THE ACTION OF A SEMISIMPLE LIE GROUP
ON ITS MAXIMAL COMPACT SUBGROUP

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ABSTRACT. In this paper we determine the structure of the minimal ideal in
the enveloping semigroup for the natural action of a connected semisimple Lie
group on its maximal compact subgroup. In particular, if $G = KAN$ is an
Iwasawa decomposition of the group G, then the group in the minimal left
ideal is isomorphic both algebraically and topologically with the normalizer
M of AN in K. Complete descriptions are given for the enveloping semigroups
in the cases $G = \mathrm{SL}(2, \mathbb{C})$ and $G = \mathrm{SL}(2, \mathbb{R})$.

1. INTRODUCTION

Throughout this paper G will be a connected semisimple Lie group with finite
centre. It therefore has an Iwasawa decomposition, that is $G = KAN$ is a product
of closed connected subgroups with K compact, A abelian and N nilpotent, and
the decomposition $g = kan$ of any element of G is unique. The prime examples
of these groups are the special linear groups $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$; here a suit-
able decomposition can be obtained by taking $K = \mathrm{SO}(n, \mathbb{R})$ in the real case and
$\mathrm{SU}(n)$ in the complex case, A to be the diagonal matrices with positive entries and
determinant 1, and N to be the upper triangular matrices with 1's on the diagonal.

The subgroup AN is closed and the homogeneous space G/AN is naturally
homeomorphic to the compact space K. G therefore has a natural action on K; if
we identify the coset gAN with $k \in K$ we can indicate the action of $g' \in G$ by

$$g' : k = gAN \mapsto g'(k) = g'gAN.$$

The restriction of this action to $K \subset G$ is just left multiplication by members of
K, that is, $k'(k) = k'k$. Thus these flows (G, K) are not only minimal (each orbit
is dense) but transitive (given $k, k' \in K$ we can find $g \in G$ with $gk = k'$).

In general, for any flow (H, X) consisting of a semigroup H acting continuously
on a compact space X we can regard each $h \in H$ as a continuous mapping $X \to X$
and then the closure of H in the compact semigroup X^X of all maps from X to
itself is a subsemigroup $\Sigma = \Sigma(H, X)$ which is compact and is called the *enveloping
semigroup* of the flow. Multiplication is continuous in Σ in the left-hand variable
($x \mapsto xy$ is continuous for each y). The structure theory for such semigroups ([2],
Theorems 1.3.11 and 1.2.16) tells us that Σ has minimal left ideals $L = L_\Sigma$. Any

two L's are isomorphic both topologically and algebraically. Each is compact. The set $E = E_\Sigma$ of idempotents in L is a left zero semigroup ($ee' = e$ for all e, e'), L has maximal subgroups $\Gamma = \Gamma_\Sigma$ (all algebraically isomorphic), and algebraically L is the direct product $E \times \Gamma$. If the flow (H, X) is minimal, then $Lx = X$ for each $x \in X$ ([2], Proposition 1.6.12). This shows that for minimal flows the action of Σ on X is determined by the semigroup product $\Sigma \times L \to L$. Thus knowledge of a minimal left ideal plays a key role in the analysis of flows.

In this paper we shall determine L for the action of G on $G/AN \cong K$. To be precise:

Theorem 1.1. *Let G be a connected semisimple Lie group with finite centre and Iwasawa decomposition $G = KAN$. Then the minimal left ideal $L_{\Sigma(G,K)}$ is homeomorphic with K. If M is the normaliser of AN in K, then the group $\Gamma_{\Sigma(G,K)}$ is isomorphic (algebraically and topologically) with M and the cardinal of the semigroup $E_{\Sigma(G,K)}$ of idempotents in $L_{\Sigma(G,K)}$ is the cardinal of K/M.*

Flows which are isomorphic with the flow on a minimal left ideal of the enveloping semigroup are discussed in Theorem IV(3.44) of [10].

This result has similarities with Theorem 9 of the paper [8]. There the group G was embedded in its largest semigroup compactification $G^{\mathcal{LUC}}$. For each compact normal subgroup N of G and each idempotent $e \in G^{\mathcal{LUC}}$ it was found that Ne had the algebraic structure $E \times \Gamma$ as above, where E was a left zero semigroup and Γ a group; moreover $\Gamma = eNe$ and $E = \{ne : n \in N \text{ and } ne = ene\}$. Theorem 1.1 is superficially the same. If e is a minimal idempotent in Σ, then $Ke = L$ is algebraically $E \times \Gamma$, $eKe = \Gamma = M$ and $E = \{ke : k \in K \text{ and } ke = eke\}$. However, here we do not have the conclusion except when the idempotent e is minimal, and the further conclusion in [8] that N can be written as a semidirect product does not hold. The difference here is that K is not normal; indeed G is only a Zappa product of K and AN.

