SUBDIVISION SCHEMES FOR ITERATED FUNCTION SYSTEMS

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ABSTRACT. We identify iterated function systems Φ and regular Borel measures μ such that the matrix subdivision process relative to a finite family $\mathcal{A}$ converges if and only if $\mathcal{A}$ satisfies certain spectral properties.

1. INTRODUCTION

Given any finite family of contractive mappings Φ on a metric space X there is a unique invariant subset Ω of X which is the union of all of its images under the contractive mappings. We suppose that there is a regular Borel measure μ on Ω such that the images of Ω under the contractive mappings partition Ω relative to μ . Repeated iterations of the contractions lead to a sequence of finer partitions of Ω . Moreover, the contractions partition any of the cells obtained at iteration level r into subcells of level $r + 1$. This refinement structure on Ω can be used to create vector fields on Ω with range in $\mathbb{R}^n$ in the following way. To this end, we require a family $\mathcal{A}$ of $n \times n$ matrices of the same cardinality as Φ . Each mapping in Φ is associated with a unique matrix in $\mathcal{A}$. We start with a constant vector field $\mathbf{h}$ on Ω and break it into a piecewise constant function whose value on the cell $\phi(\Omega)$ is $\mathbf{A}\mathbf{h}$ where $\phi \in \Phi$ and $\mathbf{A} \in \mathcal{A}$ are associated. We iterate this procedure to get a sequence of piecewise constant functions on finer and finer partitions of Ω . This process is referred to as *matrix subdivision* and is important in both geometric modeling and wavelet analysis [1, 4, 5, 6, 7, 8]. If it converges, it produces a *refinable function* relative to $\mathcal{A}$ and Φ . Our purpose here is to study it in greater generality than previously available. Our goal is to identify conditions on Φ and the measure μ such that a useful characterization of the convergence of the subdivision process to a function in $L_p(\Omega, \mu)$ can be given in terms of the spectral properties of $\mathcal{A}$.

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This paper is organized into four sections. Section 2 is devoted to a description of subdivision schemes and the definition of convergence. A few simple observations are also made on convergent subdivision schemes. In Section 3 we define and discuss the concept of μ -uniformity of a family of contractions for any μ -measurable set. Finally, in Section 4, we establish a characterization of convergent subdivision schemes and prove that the subdivision scheme converges exponentially fast and that its limit function enjoys Hölder regularity.

2. CONTRACTIONS AND SUBDIVISION

For any integer $n > 0$ we denote the set $\{0, 1, \dots, n-1\}$ by $\mathbb{Z}_n$ and use $\mathbb{Z}_\infty = \{0, 1, \dots\}$. Let X be a complete metric space, and let $\Phi = \{\phi_\ell : \ell \in \mathbb{Z}_m\}$ be a finite set of contractions on X . Recall the following facts from [2]. There exists a unique closed and bounded set Ω such that

$$(2.1) \quad \Omega = \bigcup_{\ell \in \mathbb{Z}_m} \phi_\ell(\Omega).$$

To any vector $\mathbf{j} = (j_k : k \in \mathbb{Z}_r) \in \mathbb{Z}_m^r$, that is, $j_k \in \mathbb{Z}_m$, $k \in \mathbb{Z}_r$, we associate the composite mapping

$$(2.2) \quad \phi_{\mathbf{j}} = \phi_{j_0} \circ \dots \circ \phi_{j_{r-1}}$$

and observe that

$$(2.3) \quad \Omega = \bigcup_{\mathbf{j} \in \mathbb{Z}_m^r} \phi_{\mathbf{j}}(\Omega).$$

Also, for any nonempty set $\Gamma \subseteq X$ we define the set-valued mapping

$$\mathcal{S}(\Gamma) = \bigcup_{\ell \in \mathbb{Z}_m} \phi_\ell(\Gamma)$$

and, for $r \in \mathbb{Z}_\infty$, its iterates $\mathcal{S}^{r+1}(\Gamma) = \mathcal{S}(\mathcal{S}^r(\Gamma))$. It is known that

$$(2.4) \quad \lim_{r \rightarrow \infty} \mathcal{S}^r(\Gamma) = \Omega$$

in the Hausdorff metric [2].

Let $x_{\mathbf{j}} \in X$, $\mathbf{j} \in \mathbb{Z}_m^r$, $r \in \mathbb{Z}_\infty$, be the fixed point of the contraction $\phi_{\mathbf{j}}$. It is proved in [2] that Ω is compact and is the closure of the set

$$(2.5) \quad F_\Phi = \{x_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty\}.$$

For every (infinite) vector $\mathbf{j} = (j_\ell : \ell \in \mathbb{Z}_\infty) \in \mathbb{Z}_m^\infty$ and $r \in \mathbb{Z}_\infty$ we form the (finite) vectors $\mathbf{j}^r = (j_\ell : \ell \in \mathbb{Z}_r)$ and recall that $\lim_{r \rightarrow \infty} x_{\mathbf{j}^r}$ exists. We shall denote this limit by $x_{\mathbf{j}}$. Moreover, for every $x, y \in \Omega$ there exists a $\mathbf{j} \in \mathbb{Z}_m^\infty$ such that $x = \lim_{r \rightarrow \infty} \phi_{\mathbf{j}^r}(y)$.

