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# A COMPACT SET WITH NONCOMPACT DISC-HULL

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ABSTRACT. The disc-hull of a set is the union of the set and all $H^\infty$ discs whose boundaries lie in the set. We give an example of a compact set in $\mathbb{C}^2$ whose disc-hull is not compact, answering a question posed by P. Ahern and W. Rudin.

The polynomial hull of a compact set $X \subset \mathbb{C}^n$ is the set $\widehat{X}$ of all points $x \in \mathbb{C}^n$ at which the inequality $|P(x)| \le \max\{|P(z)| : z \in X\}$ holds for every polynomial $P$. Let $U$ denote the unit disc in $\mathbb{C}$. In [1] P. Ahern and W. Rudin introduced the following definition.

"If $\Phi : U \to \mathbb{C}^n$ is a non-constant map whose components are in $H^\infty(U)$, its range $\Phi(U)$ is called an $H^\infty$-disc, parametrized by $\Phi$. If $\lim_{r \nearrow 1}(\Phi(re^{i\theta})) \in X$ for almost all $e^{i\theta}$ on the unit circle $T$, then $\Phi(U)$ is an $H^\infty$-disc whose boundary lies in $X$."

They further define the disc-hull $D(X)$ to be the union of $X$ and all $H^\infty$-discs whose boundaries lie in $X$. Because of the maximum principle, $D(X) \subset \widehat{X}$. One of the questions posed in [1] (see p. 25) is whether the disc-hull $D(X)$ is always compact for a compact set $X \subset \mathbb{C}^n$.

Below we answer this question negatively by constructing a counter-example in $\mathbb{C}^2$.

1. Define $\omega = \{z \in U : \operatorname{Re} z > \frac{1}{2}\}$. Let $\varphi : \overline{U} \to \overline{\omega}$ be the Riemann map satisfying $\varphi(\pm i) = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ and $\varphi(1) = 1$. Therefore $\operatorname{Re} \varphi(e^{i\theta}) = \frac{1}{2}$ for $\operatorname{Re} e^{i\theta} \le 0$. Also, $0 \notin \varphi(\overline{U})$, $|\varphi(0)| < 1$, and hence $\lim_{n\to\infty} \varphi^n(0) = 0$.

2. Let $X = \{(\zeta, \eta) \in \mathbb{C}^2 : \zeta \in T, \eta \in \Gamma_\zeta\}$, where the fiber $\Gamma_\zeta$ is defined as follows. $\Gamma_\zeta = T$ for $\operatorname{Re} \zeta > 0$, $\Gamma_\zeta = \overline{U}$ for $\zeta = \pm i$, and $\Gamma_\zeta = \{\varphi^n(\zeta) : n \in \mathbf{N}\} \cup \{0\}$ for $\operatorname{Re} \zeta < 0$.

One can check that the complement $\mathbb{C}^2 \backslash X$ of $X$ is an open set and, since $X$ is also bounded, it is compact. One can also notice that $X$ is connected.

3. For each $n$ consider

$$\Phi_n(z) = (z, \varphi^n(z)) : U \to \mathbb{C}^2.$$

By construction, $\Phi_n(T) \subset X$, so, $\Phi_n(U) \in D(X)$. One can see that $\lim_{n\to\infty} \Phi_n(0) = (0,0)$; therefore $(0,0) \in \overline{D(X)}$.

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4. To complete the example we now need to show that $(0,0) \notin D(X)$. If not, then there is an $H^\infty$-disc $\Phi(U)$, $\Phi(z) = (\alpha(z), \beta(z)) : U \to \mathbb{C}^2$, such that $\lim_{r\nearrow 1}(\Phi(rt)) \in X$ for almost all $t \in T$ and $(0,0) \in \Phi(U)$. Without loss of generality we may assume that $\Phi(0) = (\alpha(0), \beta(0)) = (0,0)$ (one can consider $\Phi \circ \psi$ in place of $\Phi$ for a suitable Möbius transformation $\psi$). Since by construction $\lim_{r\nearrow 1}(\alpha(re^{i\theta})) \in T$ for almost all $e^{i\theta}$, $\alpha(z)$ is a nonconstant inner function.

For any function $u(z) \in H^\infty(U)$ the radial $\lim_{r\nearrow 1}(u(re^{i\theta}))$ exists for almost all $t = e^{i\theta} \in T$. Let $u(t)$ denote the corresponding limit. The following property of an $H^\infty$ function $u(z)$ (see [2], p. 339) will be used:

$$\text{If } u(t) = 0 \text{ for } t \in T' \subset T, \text{ and } T' \text{ has positive measure,} \\ \text{then } u(z) = 0 \text{ for all } z \in U. \tag{1}$$

For the function $\alpha(z)$ we introduce the set

$$T_0 = \{t \in T : \alpha(t) \text{ exists and } \alpha(t) \in T\},$$

so $T_0$ is almost all of $T$. Consider the following sets:

$$\begin{aligned} S^- &= \alpha^{-1}\{e^{i\theta} : \operatorname{Re} e^{i\theta} < 0\} \cap T_0, \\ S^+ &= \alpha^{-1}\{e^{i\theta} : \operatorname{Re} e^{i\theta} > 0\} \cap T_0, \\ S^0 &= \alpha^{-1}\{e^{i\theta} : \operatorname{Re} e^{i\theta} = 0\} \cap T_0. \end{aligned}$$

Our main goal now is to prove that $S^-$ has positive measure. Notice that $S^- \cup S^+ \cup S^0 = T_0$ which has full circle measure.

The set $\alpha(S^0)$ consists of two points and if $S^0$ had positive measure, then according to (1), $\alpha(z)$ would be constant. Therefore, $S^0$ has measure 0. If $S^+$ had the full measure, then $\operatorname{Re}\alpha(0)$ would be positive since it is the average of its values on $T$, but $\alpha(0) = 0$. Therefore, the measure of $S^-$ is positive.

Introduce now the following functions: $u_p(z) = \beta(z) - \varphi^p(\alpha(z))$ for $p = 1, 2, ...$; $u_0(z) = \beta(z)$. All of them are in $H^\infty(U)$. Define $S_p = \{t \in T : u_p(t) = 0\}$. One can see that by construction almost all points of $S^-$ lie in $\bigcup S_p$. Therefore there exists a $q$ such that $S_q$ has positive measure.

If $q = 0$, then by (1), $\beta(z) = u_0(z) = 0$ on $U$. This implies that $X$ (containing almost all of $\Phi(T)$) contains almost all points $(e^{i\theta}, 0)$. This is impossible since for all $\operatorname{Re} e^{i\theta} > 0$, the point $(e^{i\theta}, 0) \notin X$.

Therefore $q > 0$. So, by (1), $u_q(z) = \beta(z) - \varphi^q(\alpha(z)) = 0$ for all of $U$. Now $0 = \beta(0) = \varphi^q(\alpha(0))$, and $\varphi(0) = 0$, contradicting $0 \notin \varphi(\overline{U})$.

*Remark.* One can see that the entire disc $(U, 0) \subset \widehat{X}$ and all the points of this disc belong to $\overline{D(X)}$ but not to $D(X)$.

## Added after posting

After the galley proofs were returned, the authors were informed by J. Globevnik that a different counterexample was published by Herb Alexander in "A disc-hull in $\mathbb{C}^2$", Proc. Amer. Math. Soc. **120** (1994), 1207–1209.

## References


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