It is often hard to obtain detailed information about semigroup compactifications of a group G, and in particular about the decomposition of a minimal left ideal in the form $L = E \times \Gamma$. One particular compactification which has been much studied is the largest semigroup compactification $G^{\mathcal{LUC}}$ ('largest' meaning that every other compactification is a continuous homomorphic image of $G^{\mathcal{LUC}}$). One natural question is whether the semigroup E of idempotents is closed for $G^{\mathcal{LUC}}$. This was shown not to be so in the case $G = \mathbb{Z}$ by J. Auslander (unpublished). We shall obtain this result for groups in our class when M is finite, and also for the group $\mathrm{SL}(2, \mathbb{C})$ for which M is infinite. Another question is how big the group Γ must be, and indeed whether it must be non-trivial. The latter question appears to be open in the case in which G is minimally almost periodic (that is, the largest topological group compactification of G is just a singleton). Since the groups $\mathrm{SL}(n, \mathbb{R})$ are minimally almost periodic, it is of interest that our present approach gives a little more information. We need a simple result: a surjective homomorphism from one compact semigroup onto another sends a maximal subgroup in a minimal left ideal of the first onto a corresponding object in the second (a proof can be found in Lemma 2.3 of [7]). Theorem 1.1 then yields immediately

Corollary 1.2. *Let G be as in Theorem 1.1 and let Γ be the maximal subgroup in a minimal left ideal of $G^{\mathcal{LUC}}$. Then M is a homomorphic image of Γ.*

Illustrations of the complexity of M can be found in [6], p. 324 and also Chap. VII, especially §5, and [11], pp. 28–29. For the examples $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ considered in §3, $M = \{\pm 1\}$ and $\mathbb{T}$ respectively.

In §3 we illustrate the general theory by looking at the particular groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ in more detail. We find that the enveloping semigroups are remarkably small here.

2. The proof of Theorem 1.1

To prove Theorem 1.1 we need to consider some flows related to $(G, G/AN \cong K)$. The product AN is a closed subgroup of G and is isomorphic (algebraically and topologically) to the semi-direct product of A and N, with N normal in AN. Since $AN \subset G$, we have $\Sigma(AN, G/AN) \subset \Sigma(G, G/AN)$. Again, MAN is a closed subgroup of G; it is the normalizer of N in G. We can therefore consider the flows $(G, G/MAN)$ and $(AN, G/MAN)$. The flow $(G, G/MAN)$ has been studied by many others (see Glasner [4], Warner [11]) with other aims in view. The key result for us is due to Moore [9] (see also de Vries [10], VI(6.1)5 and VI(1.8)); in the language of flows it implies that

Theorem 2.1. *The flow $(G, G/MAN)$ is proximal.*

The meaning of *proximal* is that, for any pair $x, y \in G/MAN$, there is a net (g_i) in G such that $\lim_i g_i(x) = \lim_i g_i(y)$. Since $\Sigma(G, G/MAN)$ is compact and contains a dense image of G, it means the same thing to say that there is $z \in \Sigma(G, G/MAN)$ with $z(x) = z(y)$ ([10], Theorem IV(3.22)). For the proof of Theorem 2.1, see the references cited above or [10], VI(6.5).

Theorem 2.1 has the following corollary which appears in [10], IV(6.8)3.

Corollary 2.2. *The flow $(AN, G/MAN)$ is proximal.*

Here is a sketch of its proof. Given $x, y \in G/MAN$, there is a net $(k_i a_i n_i)$ in $KAN = G$ with $\lim_i k_i a_i n_i(x) = \lim_i k_i a_i n_i(y)$. Compactness allows us to assume $k_i \to k \in K$ and $a_i n_i \to z \in \Sigma(AN, G/MAN)$. Ellis's joint continuity theorem ([2], Appendix B) tells us $k_i a_i n_i \to kz$, whence $kz(x) = kz(y)$ and as k is invertible, $z(x) = z(y)$.