Suppose that μ is a regular Borel measure on Ω with $0 < \mu(\Omega) < \infty$; cf. [9]. For $\Gamma \subseteq \Omega$, we denote by $L_p(\Gamma, \mu)$, $1 \leq p < \infty$, the spaces of p -integrable functions $f : \Gamma \rightarrow \mathbb{R}$ for which

$$\|f\|_{L_p(\Gamma, \mu)} := \left(\int_\Gamma |f(x)|^p d\mu(x) \right)^{1/p}$$

is finite. Mostly, we will write $\|f\|_{L_p(\Gamma)}$ and in the case that $\Gamma = \Omega$ use $\|f\|_p$. Moreover, we will use $L_p^{s \times t}(\Gamma, \mu)$ for all $\mathbb{R}^{s \times t}$ matrix-valued functions on Γ , that

is, $\mathbf{F} = (f_{\ell k} : \ell \in \mathbb{Z}_s, k \in \mathbb{Z}_t)$, such that $f_{\ell k} \in L_p(\Gamma, \mu)$, $\ell \in \mathbb{Z}_s$, $k \in \mathbb{Z}_t$, and equip it with the norm

$$\|\mathbf{F}\|_{L_p^{s \times t}(\Gamma, \mu)} = \sum_{\ell \in \mathbb{Z}_s} \sum_{k \in \mathbb{Z}_t} \|f_{\ell k}\|_{L_p(\Gamma, \mu)}.$$

As above, we use the abbreviated notation $\|\mathbf{F}\|_{L_p(\Gamma)}$, $\|\mathbf{F}\|_p$ and also $L_p^s(\Gamma, \mu)$ for the space $L_p^{s \times 1}(\Gamma, \mu)$. In this way, the norm of a vector field $\mathbf{f} = (f_j : j \in \mathbb{Z}_s) \in L_p^s(\Gamma, \mu)$ is defined as the norm of $\mathbf{f}$, interpreted as an $s \times 1$ matrix-valued function. For any two matrices $\mathbf{A} \in \mathbb{R}^{n \times s}$, $\mathbf{B} \in \mathbb{R}^{t \times n}$ it follows from Minkowski's inequality that

$$\|\mathbf{A}\mathbf{F}\mathbf{B}\|_{L_p^{n \times n}(\Gamma)} \leq \|\mathbf{A}^T\| \|\mathbf{B}\| \|\mathbf{F}\|_{L_p^{s \times t}(\Gamma)},$$

where

$$\|\mathbf{A}\| := \max_{j \in \mathbb{Z}_n} \sum_{k \in \mathbb{Z}_s} |A_{jk}|.$$

We assume that the measure μ and the family Φ of contractive mappings have the following properties. There exist positive constants C, D such that for all $\ell \in \mathbb{Z}_m$, $\Gamma \subseteq \Omega$ and $f \in L_p(\Gamma, \mu)$,

$$(2.6) \quad C \|f\|_{L_p(\phi_\ell(\Gamma), \mu)} \leq \|f \circ \phi_\ell\|_{L_p(\Gamma, \mu)} \leq D \|f\|_{L_p(\phi_\ell(\Gamma), \mu)}.$$

Moreover, we require that

$$(2.7) \quad \phi_\ell \text{ is injective,} \quad \ell \in \mathbb{Z}_m,$$

and

$$(2.8) \quad \mu(\phi_\ell(\Omega) \cap \phi_k(\Omega)) = 0, \quad \ell, k \in \mathbb{Z}_m, \ell \neq k.$$

For $\Gamma \subseteq \Omega$ and $\ell \in \mathbb{Z}_m$, equation (2.6) implies that

$$(2.9) \quad C^p \mu(\phi_\ell(\Gamma)) \leq \mu(\Gamma) \leq D^p \mu(\phi_\ell(\Gamma)).$$

In particular, we obtain that $\mu(\phi_j(\Omega)) > 0$, $\mathbf{j} \in \mathbb{Z}_m^r$, $r \in \mathbb{Z}_\infty$. Moreover, from (2.6) and (2.8) it follows that

$$\mu(\phi_j(\Omega) \cap \phi_k(\Omega)) = 0, \quad \mathbf{j}, \mathbf{k} \in \mathbb{Z}_m^r, \mathbf{j} \neq \mathbf{k}, \quad r \in \mathbb{Z}_\infty.$$

We shall make use of the family of *pushforward measures* $\{\mu_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}_m^r, r \in \mathbb{Z}_\infty\}$ defined by setting, for any μ -measurable set $\Gamma \subseteq \Omega$,

$$\mu_{\mathbf{j}}(\Gamma) = \mu(\phi_{\mathbf{j}}(\Gamma)).$$

Since for all $f \in L_p(\Omega)$ we have that

$$\|f \circ \phi_{\mathbf{j}}\|_{L_p(\Gamma, \mu_{\mathbf{j}})} = \|f\|_{L_p(\phi_{\mathbf{j}}(\Gamma), \mu)},$$

condition (2.6) says, for $\ell \in \mathbb{Z}_m$, that on the set $\phi_\ell(\Omega)$ the $L_p(\Gamma, \mu)$ and $L_p(\phi_\ell(\Gamma), \mu_\ell)$ norms are equivalent.

Let $\mathcal{A} = \{\mathbf{A}_\ell : \ell \in \mathbb{Z}_m\}$ be a finite collection of $n \times n$ matrices. For every $\mathbf{j} = (j_k : k \in \mathbb{Z}_r) \in \mathbb{Z}_m^r$ and $r \in \mathbb{Z}_\infty$, we define the matrices $\mathbf{A}_{\mathbf{j}} = \mathbf{A}_{j_{r-1}} \cdots \mathbf{A}_{j_0}$ and the piecewise constant matrix-valued function $\mathbf{A}^{[r]} \in L_p^{n \times n}(\Omega)$

$$(2.10) \quad \mathbf{A}^{[r]} = \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \mathbf{A}_{\mathbf{j}} \chi_{\phi_{\mathbf{j}}(\Omega)}, \quad r \in \mathbb{Z}_\infty,$$

where, for $\Gamma \subseteq \Omega$, we write χ_Γ for the characteristic function of Γ . Note that the order of the matrices in the product (2.10) is the *reverse* of the order of the

mappings in the composition (2.2). From the upper bound in equation (2.6) and the injectivity of ϕ_ℓ , $\ell \in \mathbb{Z}_m$, it follows, for $r \in \mathbb{Z}_\infty$ and $k \in \mathbb{Z}_m$, that

$$(2.11) \quad \mathbf{A}^{[r+1]} \circ \phi_k = \mathbf{A}^{[r]} \mathbf{A}_k, \quad \mu\text{-a.e.}$$

Given any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we denote by $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{n \times n}$ their *tensor product*

$$\mathbf{a} \otimes \mathbf{b} = (a_\ell b_k : \ell, k \in \mathbb{Z}_n)$$

and use $\mathbf{1}$ for the vector in $\mathbb{R}^n$ whose entries are all 1.

