 0 or K by the maximality of H . If $C_K(H) = K$, then $G = H \oplus K$ with $(|H|, |K|) = 1$. Since both H and K are I-E groups, G is an I-E group by [10, Corollary 2.2].

So assume $C_K(H) = 0$. If H is also a minimal normal subgroup, then it is unique and so $p \in I(G)$ by Lemma 2.9. Suppose H is not minimal. Let $0 = H_n \subseteq H_{n-1} \subseteq \cdots \subseteq H_0 = H$ be a principal series of H . Since H is an I-E group, each H_i is a fully invariant subgroup of H and thus a normal subgroup of G for all $i = 0, 1, \dots, n$. Therefore H_{n-1} is a minimal normal subgroup of G .

We now want to show that $p_{n-1}: H_{n-1} + K \rightarrow K$ is in $I(H_{n-1} + K)$. Since H_{n-1} is a minimal normal subgroup of G , $H_{n-1} = \bigoplus_{i=1}^m S_i$ where the S_i are mutually isomorphic simple groups. Moreover $K \cong G/H$ is also a simple group by maximality of H . Let $G_{n-i} = H_{n-i} + K$ for all $i = 1, 2, \dots, n$. The centralizer $C_K(H_{n-1})$ is a normal subgroup of K , and so must be 0 or K .

Case I: Assume $C_K(H_{n-1}) = K$.

Let s be the order of H_{n-1} and let t be the order of K . Since s, t are relatively prime, there exist integers u, v such that $us + vt = 1$. Now for all $h \in H_{n-1}$, $k \in K$, $us(h + k) = ush + usk = (1 - vt)k = k$. Therefore $p_{n-1} = us1 \in I(G_{n-1})$, where 1 denotes the identity map, is the desired projection from G_{n-1} to K .

Case II: Assume $C_K(H_{n-1}) = 0$.

If H_{n-1} is a minimal normal subgroup of G_{n-1} , it is unique and so $p_{n-1}: G_{n-1} \rightarrow K$ is in $I(G_{n-1})$ by Lemma 2.9. If H_{n-1} is not minimal, then without loss of generality, we assume $Q = S_1 \oplus S_2 \oplus \cdots \oplus S_r$ with $r < m$ a minimal normal subgroup of G_{n-1} . By repeating the argument in Case I, we may assume Q is a minimal normal subgroup of $Q + K$ and $C_K(Q) = 0$. Therefore the projection $q_1: Q + K \rightarrow K$ is in $I(Q + K)$.

Applying the above arguments inductively on the group $\bar{G}_{n-1} = G_{n-1}/Q \cong S_{r+1} \oplus S_{r+2} \oplus \cdots \oplus S_m + K$, there exists $q_2 \in I(G_{n-1})$ such that for all $h \in H_{n-1}$, $k \in K$, we have $(h + k)q_2 = c + k$ for some $c \in Q$. A routine argument yields that $p_{n-1} = q_2q_1 \in I(G_{n-1})$ is the desired projection from G_{n-1} to K .

Now, let $\bar{G} = G/H_{n-1}$. Then $\bar{H}_{n-2}$ is a minimal normal subgroup of $\bar{G}$. Similar reasoning as in Cases I and II above yields that $\bar{p}_{n-2}: G_{n-2}/H_{n-1} \rightarrow K/H_{n-1}$ is in $I(G_{n-2}/H_{n-1})$. Hence there exist maps p_{n-1} and $\nu \in I(G_{n-2})$ such that for all $x \in H_{n-1}$, $h \in H_{n-2}$, and $k \in K$, we have $(x + k)p_{n-1} = k$ and $(h + k)\nu = y + k$ for some $y \in H_{n-1}$. Therefore $(h + k)\nu p_{n-1} = (y + k)p_{n-1} = k$. So $p_{n-2} = \nu p_{n-1} \in I(G_{n-2})$ is the desired projection from G_{n-2} to K .

Inductively, we conclude that the projection $p = p_0: G_0 = G \rightarrow K$ is in $I(G)$. This completes the proof. $\square$

Note that when coprimeness is not assumed in Proposition 2.10, there is no obvious evidence to ensure the lifting of the I-E property from a maximal normal subgroup H to G . In Example 2.2, A_5 is maximal in G , but G is not an I-E group. On the other hand, the I-E condition cannot be inherited by a maximal normal subgroup, in general, as shown in Example 2.8.