The left action of M on $G/AN \cong K$ is just the group multiplication of elements of K on the left by elements of M. However, M also has a continuous right action on G/AN which is essentially just multiplication on the right by elements of M: for $g \in G$, $m \in M$ we have

$$gANm = gmAN$$

since M is in the normalizer of AN. Notice that this formula shows that the right action of M commutes with the (left) action of G; as a consequence, it also commutes with the (left) action of $\Sigma(G, G/AN)$. When G/AN is identified with K, the right action is, of course, just right multiplication:

$$(k, m) \mapsto km, \quad K \times M \to K,$$

and we have

$$(*) \qquad (y(k))m = y(km), \quad y \in \Sigma(G, G/AN),\ k \in K,\ m \in M.$$

We next see that the flows $(AN, G/AN)$ and $(AN, G/MAN)$ have fixed points.

Proposition 2.3. *The fixed points of $(AN, G/AN)$ are the cosets mAN with $m \in M$ (or when G/AN is identified with K, simply the elements of M). The flow $(AN, G/MAN)$ has exactly one fixed point, the coset MAN.*

Proof. For $m \in M$ and $an \in AN$ we find $an(mAN) = anANm = ANm = mAN$, so mAN is a fixed point. Conversely, let $k \in K \cong G/AN$ be fixed by AN. Then $ankAN = kAN$ for all $an \in AN$. In particular $ank \in kAN$, whence $k^{-1}ANk \subseteq AN$. Now $\{k' \in K : k'^{-1}ANk' \subseteq AN\}$ is a closed subsemigroup of the compact group K, and therefore it contains k^{-1} whenever it contains k (see [3], Theorem 1.10 ff.). Therefore $k^{-1}ANk = AN$, so that k is a normalizer of AN lying in K, that is $k \in M$.

It follows that the coset MAN is a fixed point of $(AN, G/MAN)$. Obviously a proximal flow can have no more than one fixed point, so our final assertion follows from Theorem 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We now begin to consider the enveloping semigroups of our flows.

Proposition 2.4. *The enveloping semigroup $\Sigma(AN, G/MAN)$ has a (semigroup) zero.*

Proof. We know (Proposition 2.3) that the coset $x_0 = MAN$ is a fixed point for the flow, and therefore a fixed point for all maps in $\Sigma(AN, G/MAN)$. We show that the map $z_0 : G/MAN \to G/MAN$ defined by $z_0(x) = x_0$ ($x \in G/MAN$) belongs to $\Sigma(AN, G/MAN)$; it is then obvious that it is a semigroup zero. By the compactness of $\Sigma(AN, G/MAN)$ it is enough to show that for each finite subset $\{x_1, \ldots, x_n\}$ of G/MAN there is $z \in \Sigma(AN, G/MAN)$ with $z(x_i) = x_0$ ($1 \leq i \leq n$). By proximality (Corollary 2.2) we can find $z_1 \in \Sigma(AN, G/MAN)$ with $z_1(x_1) = z_1(x_0) = x_0$. We can then find $z_2 \in \Sigma(AN, G/MAN)$ with $z_2(z_1(x_2)) = z_2(x_0) = x_0$; notice also that $z_2(z_1(x_1)) = z_2(x_0) = x_0$. Continuing in this way we find $z_1, \ldots, z_n$ with $(z_n \ldots z_1)(x_i) = x_0$, $1 \leq i \leq n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

This zero is obviously the (unique) minimal left ideal of $\Sigma(AN, G/MAN)$. We next find the minimal left ideals of $\Sigma(AN, G/AN)$.

Proposition 2.5. *The set*

$$I = \{w \in \Sigma(AN, G/AN) : w(G/AN) \subseteq \{mAN : m \in M\}\}$$

is non-empty and is the set of right zeros of $\Sigma(AN, G/AN)$. For each $w \in I$, the singleton $\{w\}$ is a minimal left ideal in $\Sigma(AN, G/AN)$, and all minimal left ideals are of this form.

Proof. The natural map $\varphi : G/AN \to G/MAN$, $gAN \mapsto gMAN$, maps the orbits of the right action of M on G/AN to points, and so it is a flow homomorphism $(AN, G/AN) \to (AN, G/MAN)$. It therefore lifts to a continuous semigroup homomorphism $\varphi_\Sigma : \Sigma(AN, G/AN) \to \Sigma(AN, G/MAN)$.

Take a net $(a_i n_i)$ in AN which converges in $\Sigma(AN, G/MAN)$ to its zero z_0. Then there is a subnet $(a_j n_j)$ of $(a_i n_i)$ which converges in $\Sigma(AN, G/AN)$ to some element, say w. For any $k \in G/AN$,

$$\varphi(w(k)) = \lim_j \varphi(a_j n_j(k)) = \lim_j \varphi_\Sigma(a_j n_j)(\varphi(k)) = z_0(\varphi(k)) = MAN.$$

Thus $I \neq \emptyset$. Since each element mAN is fixed in G/AN for the action of AN, and so also for the action of $\Sigma(AN, G/AN)$, we see that each element of I is a right zero in $\Sigma(AN, G/AN)$, and so forms a minimal left ideal.