Definition 2.1. A finite collection $\mathcal{A}$ of $n \times n$ matrices *admits a $L_p(\Omega, \mu)$ convergent subdivision scheme* if there exists a nontrivial vector field $\mathbf{f} \in L_p^n(\Omega, \mu)$ such that

$$(2.12) \quad \lim_{r \rightarrow \infty} \mathbf{A}^{[r]} = \mathbf{1} \otimes \mathbf{f}.$$

Whenever the subdivision scheme converges we call $\mathbf{f}$ its limit.

Later, we shall use the fact that the set of matrices

$$\mathbf{A}_\ell := \mathbf{H} := \frac{1}{n} \mathbf{1} \otimes \mathbf{1}, \quad \ell \in \mathbb{Z}_m,$$

admit an $L_p(\Omega, \mu)$ convergent subdivision scheme with limit $\mathbf{h} := n^{-1} \mathbf{1}$.

Proposition 2.2. *If $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$, then, for every $\ell \in \mathbb{Z}_m$, we have that*

$$(2.13) \quad \mathbf{f} \circ \phi_\ell = \mathbf{A}_\ell^T \mathbf{f}, \quad \mu\text{-a.e.}$$

Proof. We fix $\ell \in \mathbb{Z}_m$. The inequalities

$$\begin{aligned} & \left\| \mathbf{1} \otimes \mathbf{f} \circ \phi_\ell - \mathbf{1} \otimes (\mathbf{A}^T \mathbf{f}) \right\|_p = \left\| \mathbf{1} \otimes \mathbf{f} \circ \phi_\ell - (\mathbf{1} \otimes \mathbf{f}) \mathbf{A}_\ell \right\|_p \\ & \leq \left\| \mathbf{1} \otimes \mathbf{f} \circ \phi_\ell - \mathbf{A}^{[r+1]} \circ \phi_\ell \right\|_p + \left\| (\mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]}) \mathbf{A}_\ell \right\|_p \\ & \leq D \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \|\mathbf{A}_\ell\| \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right\|_p \end{aligned}$$

yield the result. $\square$

We call any solution of equation (2.13) a *refinable vector field* associated to the *refinement matrices* $\mathcal{A}$. Therefore, the limit of a convergent subdivision scheme is always a refinable vector field.

We say that the measure μ is *strictly positive* on Ω if, for all $x \in \Omega$ and $\delta > 0$, the measure of the ball $B(x; \delta) = \{y : y \in \Omega, d(x, y) \leq \delta\}$ is strictly positive, that is, $\mu(B(x; \delta)) > 0$. Observe that a strictly positive measure μ satisfies, for any continuous function $f \in C(\Omega)$ and $x \in \Omega$, the inequality

$$\left| \frac{1}{\mu(B(x; \delta))} \int_{B(x; \delta)} f(y) d\mu(y) - f(x) \right| \leq \omega_\infty(f; \delta),$$

where ω_∞ denotes the modulus of continuity of f given by

$$\omega_\infty(f; \delta) = \max \{|f(x) - f(y)| : x, y \in \Omega, d(x, y) < \delta\}.$$

Since $\lim_{\delta \rightarrow 0} \omega_\infty(f; \delta) = 0$ for any $f \in C(\Omega)$ we conclude that whenever $f \in C(\Omega)$ vanishes μ -a.e. on Ω it must be identically zero on Ω . This fact leads us to the next result.

Corollary 2.3. *Suppose that μ is strictly positive on Ω and that $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$. If $\mathbf{f}$ is continuous on Ω , then, for every $x \in \Omega$,*

$$\mathbf{f}(x) \neq 0.$$

Proof. Since $\mathbf{f}$ and ϕ_ℓ , $\ell \in \mathbb{Z}_m$, are continuous we have, for any $y \in \Omega$ and $\mathbf{j} \in \mathbb{Z}_m^r$, $r \in \mathbb{Z}_\infty$, that

$$(2.14) \quad \mathbf{f}(\phi_{\mathbf{j}}(y)) = \mathbf{A}_{\mathbf{j}}^T \mathbf{f}(y).$$

Suppose that there exists a $y \in \Omega$ such that $\mathbf{f}(y) = 0$. For any $x \in \Omega$ we choose a sequence $\mathbf{j}^r \in \mathbb{Z}_m^r$, $r \in \mathbb{Z}_\infty$, such that $\lim_{r \rightarrow \infty} \phi_{\mathbf{j}^r}(y) = x$. Since $\mathbf{f}(y) = 0$ equation (2.14) and the continuity of $\mathbf{f}$ imply that

$$\mathbf{f}(x) = \lim_{r \rightarrow \infty} \mathbf{f}(\phi_{\mathbf{j}^r}(y)) = \mathbf{A}_{\mathbf{j}^r}^T \mathbf{f}(y) = 0,$$

which violates the assumption that $\mathcal{A}$ admits a convergent subdivision scheme. This contradiction proves the result. $\square$

Corollary 2.4. *Suppose that $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$. If at least two of the matrices in $\mathcal{A}$ are nonsingular, then $\mathbf{f} \neq 0$, μ -a.e.*

