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IRREDUCIBLE RESTRICTION AND ZEROS OF CHARACTERS

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Abstract. Let G be a finite group, let N be normal in G and suppose that χ is an irreducible complex character of G. Then χ_N is not irreducible if and only if χ vanishes on some coset of N in G.

1. Introduction

Let $N \triangleleft G$, where G is an arbitrary finite group, and let $\chi \in \mathrm{Irr}(G)$ be an irreducible complex character of G. In this note we give a characterization of when the restricted character χ_N is irreducible which at the same time extends Burnside's theorem on zeros of characters.

Theorem A. *Let G be a finite group and let $N \triangleleft G$. Let $\chi \in \mathrm{Irr}(G)$. Then χ_N is not irreducible if and only if χ vanishes on some coset Nx of N in G.*

When $N = 1$ (or more generally if N is abelian) Theorem A is Burnside's theorem on zeros. We mention some easy consequences of Theorem A which again extend Burnside's theorem.

Corollary B. *Let $N \triangleleft G$ with G/N a π-group and let $\chi \in \mathrm{Irr}(G)$. If χ is nonzero on the π-elements of G, then χ_N is irreducible.*

Corollary C. *Let $N \triangleleft G$ and let $H \subseteq G$ be such that $G = HN$. Let $\chi \in \mathrm{Irr}(G)$. If $\chi(h) \neq 0$ for $h \in H$, then χ_N is irreducible.*

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2. Proofs

Our proof of Theorem A is another application of the theory of character triple isomorphisms. The reader is referred to [1] for its definition and main properties.

Proof of Theorem A. Suppose first that χ_N is irreducible. Let $x \in G$. By Lemma (8.14) of [1], we have that

$$\sum_{g \in Nx} |\chi(g)|^2 = |N|\,,$$

and therefore χ cannot be zero on the coset Nx.

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Suppose now that χ_N is not irreducible. We want to find $x \in G$ such that $\chi(nx) = 0$ for all $n \in N$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Let T be the stabilizer of θ in G and by the Clifford correspondence (Theorem (6.11) of [1]), let $\psi \in \mathrm{Irr}(T)$ be such that $\chi = \psi^G$. Assume that $T < G$. Then

$$\bigcup_{g \in G} T^g \subset G\,,$$

and we let $x \in G$ lie in no G-conjugate of T. Since N is contained in every G-conjugate of T, it follows that for each element $n \in N$, the element nx is contained in no G-conjugate of T. It follows that $\chi = \psi^G$ vanishes on the entire coset Nx, by the character induction formula.

We can now assume that $T = G$. Hence χ_N is a multiple of θ. By Theorem (11.28) of [1], let (G^*, M, ν) be a character triple isomorphic to (G, N, θ) with $M \subseteq \mathbf{Z}(G^*)$. Let $\chi^* \in \mathrm{Irr}(G^*)$ correspond to χ. Let us denote by * the group isomorphism $G/N \to G^*/M$ associated to the character triple isomorphism.

By Lemma (11.24) of [1], we have that

$$e = \frac{\chi(1)}{\theta(1)} = \frac{\chi^*(1)}{\nu(1)}\,.$$

Since χ_N reduces, $e > 1$ and we conclude that χ^* is nonlinear. By Burnside's theorem (Theorem (3.15) of [1]), there exists $y \in G^*$ such that $\chi^*(y) = 0$. Now, let $V = \langle M, y\rangle$ and let $N \subseteq U \subseteq G$, where $(U/N)^* = V/N$. Since $U/N \cong V/M$ is cyclic, we see that θ has an extension $\varphi \in \mathrm{Irr}(U)$, by Corollary (11.22) of [1]. By Gallagher's theorem (Corollary (6.17) of [1]), notice that we may write

$$\chi_U = \psi\varphi$$

for some character ψ of U/N. Let us denote by ψ^* the character of V/N satisfying

$$\psi^*((wN)^*) = \psi(wN)$$

for $w \in U$. By Definition (11.23.d) of [1], we have that

$$\chi^*_V = \psi^*\varphi^*\,,$$

where φ^* is the extension of ν corresponding to φ under the character triple isomorphism. Since ν is linear, we have that φ^* is also linear.

Now, let the coset $Nx \subseteq U$ correspond to My under the character triple isomorphism. Note that we can write $\psi(x) = \psi(Nx) = \psi^*(My) = \psi^*(y)$. Now

$$0 = \chi^*(y) = \psi^*(y)\varphi^*(y)\,,$$

and the second factor is nonzero because φ^* is linear. Thus $\psi^*(y) = 0$ and hence $\psi(x) = 0$. It follows that

$$\chi(x) = \psi(x)\varphi(x) = 0\,.$$

Since x was an arbitrary element of the coset Nx, the result follows. □

Proof of Corollary B. Suppose that χ_N is not irreducible. Then there exists a coset Nx of N in G on which χ is zero. Now $Nx = Nx_\pi$ contains the π-element x_π, and this contradicts the hypothesis. □

Proof of Corollary C. If χ_N is not irreducible, there exists a coset Nx of N in G on which χ is zero. Now Nx contains some $h \in H$, and this contradicts the hypothesis. □

References

[1] M. Isaacs, *Character Theory of Finite Groups*, New York, Dover, 1994.

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