If w is a right zero, then for $k \in K$, $w(kAN)$ is fixed under the action of AN and therefore $w(kAN) \in \{mAN : m \in M\}$ by Proposition 2.3. $\square$

In our next result we consider $\Sigma_2 = \Sigma(AN, G/AN)$ as a compact subsemigroup of $\Sigma_1 = \Sigma(G, G/AN)$.

N.B. From now on we identify the quotient space G/AN with K. The reader must bear in mind that K also appears as a subgroup of G and of Σ_1; for example, in (1) of Lemma 2.8 E_w is defined as a subset of $K = G/AN$, and then appears in (2) as a subset of $K \subset \Sigma_1$.

Proposition 2.6. $\Sigma(G, K) = K\Sigma(AN, K)$ (that is, $\Sigma_1 = K\Sigma_2$).

Proof. If $z \in \Sigma_1$, there is a net $(k_i a_i n_i)$ in $KAN = G$ with $k_i a_i n_i \to z$. Compactness allows us to assume that $k_i \to k$ in K and $a_i n_i \to y$ in Σ_2. Ellis's Theorem (e.g., [1, Appendix B]) guarantees the joint continuity of the map $K \times \Sigma_1 \to \Sigma_1$ and we conclude that $z = \lim_i k_i a_i n_i = ky$. $\square$

Proposition 2.6 suggests that, since K is a group, it would be reasonable to expect that if w was a right zero of Σ_2 (so that $\{w\}$ was a minimal left ideal of Σ_2), then Kw would be a minimal left ideal in Σ_1 and would be a group algebraically. The first of these suggestions turns out to be correct but the second is false, as we shall see.

Lemma 2.7. Let w be a right zero in Σ_2.
  (1) Then Kw is a closed left ideal in Σ_1 and is homeomorphic to K.
  (2) If w' is another right zero in Σ_2, then $Kw \cap Kw' = \emptyset$.

Proof. Let $g \in G$, $k \in K$. Then $gk \in G = KAN$, so $gk = k'a'n'$ for some k', a', n'. Thus $gkw = k'a'n'w = k'w \in Kw$. Since Kw is closed and G is dense in Σ_1 it follows that Kw is a left ideal. The map $\theta : k \mapsto kw$, $K \to Kw \subset \Sigma_1$ is clearly continuous, as is $k \mapsto kw(k')$ of K onto K (for any fixed k'), and the latter map is also 1–1; this proves (1).

If $kw = k'w'$ for $k, k' \in K$, then $kw = kw^2 = k'w'w = k'w$ (Proposition 2.5), and so $k = k'$ by (1), and $w = k^{-1}kw = k^{-1}kw' = w'$; this proves (2). $\square$

Lemma 2.8. Let e be the identity of M, and let w be a right zero in Σ_2. Set
$$E_w = \{k \in K : w(k) = e\}.$$

(1) For any $m \in M$, $E_w m = \{k \in K : w(k) = m\}$, and K is the disjoint union
$$K = \bigcup_{m \in M}^{\cdot} E_w m = E_w M.$$

(2) The idempotents in Kw are given by $E_w w = \{kw : k \in E_w\}$ and form a left zero semigroup.

(3) The map $\psi : E_w \times M \to Kw$ given by $\psi(k, m) = kmw$ is bijective and is a homomorphism when $E_w \times M$ is given the left group product $(k_1, m_1)(k_2, m_2) = (k_1, m_1 m_2)$.

(4) The maximal subgroups of Kw have the form kMw for $k \in E_w$; they are closed and isomorphic (topologically and algebraically) to M.

Proof. (1) That $w(k') = m$ if and only if $k' \in E_w m$ follows from the fact that the right action of M commutes with the action of Σ_2 – see $(*)$ before Proposition 2.3 – so that $(w(k'))m^{-1} = w(k'm^{-1})$. Thus (1) is established.

(2) Let $m \in M$, the range of w. Then since $kw(E_w m) = km$, we have

$$kw(kw(E_w m)) = kw(km),$$

which is equal to $km = kw(E_w m)$ if and only if $k \in E_w$, by (1). This proves the first claim of (2), and the proof of the second claim of (2) is similar.