Proof. Let $\mathbf{A}_\ell$ and $\mathbf{A}_k$, $\ell \neq k \in \mathbb{Z}_m$, be two nonsingular matrices in $\mathcal{A}$ and set $\Gamma = \{x \in \Omega : \mathbf{f}(x) = 0\}$. From the refinement equation (2.13) and the invertibility of $\mathbf{A}_\ell$ and $\mathbf{A}_k$ we conclude that

$$\Gamma = \phi_\ell(\Gamma) = \phi_k(\Gamma)$$

and therefore, by (2.8) we obtain

$$\mu(\Gamma) \leq \mu(\phi_\ell(\Gamma) \cap \phi_k(\Gamma)) \leq \mu(\phi_\ell(\Omega) \cap \phi_k(\Omega)) = 0.$$

$\square$

3. μ -UNIFORMITY

For $\ell \in \mathbb{Z}_m$ we define the set

$$(3.1) \quad \Omega_\ell^r = \bigcup \{ \phi_{\mathbf{j}}(\Omega) : \mathbf{j} = (j_k : k \in \mathbb{Z}_{r+1}) \in \mathbb{Z}_m^{r+1}, j_r = \ell \}$$

and note that

$$\Omega = \bigcup_{\ell \in \mathbb{Z}_m} \Omega_\ell^r.$$

From (2.4) it follows that

$$\lim_{r \rightarrow \infty} \Omega_\ell^r = \Omega, \quad \ell \in \mathbb{Z}_m,$$

in the Hausdorff metric on X . We require an additional property.

Definition 3.1. We say that the family Φ of contractions is μ -uniform if for any μ -measurable set $\Gamma \subseteq \Omega$, $\mu(\Gamma) > 0$ and $\ell \in \mathbb{Z}_m$ the condition

$$(3.2) \quad \liminf_{r \rightarrow \infty} \mu(\Gamma \cap \Omega_\ell^r) > 0$$

is satisfied.

Let us first provide sufficient conditions on the family of contractions Φ and the measure μ so that Φ is μ -uniform.

Lemma 3.2. *If there exist positive constants ρ_ℓ , $\ell \in \mathbb{Z}_m$, such that $\mu_\ell = \rho_\ell \mu$, $\ell \in \mathbb{Z}_m$, then, for any $f \in L_p(\Omega)$ and $\ell \in \mathbb{Z}_m$, we have that*

$$(3.3) \quad \lim_{r \rightarrow \infty} \|f\|_{L_p(\Omega_\ell^r)} = \rho_\ell^{1/p} \|f\|_{L_p(\Omega)}.$$

Choosing any μ -measurable $\Gamma \subseteq \Omega$ and the function $f = \chi_\Gamma$ in (3.3) yields the following result.

Corollary 3.3. *If there exist positive constants ρ_ℓ , $\ell \in \mathbb{Z}_m$, such that $\mu_\ell = \rho_\ell \mu$, $\ell \in \mathbb{Z}_m$, then Φ is μ -uniform.*

A change of variables arguments implies the following result.

Corollary 3.4. *If $X = \mathbb{R}^s$, Φ consists of affine mappings and μ is the Lebesgue measure on $\mathbb{R}^s$, then Φ is μ -uniform.*

To prove Lemma 3.2 we need another ancillary fact.

Lemma 3.5. *For any $f \in C(\Omega)$ there exists a sequence φ^r , $r \in \mathbb{Z}_\infty$, of piecewise constant functions of the form*

$$(3.4) \quad \varphi^r = \sum_{j \in \mathbb{Z}_m^r} c_j \chi_{\phi_j(\Omega)}, \quad c_j \in \mathbb{R}, \quad j \in \mathbb{Z}_m^r,$$

such that

$$\lim_{r \rightarrow \infty} \|f - \varphi^r\|_{L_p(\Omega)} = 0.$$

Proof. We choose some $y \in \Omega$ and set

$$\varphi^r(f) = \sum_{j \in \mathbb{Z}_m^r} (f \circ \phi_j)(y) \chi_{\phi_j(\Omega)}.$$

Let $\delta_r = \max \{d(x, y) : x, y \in \phi_j(\Omega), j \in \mathbb{Z}_m^r\}$ and note that, since Ω is bounded and Φ is a family of contractions, $\lim_{r \rightarrow \infty} \delta_r = 0$. Note that, for any $r \in \mathbb{Z}_\infty$, we can write

$$f(x) = \sum_{j \in \mathbb{Z}_m^r} f(x) \chi_{\phi_j(\Omega)}(x), \quad x \in \Omega,$$

and therefore

$$\begin{aligned} \|\varphi^r - f\|_{L_p(\Omega)} &= \left(\int_\Omega \left| \sum_{j \in \mathbb{Z}_m^r} ((f \circ \phi_j)(y) - f(x)) \chi_{\phi_j(\Omega)}(x) \right|^p d\mu(x) \right)^{1/p} \\ &= \left(\sum_{j \in \mathbb{Z}_m^r} \int_{\phi_j(\Omega)} |(f \circ \phi_j)(y) - f(x)|^p d\mu(x) \right)^{1/p} \\ &\leq \omega_\infty(f; \delta_r) (\mu(\Omega))^{1/p}. \end{aligned}$$

Thus, the result follows. $\square$

Proof of Lemma 3.2. For $r \in \mathbb{Z}_\infty$ let φ^r be a function of the form (3.3). Associated to any $j = (j_k : k \in \mathbb{Z}_r)$ we introduce the constant $\rho_j = \prod_{k \in \mathbb{Z}_r} \rho_{j_k}$ and note that $\rho_j = \frac{\mu_j(\Omega)}{\mu(\Omega)}$. For $\ell \in \mathbb{Z}_m$ we have that

$$\|\varphi^r\|_{L_p(\Omega_\ell^r)}^p = \sum_{j \in \mathbb{Z}_m^r} |c_j|^p \rho_j \rho_\ell \mu(\Omega) = \rho_\ell \sum_{j \in \mathbb{Z}_m^r} |c_j|^p \rho_j \mu(\Omega) = \rho_\ell \|\varphi^r\|_{L_p(\Omega)}^p.$$