In the final result of this section, we do not assume the relatively prime condition on the order of H and K .

Theorem 2.11. *Let G be a finite group and let G be a semidirect sum of a fully invariant subgroup H and a subgroup K . Suppose H is a unique minimal normal subgroup of G . If H and K are both I-E groups, then G is an I-E group.*

Proof. Note that the assumption on H implies the projection map $p \in I(G)$ by Lemma 2.9.

We first consider the case when H is nonabelian. Since H is a minimal normal subgroup of G , H is characteristically simple and so the automorphism near-ring $A(H) = M_0(H)$ by [18, Theorem 10.11]. Moreover, the hypothesis that H is an I-E group together with $A(H) = M_0(H)$ implies $I(H) = M_0(H)$ and thus H is a finite simple nonabelian group by a result of Frölich [8].

Since H is normal in G , the centralizer $C_G(H)$ is a normal subgroup of G . By uniqueness of H , $C_G(H)$ must be contained in H and therefore $C_G(H) = 0$ because H is simple nonabelian.

Let $\alpha \in \text{End}(G)$ be arbitrary. If $Gp\alpha(1-p) = K\alpha(1-p) = 0$, then $p\alpha(1-p) = 0 \in I(G)$. If $K\alpha(1-p) \neq 0$, let $W = \{k \in K \mid k\alpha(1-p) \neq 0\}$. Observe that $1 - \rho_a \in I(G, H)$ for all $a \in H$ where ρ_a is the inner automorphism of G induced by element a . Let $k \neq 0 \in K$. If $k(1 - \rho_a) = 0$ for all $a \in H$, then $k \in C_G(H) = 0$ and thus there must exist some $b \in H$ with respect to k such that $k(1 - \rho_b) \neq 0$. Hence the hypothesis for Theorem 10.24 in [18] holds for the set $K \setminus \{0\}$. By applying [18, Theorem 10.24], there exists $\eta \in I(G)$ such that

$$(2.1) \quad k\eta = \begin{cases} k\alpha(1-p), & \text{if } k \in W; \\ 0, & \text{if } k \in K \setminus W. \end{cases}$$

It is now easy to verify that $p\alpha(1-p) = p\eta \in I(G)$ for all $\alpha \in \text{End}(G)$. By Theorem 2.1, G is an I-E group.

Now consider the case when H is abelian. The assumption that H is I-E and finite abelian implies that H is a cyclic group of prime order. Moreover, that H is a unique minimal normal subgroup of G implies on the one hand the centralizer $C_G(H) = H$ because $C_G(H)$ is a normal subgroup of G , and on the other hand, any two nonzero normal subgroups H_1, H_2 of G satisfying the commutator $[H_1, H_2] = 0$ must have $H_1 = H_2 = H$.

It is now not difficult to verify that the hypothesis for Theorem 3.5 in [4] holds for the set $K \setminus \{0\}$. By using this theorem, there exists an $\eta \in I(G)$ such that, for any $\alpha \in \text{End}(G)$, we have $k\alpha(1-p) = k\eta$ for all $k \in K$. Hence $p\alpha(1-p) = p\eta \in I(G)$ and thus G is an I-E group by Theorem 2.1. $\square$

As a quick application of the above result, we immediately have the symmetric group S_n with $n \geq 5$ and the dihedral group D_n of order $2n$ with n odd are all I-E groups.

3. DIRECT SUMS OF I-E GROUPS

Proposition 3.1. *Let G be direct sum of normal subgroups H and K of G . Then:*

- (1) *If G is an I-E group, then H and K are both I-E groups. In particular, all direct components of an I-E group are I-E groups.*
- (2) *Suppose that both H and K are fully invariant subgroups of G . If H and K are both I-E groups, then G is an I-E group.*

Proof. (1) Since endomorphisms of H and K extend to endomorphisms of G and elements of $I(G)$ restricted to H and K yield elements of $I(H)$ and $I(K)$, it follows that H and K will be I-E groups if G is an I-E group.