(3) From (1) and Lemma 2.7, ψ is obviously bijective. If $k_1, k_2 \in E_w$ and $m_1, m_2 \in M$, then arguing as in (2) we have $k_2 m_2 w(E_w m) = k_2 m_2 m$ which yields

$$(k_1 m_1 w)(k_2 m_2 w) = k_1 m_1 m_2 w;$$

but this is just $\psi(k_1, m_1)\psi(k_2, m_2) = \psi(k_1, m_1 m_2) = \psi((k_1, m_1)(k_2, m_2))$ as required.

(4) From (3) the maximal subgroups of Kw are of the form $\psi(k, M) = kMw$ and they are isomorphic to M. They are homeomorphic with M from Lemma 2.7 (1). □

Proof of Theorem 1.1. We must show that the left ideal Kw (Lemma 2.7) is minimal. But this is immediate from Lemma 2.8 (3), since any semigroup of the form $E \times \Gamma$ where E is a left zero semigroup and Γ is a group contains no proper left ideals. □

Corollary 2.9. *If M is finite, the set of idempotents in each minimal left ideal of Σ_1 is not closed.*

Proof. The sets $E_w m$ ($m \in M$) form a finite partition of K under our hypotheses, and if one of these sets is closed they all are. But this means that each is also open (in K). Since K is connected, this cannot happen. The map $K \to Kw$ is a homeomorphism (Lemma 2.7(1)), so the set $E_w w$ of idempotents in the minimal left ideal Kw_0 is not closed. □

Finally in this section we look at the semigroup $\Sigma(G, G/MAN)$. We are able to say more about its ideal structure. We want (a variant of) Proposition 2.6 in this context. To state it properly, we must recognize that the map $G \to \Sigma(G, G/MAN)$ might not be 1–1 on K. However, each $k \in K$ does act on $\Sigma(G, G/MAN)$; for $s \in \Sigma(G, G/MAN)$ and $x \in G/MAN$ we write $k(s)(x) = k(s(x))$.

Proposition 2.10. *We have $\Sigma(G, G/MAN) = K(\Sigma(AN, G/MAN))$. If z_0 is the zero of $\Sigma(AN, G/MAN)$, then the minimal right ideals of $\Sigma(G, G/MAN)$ are the singletons of the form $\{k(z_0)\}$ for $k \in K$. The minimal ideal of $\Sigma(G, G/MAN)$ is $K(z_0)$. It is a left zero semigroup and is homeomorphic to K/M.*

Proof. As in the proof of Proposition 2.6 we see that any element of $\Sigma(G, G/MAN)$ is of the form $k(s)$ ($k \in K$, $s \in \Sigma(AN, G/MAN)$).

For each $s \in \Sigma(G, G/MAN)$, $x \in G/MAN$ we have $(z_0 s)(x) = z_0(s(x)) = x_0$ (with the notation of Proposition 2.4). Therefore z_0 is a left zero in $\Sigma(G, G/MAN)$. Thus $\{z_0\}$ is a minimal right ideal, and hence all minimal right ideals are singletons. From the beginning of this proof they will be of the form $\{k(z_0)\}$ for $k \in K$. The union of all these right ideals, that is $K(z_0)$, is the unique minimal left ideal and also the minimal two-sided ideal of $\Sigma(G, G/MAN)$. $K(z_0)$ is a continuous image of K and so compact, and the map $K \to K(z_0)$ has kernel $\{k : k(z_0) = z_0\} = \{k : kMAN = MAN\} = M$, so that $K(z_0)$ is homeomorphic with K/M. □

3. The examples $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$

In this section in order to illuminate the general theory of §2 we discuss the enveloping semigroup $\Sigma_1 = \Sigma(G,K) = \Sigma(G,G/AN)$ when $G = \mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$. It is remarkable that fairly complete descriptions of these semigroups can be given and that they are in many ways uncomplicated. One point to notice is that for $\mathrm{SL}(2,\mathbb{C})$, even though the group M is not finite (so that Corollary 2.9 does not apply), the set of idempotents in a minimal left ideal of the enveloping semigroup is not closed.

3.1. We begin by describing a particular compact semigroup $\mathcal{S}$ (which will turn out to be isomorphic with $\Sigma_2 \setminus AN = \Sigma(AN,K) \setminus AN$ for $\mathrm{SL}(2,\mathbb{C})$).