Next, we choose $g \in C(\Omega)$. By Lemma 3.5, given any $\varepsilon > 0$ there is an $r_0 \in \mathbb{Z}_\infty$ such that for all $r > r_0$ we have that $\|\varphi^r(g) - g\|_{L_p(\Omega)} < \varepsilon/2$ and therefore, since $\rho_\ell \leq 1$, $\ell \in \mathbb{Z}_m$,

$$\begin{aligned} \|g\|_{L_p(\Omega_\ell^r)} &\leq \|\varphi^r(g)\|_{L_p(\Omega_\ell^r)} + \|g - \varphi^r(g)\|_{L_p(\Omega)} \leq \rho_\ell^{1/p} \|\varphi^r(g)\|_{L_p(\Omega)} + \frac{\varepsilon}{2} \\ &\leq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} + \varepsilon. \end{aligned}$$

In exactly the same way we also obtain

$$\|g\|_{L_p(\Omega_{\ell,r})} \geq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} - \varepsilon$$

which proves (3.3) for continuous functions. For $f \in L_p(\Omega)$ we let $\varepsilon > 0$ and pick $g \in C(\Omega)$ such that $\|f - g\|_{L_p(\Omega)} < \varepsilon/3$. By what we just proved, we can also choose an $r_0 \in \mathbb{Z}_\infty$ such that, for any $r > r_0$ and $\ell \in \mathbb{Z}_m$,

$$\left| \|g\|_{L_p(\Omega_\ell^r)} - \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} \right| \leq \varepsilon/3.$$

Hence, for $r > r_0$ and $\ell \in \mathbb{Z}_m$ we obtain

$$\|f\|_{L_p(\Omega_\ell^r)} \leq \|g\|_{L_p(\Omega_\ell^r)} + \frac{\varepsilon}{3} \leq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} + \frac{2}{3}\varepsilon \leq \rho_\ell^{1/p} \|f\|_{L_p(\Omega)} + \varepsilon,$$

and likewise

$$\|f\|_{L_p(\Omega_\ell^r)} \geq \rho_\ell^{1/p} \|g\|_{L_p(\Omega)} - \varepsilon,$$

which proves the lemma. $\square$

Based on property (3.2) we have the following result.

Lemma 3.6. *If Φ is μ -uniform in Ω , then, for any $f \in L_p(\Omega)$ for which there exists $\ell \in \mathbb{Z}_m$ such that*

$$(3.5) \quad \lim_{r \rightarrow \infty} \|f\|_{L_p(\Omega_\ell^r)} = 0,$$

it follows that $f = 0$, μ -a.e.

Proof. Let $f \in L_p(\Omega)$ satisfy (3.5) and suppose that $f \neq 0$. Then there exists a constant $\delta > 0$ such that $\mu(F_\delta) > 0$, where

$$F_\delta := \{x : |f(x)| > \delta\}.$$

Therefore, we obtain

$$\|f\|_{L_p(\Omega_\ell^r)} \geq \delta (\mu(F_\delta \cap \Omega_\ell^r))^{1/p}.$$

Since Φ is μ -uniform, this yields the contradiction

$$0 = \liminf_{r \rightarrow \infty} \|f\|_{L_p(\Omega_\ell^r)} \geq \delta \liminf_{r \rightarrow \infty} (\mu(F_\delta \cap \Omega_\ell^r))^{1/p} > 0.$$

$\square$

4. CONVERGENCE OF SUBDIVISION SCHEMES

Our first result in this section provides a necessary condition for the convergence of the matrix subdivision scheme.

Proposition 4.1. *If Φ is μ -uniform and $\mathcal{A}$ admits a convergent subdivision scheme, then*

$$(4.1) \quad \mathbf{A}_\ell \mathbf{1} = \mathbf{1}, \quad \ell \in \mathbb{Z}_m.$$

Proof. For any $\mathbf{j} \in \mathbb{Z}_m^{r+1}$ and μ -a.e. $x \in \phi_{\mathbf{j}}(\Omega)$ we conclude from the definition (2.10) that $\mathbf{A}^{[r+1]}(x) = \mathbf{A}_{\mathbf{j}}$. Hence, for $\ell \in \mathbb{Z}_m$ and $r \in \mathbb{Z}_\infty$ we get that

$$(4.2) \quad \mathbf{A}^{[r+1]} \Big|_{\Omega_\ell^r} = \mathbf{A}_\ell \mathbf{A}^{[r]} \Big|_{\Omega_\ell^r}, \quad \mu\text{-a.e.},$$

and consequently we obtain

$$\begin{aligned} \|(\mathbf{1} - \mathbf{A}_\ell \mathbf{1}) \otimes \mathbf{f}\|_{L_p(\Omega_\ell^r)} &= \|\mathbf{1} \otimes \mathbf{f} - \mathbf{A}_\ell (\mathbf{1} \otimes \mathbf{f})\|_{L_p(\Omega_\ell^r)} \\ &\leq \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \left\| \mathbf{A}^{[r+1]} - \mathbf{A}_\ell (\mathbf{1} \otimes \mathbf{f}) \right\|_{L_p(\Omega_\ell^r)} \\ &= \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \left\| \mathbf{A}_\ell (\mathbf{A}^{[r]} - \mathbf{1} \otimes \mathbf{f}) \right\|_{L_p(\Omega_\ell^r)} \\ &\leq \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r+1]} \right\|_p + \left\| \mathbf{A}_\ell^T \right\| \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right\|_p. \end{aligned}$$

Since the upper bound in the last inequality converges to zero for $r \rightarrow \infty$, Lemma 3.6 proves the result. $\square$

To state the next result we use $\langle \cdot, \cdot \rangle$ for the standard inner product on $\mathbb{R}^n$.