(2) Let $\alpha \in \text{End}(G)$. Since H is fully invariant, α restricted to H gives an endomorphism of H . Assuming that H is an I-E group, we may represent $\alpha|_H$ as a finite sum of inner automorphisms of H . Therefore $\alpha|_H \in I(H)$. Similarly, $\alpha|_K \in I(K)$. Let $\mu \in I(H)$, $\nu \in I(K)$ such that $\alpha|_H = \mu|_H$ and $\alpha|_K = \nu|_K$. Note that here $\mu|_K = 1|_K$ and $\nu|_H = 1|_H$ where 1 is the identity map on G . It is then routine to verify that $\alpha = \mu - 1 + \nu \in I(G)$. $\square$

From Proposition 3.1(1), we know that the direct summand of an I-E group is an I-E group. However, this property does not hold for a fully invariant subgroup. Observe that V_4 (i.e., Klein 4-group) is a fully invariant subgroup of the I-E group S_4 , but V_4 is not an I-E group. In Proposition 3.1(2), the requirement that both H and K be fully invariant is not superfluous as we can see in the following examples. Recall that an abelian group is I-E if and only if it is cyclic. So a direct sum of cyclic groups is not I-E if it is not cyclic.

(1) In the infinite case, consider the group G as the direct sum of the integers $\mathbb{Z}$ and the group $\mathbb{Z}_2$ of order 2. G is not an I-E group, but both $\mathbb{Z}$ and $\mathbb{Z}_2$ are I-E groups. Here $\mathbb{Z}$ is not a fully invariant subgroup of G , but $\mathbb{Z}_2$ is an I-E group.

(2) In the finite case, consider V_4 as the direct sum of two copies of $\mathbb{Z}_2$. V_4 is not an I-E group but $\mathbb{Z}_2$ is an I-E group. However $\mathbb{Z}_2$ is not a fully invariant subgroup of V_4 .

Making use of Proposition 3.1, we can reprove the following corollary by C. G. Lyons and G. L. Peterson [10] without using Corollary 2.5.

Corollary 3.2 ([10, Corollary 2.2]). *If G is the direct sum of H and K where both H and K are I-E groups and if $(|H|, |K|) = 1$, then G is an I-E group.*

Proof. We first show that H and K will be fully invariant subgroups of G . Pick $h \in H$ and $\alpha \in \text{End}(G)$. Then $h\alpha = a + b$ for some $a \in H$ and $b \in K$. Suppose the order of H is m and the order of K is n . Then $0 = (mh)\alpha = m(h\alpha) = ma + mb = mb$, which then implies that the order of b , say $|b|$, is a factor of m . But $|b|$ must divide n and so $|b|$ is a common factor of m and n . Therefore $|b| = 1$ and $h\alpha = a \in H$. Hence H is a fully invariant subgroup of G . Similarly, K is a fully invariant subgroup of G . The result then follows from Proposition 3.1. $\square$

Corollary 3.3. *A finite nilpotent group is an I-E group if and only if all its Sylow subgroups are I-E groups.*

Proof. Recall that a finite group is nilpotent if and only if it is the direct sum of its Sylow subgroups. By Proposition 3.1 and Corollary 3.2, a finite nilpotent group is an I-E group if and only if all its Sylow subgroups are I-E groups. $\square$

Theorem 3.4. *Let G be a finite group with an abelian normal subgroup H such that the order of H is coprime to its index. If G/H and the centralizer $C_G(H)$ are I-E groups, then G is an I-E group.*

Proof. By the Schur–Zassenhaus theorem, the complement K of H in G exists. Since H is a normal subgroup of G , the centralizer $C_G(H)$ is normal in G . Note that $C_G(H)$ is the direct sum of H and $C_K(H)$. By Proposition 3.1(1), H is an I-E group. Since the order of H and K are coprime, the projection map $p : G \rightarrow K$ is in $I(G)$ by [25, Theorem 2]. Also $K \cong G/H$ is an I-E group; therefore G is an I-E group by Corollary 2.5. $\square$

As a corollary, we obtain Theorem 3.2 [10] of C. G. Lyons and G. L. Peterson [10, Theorem 3.2].

Corollary 3.5 ([10, Theorem 3.2]). *If G is the semidirect sum of a cyclic normal subgroup H and a cyclic subgroup K where $(|H|, |K|) = 1$, then G is an I-E group.*

Proof. Note that $C_K(H)$, as a subgroup of a cyclic group K , is cyclic. $C_K(H)$ is an I-E group. Because the order of H and $C_K(H)$ are relatively prime, $C_G(H) = H \oplus C_K(H)$ is an I-E group by Corollary 3.2. Hence G is an I-E group by Theorem 3.4. $\square$

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