We denote by $\overline{\mathbb{C}} = \{re^{i\theta} : 0 \leq r \leq \infty,\ -\pi < \theta \leq \pi\}$ a compactified complex plane; as usual all points $0e^{i\theta}$ are equal to 0, but the boundary $\partial\overline{\mathbb{C}} = \overline{\mathbb{C}} \setminus \mathbb{C} = \{\infty e^{i\theta} : -\pi < \theta \leq \pi\}$ is a 'circle at infinity'. The topology on $\overline{\mathbb{C}}$ is such that $re^{i\theta} \mapsto re^{i\theta}/(1+r)$ is a homeomorphism onto the closed unit disc.

The semigroup $\mathcal{S}$ will consist of two families of maps of $\overline{\mathbb{C}}$ into itself,

$$\text{(i) } \{v_b : b \in \overline{\mathbb{C}}\}, \qquad \text{(ii) } \{w_{a,b} : a \in \mathbb{C},\ b \in \overline{\mathbb{C}}\};$$

these are defined by the requirement that all of them leave the points of $\partial\overline{\mathbb{C}}$ fixed and, in addition, for $z \in \mathbb{C}$,

$$\text{(i) } v_b(z) = b, \qquad \text{(ii) } w_{a,b}(z) = \begin{cases} \infty e^{i\arg(z-a)} & \text{if } z \neq a, \\ b & \text{if } z = a. \end{cases}$$

More intuitively, when $z \neq a$, $w_{a,b}(z)$ is the point at infinity at the end of the ray drawn through z from a. It is easy to see that these maps do form a semigroup under composition. The right zeros of this semigroup are the elements with range in $\partial\overline{\mathbb{C}}$, that is the maps v_b and $w_{a,b}$ with $b \in \partial\overline{\mathbb{C}}$. Together these elements form the minimal ideal.

We give $\mathcal{S}$ the topology of pointwise convergence on $\overline{\mathbb{C}}$. For some sequences convergence is straightforward: if $a_n \to a$ and $b_n \to b$ in $\overline{\mathbb{C}}$, then $v_{a_n} \to v_a$ and for any $a \in \mathbb{C}$, $w_{a,b_n} \to w_{a,b}$. For other sequences the situation is more complicated. For example, let $a_n \to a$ in $\mathbb{C}$ with $a_n \neq a$ for all n and $\lim_n \arg(a-a_n) = \phi$. Then, for any $b \in \overline{\mathbb{C}}$, $w_{a_n,b} \to w_{a,\infty e^{i\phi}}$. An interesting aspect of this case is that the limit function does not depend on b in any way. This feature is reminiscent of others which occur in the context of minimal ideals in compact semigroups (see for example, Example 1 of [1]). We shall not go further into the details of the topology on $\mathcal{S}$, though we should mention that it is compact.

3.2. We next describe the action of AN on $K = G/AN$ for $G = \mathrm{SL}(2,\mathbb{C})$ and find Σ_2 in this case. The expression of $g \in \mathrm{SL}(2,\mathbb{C})$ in the form kan is given by

$$k = \begin{pmatrix} s & -\overline{t} \\ t & \overline{s} \end{pmatrix}, \quad a = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad n = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$$

where $|s|^2 + |t|^2 = 1\,(s,t \in \mathbb{C})$, $\varepsilon > 0$ and $\xi \in \mathbb{C}$. We shall use the abbreviations $k = (s,t)$, $a = (\varepsilon)$ and $n = (\xi)$ for these matrices. The normalizer M of AN in K consists of the diagonal elements of K, that is $(s,0)$ with $|s| = 1$.

It will be convenient to write $(s, t) \in K$ as $(s, \rho e^{i\alpha})$; then $|s|^2 + \rho^2 = 1$ so that $\rho \geq 0$ is determined by s. For $-\pi < \alpha \leq \pi$, we shall put

$$S_\alpha = \{(s, \rho e^{i\alpha}) : |s| < 1, |s|^2 + \rho^2 = 1\},$$

$\overline{S_\alpha}$ for its closure (so including points with $|s| = 1$) and $\partial\overline{S_\alpha} = \overline{S_\alpha} \setminus S_\alpha$. Note that for $\alpha \neq \gamma$, $\overline{S_\alpha} \cap \overline{S_\gamma} = \{(s, 0) : |s| = 1\} = \partial\overline{S_\alpha}$, and that this is precisely M. Thus $\partial\overline{S_\alpha}$ is independent of α.