Corollary 4.2. *If the hypothesis of Proposition 4.1 is satisfied and $\mathbf{f}$ is the limit of the subdivision scheme $\mathcal{A}$, then*

$$(4.3) \quad \langle \mathbf{f}, \mathbf{1} \rangle = 1, \quad \mu\text{-a.e.}$$

Proof. By Proposition 4.1, for $r \in \mathbb{Z}_\infty$, $\mathbf{A}^{[r]} \mathbf{1} = \mathbf{1}$, μ -a.e., and therefore

$$\mathbf{1} = \lim_{r \rightarrow \infty} \mathbf{A}^{[r]} \mathbf{1} = (\mathbf{1} \otimes \mathbf{f}) \mathbf{1},$$

which proves equation (4.3). $\square$

For the next corollary we introduce, for μ -a.e. $y \in \Omega$, $f \in L_p(\Omega)$ and $r \in \mathbb{Z}_\infty$, the following function on Ω which acts like a modulus of continuity,

$$(4.4) \quad \omega_r(f)(y) = \left(\sum_{\mathbf{j} \in \mathbb{Z}_m^r} \int_{\phi_{\mathbf{j}}(\Omega)} |f(x) - (f \circ \phi_{\mathbf{j}})(y)|^p d\mu(x) \right)^{1/p},$$

and extend it to vector fields $\mathbf{f} = (f_k : k \in \mathbb{Z}_n)$ by setting

$$\omega_r(\mathbf{f}) = \sum_{k \in \mathbb{Z}_n} \omega_r(f_k).$$

Corollary 4.3. *If $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$, then*

$$\lim_{r \rightarrow \infty} \omega_r(\mathbf{f}) = 0, \quad \mu\text{-a.e.}$$

Proof. We have μ -a.e. $y \in \Omega$ such that

$$(4.5) \quad \omega_r(\mathbf{f})(y) = \left\| \mathbf{f}^T(y) \left(\mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right) \right\|_p = \left\| \mathbf{f} - \left(\mathbf{A}^{[r]} \mathbf{f} \right)(y) \right\|_p.$$

The first identity follows readily from the fact that $\mathbf{f}$ is the limit function of a convergent subdivision scheme and thus satisfies (2.13) and (4.3). For the second identity we compute that

$$\begin{aligned} \left\| \mathbf{f} - \left(\mathbf{A}^{[r]} \mathbf{f} \right)(y) \right\|_p &= \sum_{k \in \mathbb{Z}_n} \left(\int_{\Omega} \left| f_k(x) - \left(\mathbf{A}^{[r]} \mathbf{f} \right)_k(y) \right|^p d\mu(x) \right)^{1/p} \\ &= \sum_{k \in \mathbb{Z}_n} \left(\sum_{\mathbf{j} \in \mathbb{Z}_m^{r\phi_j}(\Omega)} \int_{\Omega} \left| f_k(x) - (\mathbf{A}_{\mathbf{j}} \mathbf{f})_k(y) \right|^p d\mu(x) \right)^{1/p} \\ &= \omega_r(\mathbf{f})(y). \end{aligned}$$

□

This fact is a consequence of Theorem 4.8, however, we put it here since it has a partial converse which we state next.

Proposition 4.4. *Suppose that $\mathbf{f} \in L_p(\Omega)$ is a nontrivial vector field which is refinable relative to a family $\mathcal{A}$ of matrices, which satisfies the equation $\langle \mathbf{1}, \mathbf{f} \rangle = 1$ μ -a.e. and the conditions*

$$\lim_{r \rightarrow \infty} \omega_r(\mathbf{f}) = 0, \quad \mu\text{-a.e.},$$

and

$$\mathbb{R}^n = \text{span} \{ \mathbf{f}(y) : \mu\text{-a.e. } y \in \Omega \}.$$

Then $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$.

Proof. This fact follows conveniently from equation (4.5). □

Now we will characterize the convergence of matrix subdivision schemes. For this purpose we define the *difference operator* $\mathbf{D} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ by setting, for any $\mathbf{c} = (c_\ell : \ell \in \mathbb{Z}_n)$,

$$\mathbf{D}\mathbf{c} = (c_{\ell+1} - c_\ell : \ell \in \mathbb{Z}_{n-1}).$$

Its representation as an $(n-1) \times n$ matrix is

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}.$$

Observe the following fact.

Lemma 4.5. *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$.*

1. $\mathbf{A}\mathbf{1} = \mathbf{0}$ if and only if there is a matrix $\mathbf{B} \in \mathbb{R}^{n \times (n-1)}$ such that $\mathbf{A} = \mathbf{B}\mathbf{D}$.
2. If $\mathbf{A}\mathbf{1} = \mathbf{1}$, then there is a matrix $\mathbf{B} \in \mathbb{R}^{(n-1) \times (n-1)}$ such that $\mathbf{D}\mathbf{A} = \mathbf{B}\mathbf{D}$.

We pattern the following notion after Jia [3].

Definition 4.6. Let $\mathcal{A}$ be a finite collection of $n \times n$ matrices of cardinality m . The p -joint spectral radius of $\mathcal{A}$ relative to Φ and μ is defined as

$$(4.6) \quad \rho_p(\mathcal{A}) = \limsup_{r \rightarrow \infty} \left(\sum_{j \in \mathbb{Z}_m^r} \mu_j(\Omega) \|\mathbf{A}_j\|^p \right)^{1/rp}.$$

Theorem 4.7. Let Φ be μ -uniform. Then $\mathcal{A}$ admits an $L_p(\Omega, \mu)$ convergent subdivision scheme if and only if the following two conditions hold:

1.

$$(4.7) \quad \mathbf{A}_\ell \mathbf{1} = \mathbf{1}, \quad \ell \in \mathbb{Z}_m.$$