If we denote the image of $(s, \rho e^{i\alpha})$ under the action of $(\varepsilon)(\xi) \in AN$ by $(s', \rho'e^{i\alpha'})$ we find (using the Gram-Schmidt process as on page 117 of [5]) that

$$s' = \frac{\varepsilon^2 s + \varepsilon^2 \xi \rho e^{i\alpha}}{\sqrt{|\varepsilon^2 s + \varepsilon^2 \xi \rho e^{i\alpha}|^2 + |\rho|^2}},$$

$$\rho' e^{i\alpha'} = \frac{\rho e^{i\alpha}}{\sqrt{|\varepsilon^2 s + \varepsilon^2 \xi \rho e^{i\alpha}|^2 + |\rho|^2}}.$$

Obviously $\alpha = \alpha'$ so that S_α is an invariant subset, and the formulas show that S_α is even a minimal invariant set. Since the points of $M = \partial\overline{S_\alpha}$ are invariant, we see that $\overline{S_\alpha}$ is a compact invariant subset.

The image of S_α under the right action of $m_\gamma = (e^{-i\gamma}, 0) \in M$ is $S_{\alpha+\gamma}$. We know that this action of M commutes with the action of AN (see after Corollary 2.2) so that the action of AN on any one disc S_α determines the flow. Specifically if we fix $\alpha = 0$ and take $s \in S_\gamma$ we find $((\varepsilon)(\xi))(s) = ((\varepsilon)(\xi))(sm_\gamma^{-1})m_\gamma$. The compact semigroup Σ_2 is therefore isomorphic to $\Sigma(AN, \overline{S_0})$. (We take $\overline{S_0}$ rather than S_0 because we need a compact phase space. Recall that the points of $\partial\overline{S_0}$ are fixed for the action of AN, though not for the action of M.)

Now $\overline{S_0} = \{(s, \rho) : |s|^2 + \rho^2 = 1\}$. The map $(s, \rho) \mapsto s \mapsto s/\sqrt{1 - |s|^2}$ is a homeomorphism of $\overline{S_0}$ with the space $\overline{\mathbb{C}}$ of §3.1. Notice that $\partial\overline{S_0} = M$ maps to $\partial\overline{\mathbb{C}}$, and that the identity of M goes to $\infty e^{i0} = \infty 1 \in \partial\overline{\mathbb{C}}$.

The action on $\overline{\mathbb{C}}$ which is equivalent to the action of $(\varepsilon)(\xi)$ on $\overline{S_0}$ can be read from the formula for the action on S_0 given earlier: if $s \in S_0$ corresponds to $z \in \mathbb{C}$, then $(\varepsilon)(\xi)$ sends z to $z' = \varepsilon^2 z + \varepsilon^2 \xi = \varepsilon^2(z + \xi)$ and it leaves all $z \in \partial\overline{\mathbb{C}}$ fixed. This is (an extension to $\overline{\mathbb{C}}$ of) a standard affine action on $\mathbb{C}$.

It is fairly straightforward to find all the elements of $\Sigma(AN, \overline{\mathbb{C}})$. The possible cluster points of ε^2 are in $[0, \infty]$; the possible cluster points of $\varepsilon^2 \xi$ are either in $\mathbb{C}$ or in $\partial\overline{\mathbb{C}}$. When we are looking for elements of $\Sigma(AN, \overline{\mathbb{C}})$ we may assume that all nets involving ε^2 or $\varepsilon^2 \xi$ converge in $[0, \infty]$ or $\overline{\mathbb{C}}$. In nearly all cases we find limit maps of the form v_b (see §3.1), and we can obtain all v_b with $b \in \overline{\mathbb{C}}$. The main case in which the result is different occurs when $\varepsilon^2 \to \infty$ and $\varepsilon^2 \xi \to \infty e^{i\phi} \in \partial\overline{\mathbb{C}}$. We need also to assume that $\varepsilon^2 \xi / \varepsilon^2 = \xi$ converges to $\zeta_1 \in \mathbb{C}$ say (if $\zeta_1 \in \partial\overline{\mathbb{C}}$ we again have a limit of the form v_b), and we find the limit map is $w_{-\zeta_1, b}$ for some $b \in \overline{\mathbb{C}}$. Again all possible maps of this form can be obtained.

We conclude that the semigroup Σ_2 is isomorphic with $\Sigma(AN, \overline{\mathbb{C}}) = AN \cup \mathcal{S}$.

The size of the semigroup $\mathcal{S}$ which is adjoined to AN to form this enveloping semigroup is rather small, and consists of a restricted class of mappings. The image of $\mathbb{C}$ under an element of $\mathcal{S}$ is either a singleton or a singleton together with $\partial\overline{\mathbb{C}}$.