2. The matrices $\mathbf{B}_\ell \in \mathbb{R}^{n-1 \times n-1}$, $\ell \in \mathbb{Z}_m$, defined by $\mathbf{D}\mathbf{A}_\ell = \mathbf{B}_\ell \mathbf{D}$, satisfy

$$(4.8) \quad \rho_p(\mathcal{B}) < 1.$$

Proof. Let us assume that (4.7) and (4.8) are satisfied and show that $\mathcal{A}$ admits a convergent subdivision scheme. To this end, we consider a sequence of vector fields $\mathbf{h}_r$, $r \in \mathbb{Z}_\infty$, on Ω defined as

$$(4.9) \quad \mathbf{h}_r^T = \frac{1}{n} \mathbf{1}^T \mathbf{A}^{[r]}$$

with the convention that $\mathbf{A}^{[0]} = \mathbf{I}$, the identity matrix. Now, for $r \in \mathbb{Z}_\infty$ and $\mathbf{c} \in \mathbb{R}^n$, using the vector $\mathbf{j}^r = (j_k : k \in \mathbb{Z}_r)$, we obtain the equation

$$\begin{aligned} \|\langle \mathbf{c}, \mathbf{h}_{r+1} - \mathbf{h}_r \rangle\|_{L_p(\Omega)}^p &= \sum_{j \in \mathbb{Z}_m^{r+1}} \|\langle \mathbf{c}, \mathbf{h}_{r+1} - \mathbf{h}_r \rangle\|_{L_p(\phi_j(\Omega), \mu)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} \|\langle \mathbf{c}, \mathbf{h}_{r+1} \circ \phi_j - \mathbf{h}_r \circ \phi_j \rangle\|_{L_p(\Omega, \mu_j)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} \left\| \left\langle \mathbf{c}, \mathbf{A}_{j_r}^T (\mathbf{A}_{j_r} - \mathbf{H})^T \mathbf{1} \right\rangle \right\|_{L_p(\Omega, \mu_j)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} \left\| \langle (\mathbf{A}_{j_r} - \mathbf{H}) \mathbf{A}_{j_r} \mathbf{c}, \mathbf{1} \rangle \right\|_{L_p(\Omega, \mu_j)}^p \\ &= \sum_{j \in \mathbb{Z}_m^{r+1}} n^{-p} \left| \langle (\mathbf{A}_{j_r} - \mathbf{H}) \mathbf{A}_{j_r} \mathbf{c}, \mathbf{1} \rangle \right|^p \mu_j(\Omega). \end{aligned}$$

Observe for any $k \in \mathbb{Z}_m$ that $(\mathbf{A}_k - \mathbf{H}) \mathbf{1} = \mathbf{0}$. Thus, there exists a matrix $\mathbf{C}_k \in \mathbb{R}^{n \times n-1}$ such that $\mathbf{A}_k - \mathbf{H} = \mathbf{C}_k \mathbf{D}$ and we conclude, for $\mathbf{j} \in \mathbb{Z}_m^r$ and $k \in \mathbb{Z}_m$, that

$$(\mathbf{A}_k - \mathbf{H}) \mathbf{A}_j = \mathbf{C}_k \mathbf{D} \mathbf{A}_j = \mathbf{C}_k \mathbf{B}_j \mathbf{D}.$$

Consequently there exists a positive constant α such that

$$\|\langle \mathbf{c}, \mathbf{h}_{r+1} - \mathbf{h}_r \rangle\|_p^p \leq \alpha \|\mathbf{c}\|_p^p \sum_{j \in \mathbb{Z}_m^r} \mu_j(\Omega) \|\mathbf{B}_j\|^p.$$

Thus, for any σ such that $\rho_p(\mathcal{B}) < \sigma < 1$ there exists an $r_0 \in \mathbb{Z}_\infty$ and a positive constant β such that

$$(4.10) \quad \|\mathbf{h}_{r+1} - \mathbf{h}_r\|_p \leq \beta \sigma^r, \quad r \geq r_0.$$

Therefore, $\mathbf{h}_r$, $r \in \mathbb{Z}_\infty$, is a Cauchy sequence in $L_p^n(\Omega)$ which necessarily converges to some function $\mathbf{f} \in L_p^n(\Omega)$.

It remains to verify that $\mathbf{f}$ is indeed the limit of the subdivision scheme. For this purpose, we observe that, for $r \in \mathbb{Z}_\infty$,

$$(4.11) \quad \left\| D\mathbf{A}^{[r]} \right\|_p = \left\| \mathbf{B}^{[r]} D \right\|_p \leq \|D\| \left\| \mathbf{B}^{[r]} \right\|_p = \|D\| \left(\sum_{\mathbf{j} \in \mathbb{Z}_m^r} \mu_{\mathbf{j}}(\Omega) \|\mathbf{B}_{\mathbf{j}}\|^p \right)^{1/p}.$$

Hence, we conclude that

$$(4.12) \quad \lim_{r \rightarrow \infty} \left\| D\mathbf{A}^{[r]} \right\|_p = 0.$$

In other words, we have shown, for any $\mathbf{y} \in \mathbb{R}^n$ such that $\langle \mathbf{y}, \mathbf{1} \rangle = 0$, that

$$(4.13) \quad \lim_{r \rightarrow \infty} \mathbf{y}^T \mathbf{A}^{[r]} = \mathbf{y}^T (\mathbf{1} \otimes \mathbf{f}).$$

Moreover, (4.9) yields for all $r \in \mathbb{Z}_\infty$ that equation (4.13) holds with $\mathbf{y}$ replaced by $\mathbf{1}$ and therefore we have confirmed that $\mathbf{f}$ is the limit of the subdivision scheme $\mathcal{A}$.