3.3. We now consider the structure of the minimal left ideals in $\Sigma_1 = K\Sigma_2$. All are isomorphic and of the form Kw where w is a right zero of Σ_2. As in §2, we write $K = M \cup \bigcup_{\pi < \alpha \leq \pi} S_\alpha$. The right action of M permutes the S_α's.

In §3.2, what precisely was determined was $\Sigma(AN, \overline{S_0})$. A map $v \in \Sigma(AN, \overline{S_0})$ extends uniquely to $\tilde{v} \in \Sigma_2$ through the formula

$$\tilde{v}(k) = (v(km_{-\alpha}))m_\alpha \quad \text{if } k \in S_\alpha = S_0 m_\alpha$$

(of course $\tilde{v}(k) = k$ for all $k \in M$). This formula means that the right zeros of Σ_2 are the maps $\tilde{v}_b$ and $\tilde{w}_{a,b}$ with $b \in M$.

We now determine some of the idempotents in Σ_1. According to Lemma 2.8, $k\tilde{v}_{m_\alpha}$ is an idempotent iff $\tilde{v}_{m_\alpha}(k) = 1$ (the identity of M). We consider two cases. First, if $k \in M$, then $\tilde{v}_{m_\alpha}(k) = k$, so $k\tilde{v}_{m_\alpha}$ is an idempotent if and only if $k = 1$, that is $k\tilde{v}_{m_\alpha} = \tilde{v}_{m_\alpha}$. Secondly when $k \in S_\gamma$, $\tilde{v}_{m_\alpha}(k) = (v_{m_\alpha}(km_{-\gamma}))m_\gamma = m_\alpha m_\gamma$, so $k\tilde{v}_{m_\alpha}$ is an idempotent if and only if $\gamma = -\alpha$; this means all elements $k\tilde{v}_{m_\alpha}$ with $k \in S_{-\alpha}$ are idempotents. Thus $E_{\tilde{v}_{m_\alpha}} = S_{-\alpha} \cup \{1\}$, and $K\tilde{v}_{m_\alpha} = E_{\tilde{v}_{m_\alpha}} M$ (which is isomorphic with $E_{\tilde{v}_{m_\alpha}} \times M$).

An immediate consequence of the above formulation is that *the idempotents in $K\tilde{v}_{m_\alpha}$ do not form a closed set.* Indeed the closure $\overline{S}_{-\alpha}\tilde{v}_{m_\alpha}$ of $S_{-\alpha}\tilde{v}_{m_\alpha}$ contains $M\tilde{v}_{m_\alpha}$, but the only idempotent member of $M\tilde{v}_{m_\alpha}$ is $\tilde{v}_{m_\alpha}$ itself.

3.4. The description of the enveloping semigroup for $G = \mathrm{SL}(2, \mathbb{R})$ is much simpler. First, K is just the circle $\mathbb{T}$, which we identify with $[0, 2\pi)$. The subgroup M is $\{0, \pi\}$. The action of an element of K on $\mathbb{T}$ is just a rotation. The elements of $\Sigma(G, K) \setminus G$ can be described as follows. Take $0 \leq \phi < \pi$. Partition $\mathbb{T}$ into $\{\phi\}$, $(\phi, \phi + \pi)$, $\{\phi + \pi\}$ and $(\phi + \pi, \phi + 2\pi)$. Take $\psi_1, \psi_2 \in [0, 2\pi)$ and define maps U_{ϕ, ψ_1, ψ_2} by

$$U_{\phi, \psi_1, \psi_2}(\phi) = \psi_1, \qquad U_{\phi, \psi_1, \psi_2}(\phi + \pi) = \psi_1 + \pi,$$
$$U_{\phi, \psi_1, \psi_2}((\phi, \phi + \pi)) = \{\psi_2\}, \quad U_{\phi, \psi_1, \psi_2}((\phi + \pi, \phi + 2\pi)) = \{\psi_2 + \pi\}.$$

These maps lie in the minimal ideal when their images consist of just two points (so either $\psi_1 = \psi_2$ or $\psi_1 = \psi_2 + \pi$). In the latter situation, $U_{\phi, \psi, \psi}$ is idempotent when $\psi \in [\phi, \phi + \pi)$. Otherwise, $U_{\phi, \psi, \psi}([\phi, \phi + \pi)) \subseteq [\phi + \pi, \phi + 2\pi)$ whence $U_{\phi, \psi, \psi}$ is not idempotent but $U^2_{\phi, \psi, \psi}$ is. That these idempotents do not form a closed set is seen by letting $\psi \nearrow \phi + \pi$ (as well as from Corollary 2.9).

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