For the converse, we assume that $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$. Then, for any $\mathbf{j} \in \mathbb{Z}_m^r$ and $k, \ell \in \mathbb{Z}_n$, we have that

$$\begin{aligned} \mu_{\mathbf{j}}(\Omega) \left\| (D\mathbf{A}_{\mathbf{j}})_{k,\ell} \right\|^p &= \left\| (\mathbf{A}_{\mathbf{j}})_{k+1,\ell} - (\mathbf{A}_{\mathbf{j}})_{k,\ell} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega), \mu)}^p \\ &\leq 2^p \left(\left\| (\mathbf{A}_{\mathbf{j}})_{k+1,\ell} - f_\ell \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p + \left\| (\mathbf{A}_{\mathbf{j}})_{k,\ell} - f_\ell \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p \right) \\ &= 2^p \left(\left\| (\mathbf{A}^{[r]})_{k+1,\ell} - f_\ell \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p + \left\| (\mathbf{A}^{[r]})_{k,\ell} - f_\ell \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p \right). \end{aligned}$$

Therefore, there exists a positive constant γ such that

$$\mu_{\mathbf{j}}(\Omega) \|\mathbf{B}_{\mathbf{j}} D\|^p \leq \gamma \left\| \mathbf{A}^{[r]} - \mathbf{1} \otimes \mathbf{f} \right\|_{L_p(\phi_{\mathbf{j}}(\Omega))}^p.$$

Consequently, summing both sides of this inequality over $\mathbf{j}$ and taking into account the fact that D is surjective, we conclude that there is a positive constant η such that

$$(4.14) \quad \sum_{\mathbf{j} \in \mathbb{Z}_m^r} \mu_{\mathbf{j}}(\Omega) \|\mathbf{B}_{\mathbf{j}} D\|^p \leq \eta \left\| \mathbf{A}^{[r]} - \mathbf{1} \otimes \mathbf{f} \right\|_p^p$$

which proves the claim. $\square$

We finally show that a convergent subdivision scheme converges exponentially fast and that its limit enjoys Hölder regularity.

Theorem 4.8. *If $\mathcal{A}$ admits a convergent subdivision scheme with limit $\mathbf{f}$, then*

$$\lim_{r \rightarrow \infty} \omega_r^{1/r}(\mathbf{f}) \leq \rho_p(\mathcal{B}), \quad \mu\text{-a.e.}$$

Proof. For any σ with $\rho_p(\mathcal{B}) < \sigma < 1$ there exist, by (4.10), numbers $r_0 \in \mathbb{Z}_\infty$ and $\alpha > 0$ such that, for $r > r_0$,

$$(4.15) \quad \|\mathbf{f} - \mathbf{h}_r\|_p = \alpha \sigma^r.$$

Let $\mathbf{P} := \mathbf{H} - \mathbf{I}$ and observe that $\mathbf{P}\mathbf{1} = 0$. Hence, there exists a matrix $\mathbf{Q} \in \mathbb{R}^{n \times (n-1)}$ such that $\mathbf{P} = \mathbf{Q}D$. Using (4.9) and (4.11) we find an $r_1 \in \mathbb{Z}_\infty$ and $\beta > 0$ such that, for $r \geq r_1$,

$$(4.16) \quad \left\| \mathbf{1} \otimes \mathbf{h}_r - \mathbf{A}^{[r]} \right\|_p = \left\| \mathbf{P} \mathbf{A}^{[r]} \right\|_p = \left\| \mathbf{Q} D \mathbf{A}^{[r]} \right\|_p \leq \beta \sigma^r(\mathcal{B}).$$

Thus, there is a constant $\gamma > 0$ such that, for $r \geq \max\{r_0, r_1\}$,

$$\left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right\|_p \leq \left\| \mathbf{1} \otimes \mathbf{f} - \mathbf{1} \otimes \mathbf{h}_r \right\|_p + \left\| \mathbf{1} \otimes \mathbf{h}_r - \mathbf{A}^{[r]} \right\|_p \leq \gamma \sigma^r.$$

Hence, we have, for $r \geq \max\{r_0, r_1\}$, that

$$\omega_r(\mathbf{f})(y) = \left\| \mathbf{f}^T(y) \left(\mathbf{1} \otimes \mathbf{f} - \mathbf{A}^{[r]} \right) \right\|_p \leq \gamma \sum_{k \in \mathbb{Z}_n} |f_k(y)| \sigma^r.$$

Since $\mathbf{f} \in L_p(\Omega)$, it is finite μ -a.e. and hence, for μ -a.e. $y \in \Omega$ we conclude that

$$\lim_{r \rightarrow \infty} \omega_r^{1/r}(\mathbf{f})(y) \leq \sigma.$$

However, σ was chosen arbitrarily greater than $\rho_p(\mathcal{B})$. This proves the result. $\square$

REFERENCES

1. Z. Chen, C. A. Micchelli, and Yuesheng Xu. A construction of interpolating wavelets on invariant sets. *Math. Comp.*, **68** (1999), 1569–1587. MR **99m**:65017
2. J. E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. J.*, **30** (1981), 713–747. MR **82h**:49026
3. R.-Q. Jia. Subdivision schemes in l_p spaces. *Advances Comput. Math.*, **3** (1995), 309–341. MR **96d**:65028
4. C. A. Micchelli and H. Prautzsch. Uniform refinement of curves. *Linear Algebra Appl.*, **114** (1989), 841–870. MR **90k**:65088
5. C. A. Micchelli, T. Sauer, and Yuesheng Xu. A construction of refinable sets for interpolating wavelets. *Results in Mathematics*, **34** (1998), 359–372. MR **99h**:42063
6. C. A. Micchelli and Yuesheng Xu. Using the matrix refinement equation for the construction of wavelets on invariant sets. *Appl. Comp. Harmonic Anal.*, **1** (1994), 391–401. MR **96k**:42044
7. C. A. Micchelli and Yuesheng Xu. Reconstruction and decomposition algorithms for biorthogonal multiwavelets. *Multidimensional Systems and Signal Processing*, **8** (1997), 31–69. MR **98g**:94007
8. C. A. Micchelli. *Mathematical Aspects of Geometric Modeling*, volume 65 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. SIAM, 1995. MR **95i**:65036
9. K. R. Parthasarathy. *Probability Measures on Metric Spaces*. Academic Press, 1967. MR **37**:2271

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